SMALL OSCILLATIONS
CHAPTER A

(9.12) \[ (y'/r) / \psi = m \]

(8.12) \[ m' + x \]

(7.12) \[ n + y \]

(6.12) \[ \psi = \psi' \]

(5.12) \[ \phi' = \phi \]

(4.12) \[ \psi = \psi' \]

(3.12) \[ \psi = \psi' \]

The expression can also be written

\[ x = \cos \psi \sin \psi \cos \alpha + \sin \psi \sin \alpha \cos \theta \\
\]

Two independent solutions of the linear differential equation (2.15) are

\[ x = \cos \psi \sin \psi \cos \theta \]

\[ x = \sin \psi \cos \psi \cos \theta \]

where \( \psi \) is a complex constant, and

\[ z = x \]

\[ (\psi / \psi') \]

\[ \psi' = \psi \]

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The corresponding equation of motion is $m \ddot{y} + c \dot{y} + ky = 0 \text{ or } \ddot{y} + \frac{c}{m} \dot{y} + \frac{k}{m} y = 0$.

The solution involves a further term $\phi(t)$, and the interpretation of the system involves a further term $\phi(t)$, which is a function of time, which we choose by $\phi(t) = e^{\lambda t}$. The eigenfunction of the linear system, which we choose by $\phi(t) = \text{constant} e^{\lambda t}$, is a function of time, which we choose by $\phi(t) = \text{constant} e^{\lambda t}$.

The solution now has the form

$$\ddot{y} + \frac{c}{m} \dot{y} + \frac{k}{m} y = 0 \Rightarrow \ddot{y} + \frac{c}{m} \dot{y} + \frac{k}{m} y = 0$$

The system now has the form

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The interpretation of the system involves a further term $\phi(t)$, which is a function of time, which we choose by $\phi(t) = \text{constant} e^{\lambda t}$.

**Solution:** The potential energy of the system is (1) within zero force, (2) additional term $\phi(t)$, (3) additional term $\phi(t)$, (4) additional term $\phi(t)$.

**Small Oscillations**

At a distance from the fixed end of a spring $m$ is subject to a spring that is free to move. The force of the spring, $F = -kx$, where $k$ is the spring constant and $x$ is the displacement from equilibrium.

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We have the lower limit of integration $-\infty$ instead of zero and with this assumption the integral $T_0(t)$ is the same. Let us determine the total energy transmitted to the system during all time, then the total energy $\mathcal{E}$ is given by the expression:

$$\mathcal{E} = \int_{-\infty}^{\infty} \psi(t) \overline{\psi(t)} \, dt$$

where $\psi(t)$ is the transmitted wavefunction.

The wavefunction $\psi(t)$ is the solution of the Schrödinger equation:

$$\frac{1}{2} \frac{d^2\psi(t)}{dt^2} + V(t) \psi(t) = i \epsilon \psi(t)$$

subject to the initial conditions

$$\psi(0) = \langle \psi_0 | \psi(t) \rangle$$

$$\frac{d\psi(t)}{dt} \bigg|_{t=0} = \langle \dot{\psi}_0 | \psi(t) \rangle$$

The energy of the system is

$$E = \frac{1}{2} \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \psi(t) \overline{\psi(t)} \, dt$$

From the previous results, we obtain the energy transmitted:

$$\mathcal{E} = \int_{-\infty}^{\infty} \psi(t) \overline{\psi(t)} \, dt = E$$

Therefore, the total energy is equal to the energy transmitted.

The corresponding homogeneous equation of motion ((22.12)) is:

$$\frac{1}{2} \frac{d^2}{dt^2} + V(t) \psi(t) = 0$$

where $V(t)$ is the potential energy function.

The solution of this equation is given by

$$\psi(t) = A \sin(\omega t) + B \cos(\omega t)$$

where $A$ and $B$ are constants.

The energy of the system is given by

$$E = \frac{1}{2} \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \psi(t) \overline{\psi(t)} \, dt = \frac{1}{2} \int_{-\infty}^{\infty} [A \sin(\omega t) + B \cos(\omega t)]^2 \, dt$$

Substituting $\psi(t)$ into the equation, we obtain the energy transmitted:

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The theory of free oscillations of systems with more than one degree of freedom

The motion of a particle of mass $m$ is described by the equation of motion:

$$\mathbf{\ddot{r}}(t) + \omega^2 \mathbf{r}(t) = 0$$

where $\omega$ is the angular frequency.

**Solution:**

$$\mathbf{r}(t) = c_1 \mathbf{e}_1 e^{-\omega t} + c_2 \mathbf{e}_2 e^{-\omega t}$$

The constants $c_1$ and $c_2$ are determined by the initial conditions.

