Solving problems with velocity-dependent forces

Charge particles in a magnetic field.

↳ better than Runge-Kutta

$$m \frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}(\vec{r})$$

for $q = -e$ for electrons

1) $$\frac{d\vec{v}}{dt} = \left(-\frac{e}{m}\right) \vec{v} \times \vec{B} = \frac{e}{m} \vec{B}(r) \times \vec{v}$$

2) $$\frac{d\vec{r}}{dt} = \vec{v} \equiv \frac{\vec{p}}{m}$$

For any $f(\vec{r}, \vec{v})$: $$\frac{df}{dt} = \frac{\partial f}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial t} + \frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial \vec{v}}{\partial t}$$

$$= \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \frac{e}{m} B (\hat{B} \times \vec{v}) \cdot \frac{\partial}{\partial \vec{v}}$$

define $\omega(\vec{r}) = \frac{e}{m} B(\vec{r})$

cyclotron frequency

$$\frac{df}{dt} = \Big(\underbrace{\vec{v} \cdot \frac{\partial}{\partial r}}_{= T} + \underbrace{\omega (\hat{B} \times \vec{v}) \cdot \frac{\partial}{\partial \vec{v}}}_{= V}\Big) f$$

$$f(t) = e^{t(T+V)} f(0) \qquad = e^{t(T+V)} f$$

the approximate of

$$e^{\Delta t (T+V)} = \prod_i e^{\Delta t a_i T} e^{\Delta t b_i V}$$

As before, we have

$$e^{\epsilon T}\begin{pmatrix}\vec{r}\\ \vec{v}\end{pmatrix} = \begin{pmatrix}\vec{r}+\epsilon\vec{v}\\ \vec{v}\end{pmatrix}$$

Knowing individual Lie transforms

$$e^{\epsilon V}\begin{pmatrix}\vec{r}\\ \vec{v}\end{pmatrix} = \begin{pmatrix}\vec{r}\\ \vec{v}(\vec{r},\vec{v},\epsilon)\end{pmatrix}$$

$\theta = \omega(\vec{r})\epsilon$

$$\vec{v}(\vec{r},\vec{v},\epsilon) = \vec{v} + \sin\theta\,(\hat{B}\times\vec{v}) + (1-\cos\theta)\,\hat{B}\times(\hat{B}\times\vec{v})$$

algorithms 1a, 1b, 2a, 2b, Sym4, Sym6, .. etc.

gauge-invariant, automatic.

Also for solving problems with dissipations:

h.o. with friction:

$$\ddot{x} = \boxed{-\omega^2 x - \beta v = \dot{v}}$$

$\omega = \sqrt{\frac{k}{m}}$

$v = \frac{dx}{dt}$

$$\boxed{\dot{x} = v} \Rightarrow e^{tT}x = \underline{x + vt}$$

$$\dot{v} = -\omega^2 x \Rightarrow e^{tV}v = \underline{v + at} \qquad a = -\omega^2 x$$

$$\dot{v} = -\beta v \Rightarrow e^{tD}v = \underline{e^{-\beta t}v(0)}$$

$$f(x,v,t) = e^{t(T+V+D)}f(x,v,0)$$

How to analyze complex orbits:

Poincaré section: Kepler orbit: planar orbit in 2D.

↑ con. of angular momentum

2 momenta + 2 positions

⇒ 4 degrees of freedom

Hamiltonian:

⇒ phase-space is 4D.

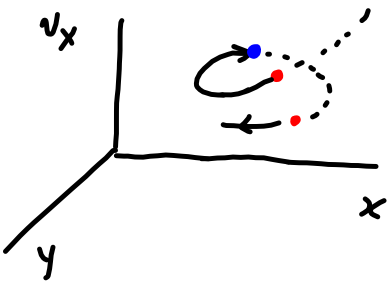
con. of energy $E = \frac{1}{2}m(v_x^2+v_y^2) + V(x,y) = \text{const}$

Solve for v_y ↪ eliminate one d.o.f.

⇒ phase-space is 3D.

is a "cut", a "section" in 3D ⇒ 2D ↙ Poincaré section

"cut" at $y=0$, a plane in x-v_x for $v_y > 0$



Poincaré section: the intersection pts of the orbit at $y=0$ with $v_y > 0$.