Solving problems with velocity-dependent forces

Charge particles in a magnetic field.

$$m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}(\vec{r})$$

for $q=-e$ for electrons

↪ better than Runge-Kutta

1) $$\frac{d\vec{v}}{dt} = \left(-\frac{e}{m}\right)\vec{v}\times\vec{B} = \frac{e}{m}\vec{B}(r)\times\vec{v}$$

2) $$\frac{d\vec{r}}{dt} = \vec{v} \equiv \frac{\vec{P}}{m}$$

For any $f(\vec{r},\vec{v})$: $$\frac{df}{dt} = \frac{\partial f}{\partial\vec{r}}\cdot\frac{\partial\vec{r}}{\partial t} + \frac{\partial f}{\partial\vec{v}}\cdot\frac{\partial\vec{v}}{\partial t}$$

$$= \vec{v}\cdot\frac{\partial f}{\partial\vec{r}} + \frac{e}{m}B(\hat{B}\times\vec{v})\cdot\frac{\partial}{\partial\vec{v}}$$

define $\omega(\vec{r}) = \frac{e}{m}B(\vec{r})$

cyclotron frequency

$$\frac{df}{dt} = \Big(\underbrace{\vec{v}\cdot\frac{\partial}{\partial r}}_{T} + \underbrace{\omega(\hat{B}\times\vec{v})\cdot\frac{\partial}{\partial\vec{v}}}_{V}\Big)f$$

$$f(t) = e^{t(T+V)}f(0) = e^{t(T+V)}f$$

the approximate of

$$e^{\Delta t(T+V)} = \prod_i e^{\Delta t a_i T}e^{\Delta t b_i V}$$

As before, we have

Knowing individual Lie transforms

$$e^{\epsilon T}\begin{pmatrix}\vec{r}\\ \vec{v}\end{pmatrix} = \begin{pmatrix}\vec{r}+\epsilon\vec{v}\\ \vec{v}\end{pmatrix}$$

$$e^{\epsilon V}\begin{pmatrix}\vec{r}\\ \vec{v}\end{pmatrix} = \begin{pmatrix}\vec{r}\\ \vec{v}(\vec{r},\vec{v},\epsilon)\end{pmatrix}$$

$\theta = \omega(\vec{r})\epsilon$

$$\vec{v}(\vec{r},\vec{v},\epsilon) = \vec{v} + \sin\theta\,(\hat{B}\times\vec{v}) + (1-\cos\theta)\,\hat{B}\times(\hat{B}\times\vec{v})$$

algorithms 1a, 1b, 2a, 2b, Sym 4, Sym 6, .. etc.

gauge-invariant, automatic.

Also for solving problems with dissipations:

h.o. with friction:

$$\ddot{x} = \boxed{-\omega^2 x - \beta v = \dot{v}} \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\boxed{\dot{x} = v} \Rightarrow e^{tT}x = \underline{x + vt} \qquad v = \frac{dx}{dt}$$

$$\dot{v} = -\omega^2 x \Rightarrow e^{tV}v = \underline{v + at} \qquad a = -\omega^2 x$$

$$\dot{v} = -\beta v \Rightarrow e^{tD}v = \underline{e^{-\beta t}v(0)}$$

$$f(x,v,t) = e^{t(T+V+D)}f(x,v,0)$$

How to analyze complex orbits:

Poincaré section: Kepler orbit: planar orbits in 2D.

↑ con. of angular momentum

2 momenta + 2 positions

⇒ 4 degrees of freedom

Hamiltonian:

⇒ phase-space is 4D.

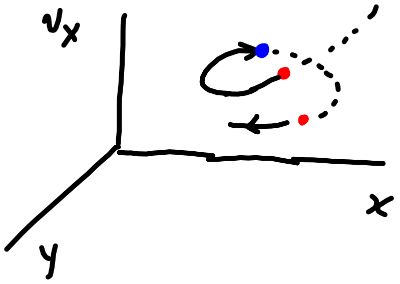
con. of energy $E = \frac{1}{2}m(v_x^2+v_y^2) + V(x,y) = \text{const}$

Solve for v_y ↳ eliminate one d.o.f.

⇒ phase-space is 3D.

is a "cut", a "section" in 3D ⇒ 2D ↙ Poincaré section

"cut" at $y=0$, a plane in $x-v_x$ for $v_y > 0$



Poincaré section: the intersection pts of the orbit at $y=0$ with $v_y > 0$.