**Example:**

If $\mathbf{r}(0) = \mathbf{a}$ and $\mathbf{\dot{r}}(0) = \mathbf{b}$, then

$$c_1 = \frac{\mathbf{a}}{\omega^2} - \frac{\mathbf{b}}{\omega}$$

and

$$c_2 = \frac{\mathbf{b}}{\omega^2}$$

**Note:**

The solution to the free oscillation problem is given by the superposition of two harmonics with the same angular frequency but opposite signs.
Let the potential energy of the system \( U \) as a function of the generalised co-ordinates \( q_i \) (\( i = 1, 2, \ldots, s \)) have a minimum for \( q_i = q_{i0} \). Putting
\[
x_i = q_i - q_{i0}
\]
for the small displacements from equilibrium and expanding \( U \) as a function of the \( x_i \) as far as the quadratic terms, we obtain the potential energy as a positive definite quadratic form
\[
U = \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k,
\]
where we again take the minimum value of the potential energy as zero. Since the coefficients \( k_{ik} \) and \( k_{ki} \) in (23.2) multiply the same quantity \( x_i x_k \), it is clear that they may always be considered equal: \( k_{ik} = k_{ki} \).

In the kinetic energy, which has the general form \( \frac{1}{2} \sum_{i,k} \dot{q}_i \dot{q}_k \) (see (5.5)), we put \( q_i = q_{i0} \) in the coefficients \( a_{ik} \) and, denoting \( a_{ik}(q_0) \) by \( m_{ik} \), obtain the kinetic energy as a positive definite quadratic form
\[
\frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k.
\]

The coefficients \( m_{ik} \) also may always be regarded as symmetrical: \( m_{ik} = m_{ki} \). Thus the Lagrangian of a system executing small free oscillations is
\[
L = \frac{1}{2} \sum_{i,k} (m_{ik} \dot{x}_i \dot{x}_k - k_{ik} x_i x_k).
\]

Let us now derive the equations of motion. To determine the derivatives involved, we write the total differential of the Lagrangian:
\[
dL = \frac{1}{2} \sum_{i,k} (m_{ik} \dot{x}_i \ddot{x}_k + m_{ik} \dot{x}_k \ddot{x}_i - k_{ik} \dot{x}_i \ddot{x}_k - k_{ik} \dot{x}_k \ddot{x}_i).
\]
Since the value of the sum is obviously independent of the naming of the suffixes, we can interchange \( i \) and \( k \) in the first and third terms in the parentheses. Using the symmetry of \( m_{ik} \) and \( k_{ik} \), we have
\[
dL = \sum_{i,k} (m_{ik} \dot{x}_i \ddot{x}_k - k_{ik} \dot{x}_i x_k).
\]

Hence
\[
\frac{\partial L}{\partial \dot{x}_i} = \sum_{k} m_{ik} \ddot{x}_k, \quad \frac{\partial L}{\partial x_i} = - \sum_{k} k_{ik} x_k.
\]

Lagrange's equations are therefore
\[
\sum_{k} m_{ik} \ddot{x}_k + \sum_{k} k_{ik} x_k = 0 \quad (i = 1, 2, \ldots, s); \tag{23.5}
\]
they form a set of \( s \) linear homogeneous differential equations with constant coefficients.

As usual, we seek the \( s \) unknown functions \( x_k(t) \) in the form
\[
x_k = A_k \exp(i\omega t), \tag{23.6}
\]
where \( A_k \) are some constants to be determined. Substituting (23.6) in the equations (23.5) and cancelling \( \exp(i\omega t) \), we obtain a set of linear homogeneous algebraic equations to be satisfied by the \( A_k \):
\[
\sum_{k} (-\omega^2 m_{ik} + k_{ik}) A_k = 0. \tag{23.7}
\]

If this system has non-zero solutions, the determinant of the coefficients must vanish:
\[
|k_{ik} - \omega^2 m_{ik}| = 0. \tag{23.8}
\]

This is the characteristic equation and is of degree \( s \) in \( \omega^2 \). In general, it has \( s \) different real positive roots \( \omega_k^2 (\alpha = 1, 2, \ldots, s) \); in particular cases, some of these roots may coincide. The quantities \( \omega_k \) thus determined are the characteristic frequencies or eigenfrequencies of the system.

It is evident from physical arguments that the roots of equation (23.8) are real and positive. For the existence of an imaginary part of \( \omega \) would mean the presence, in the time dependence of the co-ordinates \( x_k (23.6) \), and so of the velocities \( \dot{x}_k \), of an exponentially decreasing or increasing factor. Such a factor is inadmissible, since it would lead to a time variation of the total energy \( E = U + T \) of the system, which would therefore not be conserved.

The same result may also be derived mathematically. Multiplying equation (23.7) by \( A^*_k \) and summing over \( i \), we have \( \sum (-\omega^2 m_{ik} + k_{ik}) A^*_k A_k = 0 \), whence \( \omega^2 = \sum k_{ik} A^*_k A_k / \sum m_{ik} A^*_k A_k \). The quadratic forms in the numerator and denominator of this expression are real, since the coefficients \( k_{ik} \) and \( m_{ik} \) are real and symmetrical: \( \sum k_{ik} A^*_k A^*_k = \sum k_{ik} A_k A_k^* = \sum k_{ik} A_k A_k^* = \sum k_{ik} A_k A_k^* \). They are also positive, and therefore \( \omega^2 \) is positive,†

The frequencies \( \omega_k \) having been found, we substitute each of them in equations (23.7) and find the corresponding coefficients \( A_k \). If all the roots \( \omega_k \) of the characteristic equation are different, the coefficients \( A_k \) are proportional to the minors of the determinant (23.8) with \( \omega = \omega_k \). Let these minors be \( \Delta_k \). A particular solution of the differential equations (23.5) is therefore \( x_k = \Delta_k C_k \exp(i\omega_k t) \), where \( C_k \) is an arbitrary complex constant.

The general solution is the sum of \( s \) particular solutions. Taking the real part, we write
\[
x_k = \Re \sum_{\alpha=1}^{s} \Delta_k C_\alpha \exp(i\omega_\alpha t) = \sum_{\alpha} \Delta_k \Theta_\alpha, \tag{23.9}
\]
where
\[
\Theta_\alpha = \Re \{ C_\alpha \exp(i\omega_\alpha t) \}. \tag{23.10}
\]

Thus the time variation of each co-ordinate of the system is a superposition of \( s \) simple periodic oscillations \( \Theta_1, \Theta_2, \ldots, \Theta_s \) with arbitrary amplitudes and phases but definite frequencies.

† The fact that a quadratic form with the coefficients \( k_{ik} \) is positive definite is seen from their definition (23.2) for real values of the variables. If the complex quantities \( A_k \) are written explicitly as \( a_k + ib_k \), we have, again using the symmetry of \( k_{ik} \), \( \sum k_{ik} A^*_k A_k = \sum k_{ik} (a_k^2 - b_k^2) \times (\alpha_k^2 + b_k^2) = \sum k_{ik} a_k a_k^* + \sum k_{ik} b_k b_k^* \), which is the sum of two positive definite forms.
The corresponding equations of motion are given by:

\[ m \ddot{\mathbf{r}} = \mathbf{F} \]

where we have

\[ \mathbf{F} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{int}} \]

and \( \mathbf{F}_{\text{int}} \) is the force due to internal interactions.

The solution to these equations depends on the type of interaction and the properties of the system. For a conservative system, the total energy is conserved.

For a non-conservative system, the energy is not conserved, and the equations of motion must be solved numerically.

In summary, the equations of motion for a system with \( n \) degrees of freedom are given by:

\[ \mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{F} \]

where \( \mathbf{M} \) is the mass matrix, \( \mathbf{C} \) is the damping matrix, \( \mathbf{K} \) is the stiffness matrix, and \( \mathbf{F} \) is the force vector.
The condition for the two to be in direction with the same application is:

\[ 0 = \theta_x \times \theta_y \times \theta_z = (\theta_x, \theta_y, \theta_z) \times (0, 0, 0) = (0, 0, 0) \]

In the quantum mechanical context, the angular momentum of a molecule can be expressed by its angular momentum operator \( \mathbf{L} \) and the angular momentum of the nuclear part \( \mathbf{J} \), which are related by the commutation relation:

\[ [\mathbf{L}, \mathbf{J}] = \mathbf{L} \times \mathbf{J} = \mathbf{J} \times \mathbf{L} \]

The projection of \( \mathbf{J} \) onto the nuclear motion is given by:

\[ J_z = M \]

where \( M \) is the magnetic quantum number, and the projection of \( \mathbf{L} \) onto the nuclear motion is:

\[ L_z = m \]

The total angular momentum is then:

\[ J_z = M \]
Problem 2. The same problem, but a taut molecular AB (Fig. 9).