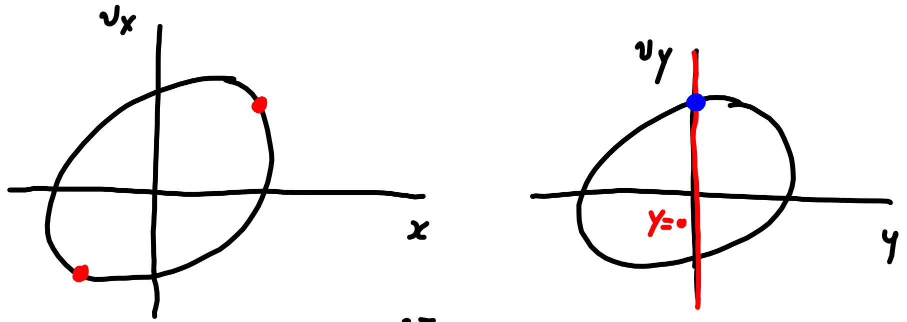
Consider the case of 2D HO

$x = x_0 \cos(\omega_x t)$ $\qquad v_x = -x_0 \omega_x \sin(\omega_x t)$

$y = y_0 \cos(\omega_y t)$ $\qquad v_y = -y_0 \omega_y \sin(\omega_y t)$

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{v_x}{x_0 \omega_x}\right)^2 = 1 \qquad \left(\frac{y}{y_0}\right)^2 + \left(\frac{v_y}{y_0 \omega_y}\right)^2 = 1$$

at $y=0$ + $v_y > 0$



$T_x = \frac{2\pi}{\omega_x}$ $\qquad T_y = \frac{2\pi}{\omega_y}$

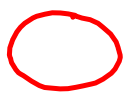
1) $T_x = T_y$ there is only one pt at the Poincaré section

2) $T_x = 2T_y$ " " two pts at the P.S.

3) $m T_x = n T_y \Rightarrow$ finite # of Points.

$\Rightarrow$ 

4) if $\frac{T_x}{T_y} = \frac{\omega_y}{\omega_x}$ is irrational



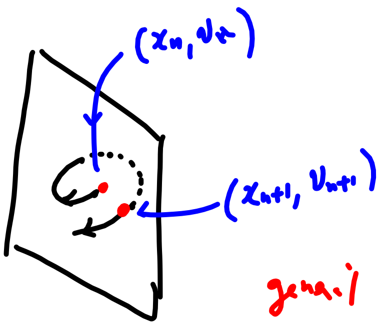
closed-loop or "Tori"

for non-integral systems, not like H.O.

5) chaos $\longrightarrow$ "fuzzy" set

The direct computation of sections.

Let (x_n, v_n) on the PS.



$\hookrightarrow (x_{n+1}, v_{n+1})$ by integrating the eq. of motion.

$$\left.\begin{aligned} x_{n+1} &= f(x_n, v_n) \\ v_{n+1} &= g(x_n, v_n) \end{aligned}\right\} \text{a mapping}$$

general property of such a mapping? on the PS

$|\det M| = 1$

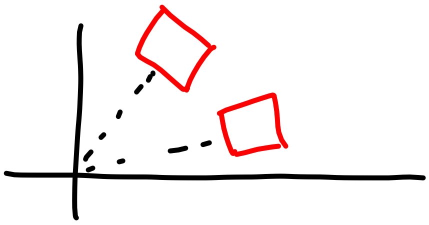
has to obey Liouville's Thm.

$$dx_n\, dv_n = \underset{\text{“}1\text{”}}{\det M}\, dx_{n+1}\, dv_{n+1}$$

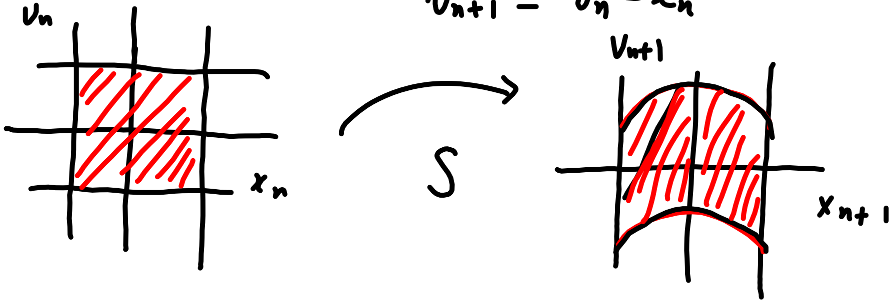
"area-preserving" map.

The study of area-preserving maps

1) Rotation R : $\begin{pmatrix} x_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_n \\ v_n \end{pmatrix}$



2) Shear S : $x_{n+1} = x_n$
$v_{n+1} = v_n - x_n^2$



Combine both: $H = RS$

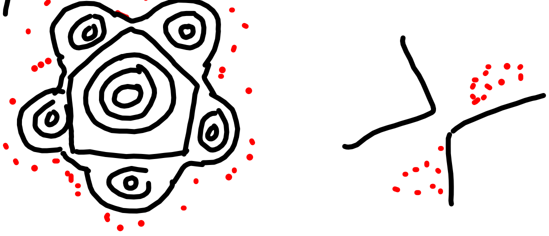
$\alpha = 76°$

Simplest Hamiltonian map.

Henon's area-preserving map.