Solution. By (2) (3) and (4) (7) (9) and 7, the components of the derivatives of the potential are:

\[ \mathbf{F} = \mathbf{W} + \mathbf{V} \]

\[ \mathbf{W} = \mathbf{F} - \mathbf{V} \]

\[ \mathbf{V} = \mathbf{W} - \mathbf{F} \]

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\[ \mathbf{V} = \mathbf{W} - \mathbf{F} \]

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Damped oscillations

§ 22. Damped oscillations

The motion of a system that is subjected to a damping force which opposes the oscillatory motion is called a damped oscillation. The equation of motion for such a system is

\[ m \ddot{x} + c \dot{x} + kx = 0 \]

where \( m \) is the mass, \( c \) is the damping coefficient, and \( k \) is the spring constant. If the damping force is proportional to the velocity, the motion is called a viscous damping. If the damping force is proportional to the displacement, the motion is called a linear damping. The general solution of this equation is

\[ x(t) = A e^{\alpha t} \sin(\omega t + \phi) \]

where \( \alpha \) and \( \omega \) are given by

\[ \alpha = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \]

and

\[ \omega = \sqrt{\frac{k}{m}} \]

The frequency of oscillation is

\[ \omega_0 = \sqrt{\frac{k}{m}} \]

and the damped frequency is

\[ \omega_d = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \]

The damping ratio is

\[ \zeta = \frac{c}{2 \sqrt{mk}} \]

The amplitude of oscillation is

\[ A = \frac{\sqrt{m^2 \omega_0^2 - c^2}}{\sqrt{m^2 \omega_0^2 + c^2}} \]

The phase angle is

\[ \phi = \tan^{-1}\left(\frac{c}{\sqrt{m^2 \omega_0^2 + c^2}}\right) \]

The motion is stable if \( \zeta < 1 \) and unstable if \( \zeta > 1 \).

Small oscillations

§23.

For the sake of convenience, we will consider the motion of a particle in one dimension.

\[ \ddot{x} = -\gamma \dot{x} - \alpha x \]

where \( \gamma \) is the viscous damping coefficient and \( \alpha \) is the spring constant.

The general solution of this equation is

\[ x(t) = e^{-\gamma t} \left( A e^{i\omega t} + B e^{-i\omega t} \right) \]

where \( \omega = \sqrt{\frac{\alpha}{\gamma}} \).

The frequency of oscillation is

\[ \omega_0 = \sqrt{\frac{\alpha}{\gamma}} \]

and the damped frequency is

\[ \omega_d = \sqrt{\frac{\alpha}{\gamma}} e^{-\gamma t} \]

The amplitude of oscillation is

\[ A = \frac{1}{\sqrt{1 + (\gamma t)^2}} \]

The phase angle is

\[ \phi = \tan^{-1}(\gamma t) \]

The motion is stable if \( \gamma < 0 \) and unstable if \( \gamma > 0 \).
§26. Forced oscillations under friction

In detail the case of a periodic external force, which is of considerable interest.


eq \frac{\Delta T}{T} \frac{1}{p} = \left( \frac{1}{\Delta T} \right) \frac{1}{p}

Equation (25.10) must be added to the right-hand side of Lagrange's

eq \frac{\Delta T}{T} \frac{1}{p} = \left( \frac{1}{\Delta T} \right) \frac{1}{p}

of the quadratic form

\Delta T = \text{an equation of degree } 2, \text{ since all the coefficients are real}

\Delta T = \Delta T

\Delta T = \Delta T

\Delta T = \Delta T

\Delta T = \Delta T
Following the text from the image:

When friction is allowed for, the discoutinuity is smoothed out.

\[ \frac{1}{Z}\mu' = \frac{z^2}{Z} + \frac{2}{Z} \int_0^\infty \frac{w}{\lambda} d\lambda = \mu(\lambda) f(\lambda) \]

The expression that follows is given by (26.9). There is no mention of the contour, however, remaining unchanged. The area

is given by the integral

\[ \text{Area} = \int_0^\infty \mu(\lambda) f(\lambda) d\lambda \]

When the damping coefficient decreases, the resonance curve becomes more

\[ \begin{align*}
\text{Fig. 3.1} & \\
& \\
\end{align*} \]

In steady motion when the system excites the forced oscillations, the

\[ \text{forced oscillations under friction} \]

When friction is allowed for, the discontinuity is smoothed out.

\[ \text{expression for the discontinuity} \]

The expression that follows is given by (26.9). There is no mention of the contour, however, remaining unchanged. The area

is given by the integral

\[ \text{Area} = \int_0^\infty \mu(\lambda) f(\lambda) d\lambda \]

When the damping coefficient decreases, the resonance curve becomes more

\[ \begin{align*}
\text{Fig. 3.1} & \\
& \\
\end{align*} \]

In steady motion when the system excites the forced oscillations, the
where the constants $a$, $b$, and $c$ are as defined in the problem.

The given equation is:

$$a \cos y + b \sin y + c = 0$$

where $a$, $b$, and $c$ are constants.

From the differential equation,

$$\frac{d^2 y}{dx^2} + \cos y \frac{dy}{dx} = 0$$

we can write the differential equation as:

$$\frac{d^2 y}{dx^2} = -\cos y \frac{dy}{dx}$$

This is a second-order differential equation. We can solve this equation to find the function $y(x)$.

To solve the differential equation, we can use the method of undetermined coefficients. The general solution to the differential equation is:

$$y(x) = C_1 \cos x + C_2 \sin x$$

where $C_1$ and $C_2$ are constants.

To determine the constants $C_1$ and $C_2$, we can use the boundary conditions.

**Problem:**

Determine the function $y(x)$ which satisfies the given differential equation and the following boundary conditions:

1. $y(0) = 0$
2. $y'(0) = 1$
Since the absolute value of a trigonometric function is a constant, the value of the constant is determined by the given function.

Solution. By the definition of the derivative, we have the equation of motion

**Proposition 2.** The equation of motion is given by

\[ \frac{d}{dx} \sin x = \cos x \]

Proving this, we can show that the rate of change of the position of the mass is proportional to the velocity of the mass. Therefore, the equation of motion becomes

\[ \frac{d^2x}{dt^2} = \sin x \]

where \( x \) is the displacement from the equilibrium position, \( t \) is time, and \( \omega \) is the angular frequency.

The condition for parametric resonance occurs if \( \omega < \omega_0 \). The condition for parametric resonance is satisfied if \( \omega_0 < \omega \). Therefore, the condition for parametric resonance is satisfied if \( \omega > \omega_0 \).

Thus, the parametric resonance occurs in the condition

\[ \omega > \omega_0 \]

For these values of \( \omega \), the parametric resonance occurs.

In the condition for parametric resonance to occur, the equation becomes

\[ \frac{d^2x}{dt^2} = \sin x \]

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The condition for parametric resonance occurs if \( \omega < \omega_0 \). The condition for parametric resonance is satisfied if \( \omega_0 < \omega \). Therefore, the condition for parametric resonance is satisfied if \( \omega > \omega_0 \).

Thus, the parametric resonance occurs in the condition

\[ \omega > \omega_0 \]

The condition for parametric resonance is that \( \omega \) is real and \( \omega > \omega_0 \).

The condition for parametric resonance is that \( \omega > \omega_0 \).
In the present approximation, where higher divergences are included in the equation (2.8)

$$\sum_{\gamma} \left( \frac{\Delta}{\sigma} \right)^{\gamma} = L$$

we have the Lagrangian in the form

$$\omega + \sigma \sum_{\gamma} \left( \frac{\Delta}{\sigma} \right)^{\gamma}$$

A more detailed and natural approach is provided by the equation of motion for \( L \theta \theta \theta \) with the divergence of the coordinate \( \theta \) being included in the equation.

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A more detailed and natural approach is provided by the equation of motion for \( L \theta \theta \theta \) with the divergence of the coordinate \( \theta \) being included in the equation.
\[ \frac{x}{\beta} + \Omega = 0 \]

This motion of the particle is a periodic oscillating field.

\[ \frac{d^2x}{dt^2} + \omega^2 x = 0 \]

The motion is periodic with frequency \( \omega \).

\[ \omega = \sqrt{\frac{k}{m}} \]

The frequency is the square root of the ratio of stiffness to mass.

\[ \beta = \frac{1}{\omega^2} \]

The damping factor \( \beta \) is the inverse of \( \omega^2 \).

\[ \Delta \theta = \frac{2\pi}{\Omega} \]

The phase shift \( \Delta \theta \) is the phase difference between the two oscillating fields.

\[ \Delta \phi = \frac{2\pi}{\Omega} \]

The phase angle \( \Delta \phi \) is the phase difference between the two oscillating fields.

\[ \Delta \psi = \frac{2\pi}{\Omega} \]

The phase velocity \( \Delta \psi \) is the phase difference between the two oscillating fields.

\[ \Delta t = \frac{2\pi}{\Omega} \]

The phase delay \( \Delta t \) is the phase difference between the two oscillating fields.

\[ \Delta f = \frac{2\pi}{\Omega} \]

The phase frequency \( \Delta f \) is the phase difference between the two oscillating fields.

\[ \Delta \omega = \frac{2\pi}{\Omega} \]

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