$x_{n+1} = x_n \cos\alpha - (v_n - x_n^2)\sin\alpha$
$v_{n+1} = x_n \sin\alpha + (v_n - x_n^2)\cos\alpha$

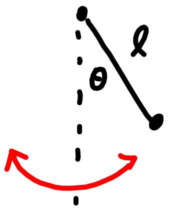
for (x, v) "small"



"fuzzy" chaotic region

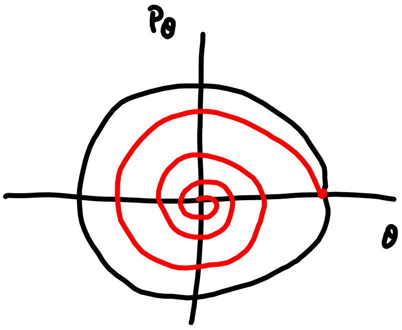
Henon's dissipative map

↳ losing energy



$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta - \gamma\frac{d\theta}{dt}$$

↖ air friction



The final, stationary state of the system is $p_\theta = 0$, $\theta = 0$

for more than one degree of freedom, the final state may not be just a or a set of fixed pts

Henon's dissipative map

$$x_{n+1} = y_n - a x_n^2 + 1$$
$$y_{n+1} = b x_n$$

$$\det\begin{pmatrix} -2a x_n & 1 \\ b & 0 \end{pmatrix} = -b$$

If $|b| < 1$, the phase-space will shrink.

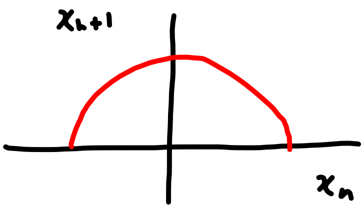
↳ it does not have a single or discrete set of fixed pts

In particular: $b = 0$ ↳ "strange attractor"

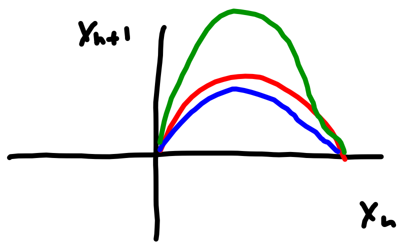
↳ self-similar

↳ fractal

$$x_{n+1} = 1 - a x_n^2$$



shift →



$$x_{n+1} = a x_n (1 - x_n)$$

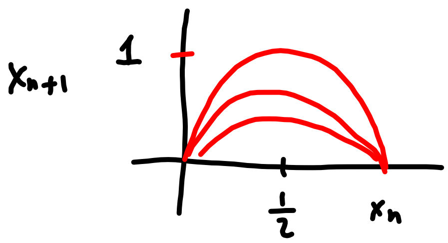
The logistic map

$$x_{n+1} = a x_n (1 - x_n) \qquad a = 4r$$

$0 < a < 4$

so that if $0 < x_n < 1$

then $0 < x_{n+1} < 1$



a mapping of $[0,1] \to [0,1]$

a model for population growth

$x_n \sim$ % of the population of a given environment.

1) when $x_n \ll 1$ $\quad x_{n+1} = a x_n$

$\Rightarrow \quad x_{n+1} = a^{n+1} x_0$ ↙ exponential growth

2) $x_{n+1} \propto 0 \sim (1 - x_n)$ if $x_n \approx 1$.

↪ decline near $x_n \approx 1$

for a given a, (growing rate)

what is the stationary population ?

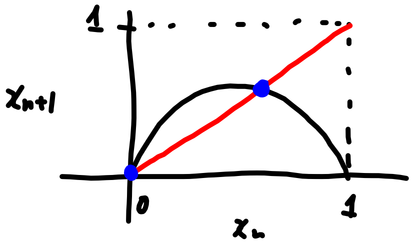
The stationary population x^*, is the fixed pt of the map

$$x^* = a x^*(1-x^*) = f(x^*)$$

$$1 = a(1-x^*)$$

$$\frac{1}{a} = 1-x^*$$

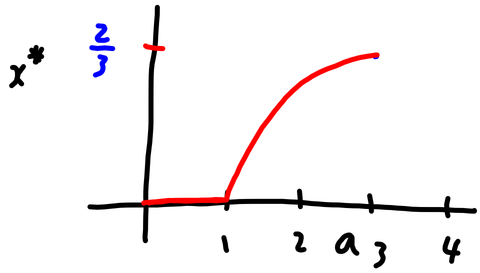
$$\Rightarrow \quad x^* = 1-\frac{1}{a}$$



two fixed pts

1) $x^* = 0$ $\quad 1 < a < 4$

2) $x^* = 1-\frac{1}{a}$ $\quad a > 1$



The stability of each fixed pt.

$$x_{n+1} = x^* + \epsilon_{n+1}$$

$$x_n = x^* + \epsilon_n$$

$$x_{n+1} = x^* + \epsilon_{n+1} = f(x^* + \epsilon_n) = f(x^*) + \epsilon_n f'(x^*)$$

$$\epsilon_{n+1} = f'(x^*)\,\epsilon_n$$

stability

$$|f'(x^*)| < 1$$

$$f(x) = a x(1-x) = ax - ax^2$$

$$f'(x) = a - 2ax$$

for $x^* = 0$; $f'(x^*) = a$

for $x^* = 1-\frac{1}{a}$ $\quad f'(x^*) = a - 2a\left(1-\frac{1}{a}\right) = a - 2a + 2 = 2 - a$

$|f'(x^*)| < 1$ for $1 < a < 3$

what happens at $a > 3$? what are the fixed pts ?