A Statistical Superfield. II. Some Further Predictions

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Abstract

In an earlier paper, an action with unconventional supersymmetry was found to reproduce standard physics at low energy, but to predict interesting new phenomena near 1 TeV and above. In the present paper, it is shown that this action results from a remarkably simple picture: a single fundamental system consisting of identical “bits” which vary randomly over all possible states. One finds that this picture leads first to a bosonic action with a randomly fluctuating potential, and then to a supersymmetric action with exactly the same form that was postulated earlier. Several new predictions are also given here (for the simplest version of the present theory): (1) The only superpartners are scalar bosons. (2) The most natural candidate for cold dark matter is therefore a spin-zero WIMP. (3) The Higgs boson has an R-parity of -1, so it can only be produced as one member of a pair of superpartners. (This prediction is associated with a reinterpretation of Yukawa couplings.) (4) All these fundamental scalar particles have an unconventional equation of motion. Finally, it is shown that the present theory leads to a consistent description in terms of quantized fields, and that the vacuum energy vanishes before supersymmetry is broken.
1 Introduction

In an earlier paper [1], the following Euclidean action was postulated:

\[ S = \int d^D x \left[ \frac{1}{2m} \partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b (\Psi^\dagger \Psi)^2 \right] \]  
(1.1)

with

\[ \Psi = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}, \quad z = \begin{pmatrix} z_b \\ z_f \end{pmatrix}. \]  
(1.2)

This action has “natural supersymmetry”, in the sense that the initial bosonic fields \( z_b \) and fermionic fields \( z_f \) are treated in exactly the same way. The only difference is that the \( z_b \) are ordinary complex numbers whereas the \( z_f \) are anticommuting Grassmann numbers.

It was demonstrated in Ref. 1 that standard physics emerges from (1.1) at energies that are far below the Planck scale, provided that specific kinds of instantons are included in the theory. For example, one obtains an \( \text{SO}(10) \) grand-unified theory, containing both the Standard Model and a natural mechanism for small neutrino masses [2-13].

In Sections 4-6 of the present paper, we will consider some further implications and predictions of (1.1). Let us begin, however, with the question of how this phenomenological action might arise from a still deeper picture.

2 Origin of Natural Supersymmetry

Suppose that the truly fundamental fields of nature are purely bosonic, but that they can be divided into two classes: a set of fields \( \Psi_b \) which can be directly observed, and a set \( \tilde{\Psi}_b \) which can be inferred only indirectly through their effect on the \( \Psi_b \). To be specific, the \( \tilde{\Psi}_b \) are assumed to randomly perturb the \( \Psi_b \) in the same way that molecules randomly perturb small particles and produce Brownian motion. These interactions are then assumed to result in an effective action of the form

\[ S_{\text{eff}} = \int d^D x \left[ \frac{1}{2m} \partial^M \Psi_b^\dagger \partial_M \Psi_b - \mu \Psi_b^\dagger \Psi_b + \tilde{V} \Psi_b^\dagger \Psi_b \right] \]  
(2.1)

where \( \mu \) is a constant and \( \tilde{V} \) is a Gaussian random variable satisfying

\[ \langle \tilde{V} \rangle = 0 \quad , \quad \langle \tilde{V}(x) \tilde{V}(x') \rangle = u(x - x'). \]  
(2.2)

The average value of a physical quantity \( F \) is given by

\[ \langle F \rangle = \left\langle \frac{\int \mathcal{D} \Psi_b \mathcal{D} \Psi_b^\dagger F e^{-S_{\text{eff}}[\Psi_b, \Psi_b^\dagger]}}{\int \mathcal{D} \Psi_b' \mathcal{D} \Psi_b'^\dagger e^{-S_{\text{eff}}[\Psi_b', \Psi_b'^\dagger]}} \right\rangle \]  
(2.3)
where $\langle \cdots \rangle$ represents an average over the perturbing potential $\tilde{V}$. The presence of the denominator makes it difficult to perform this average, but there is a trick for removing the bosonic degrees of freedom $\Psi'_b$ in the denominator and replacing them with fermionic degrees of freedom $\Psi_f$ in the numerator [14-16]: Since

\begin{equation}
\int \mathcal{D} \Psi'_b \mathcal{D} \Psi'_b e^{-S_{eff}[\Psi'_b, \Psi'_b]} = (\det A)^{-1} \tag{2.4}
\end{equation}

\begin{equation}
\int \mathcal{D} \Psi_f \mathcal{D} \Psi_f e^{-S_{eff}[\Psi_f, \Psi_f]} = \det A \tag{2.5}
\end{equation}

where $A$ represents the operator of (2.1), it follows that

\begin{equation}
\langle F \rangle = \left\langle \int \mathcal{D} \Psi'_b \mathcal{D} \Psi'_b \mathcal{D} \Psi_f \mathcal{D} \Psi_f F e^{-S_{eff}[\Psi'_b, \Psi'_b]} e^{-S_{eff}[\Psi_f, \Psi_f]} \right\rangle \tag{2.6}
\end{equation}

\begin{equation}
= \left\langle \int \mathcal{D} \Psi \mathcal{D} \Psi^\dagger F e^{-S_{eff}[\Psi, \Psi]} \right\rangle \tag{2.7}
\end{equation}

where $\Psi_b$ and $\Psi_f$ have been combined into $\Psi$ as in (1.2), and

\begin{equation}
S_{eff} [\Psi, \Psi^\dagger] = \int d^D x \left[ \frac{1}{2m} \partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + \tilde{V} \Psi^\dagger \Psi \right]. \tag{2.8}
\end{equation}

For a Gaussian random variable $v$ whose mean is zero, the result

\begin{equation}
\langle e^{-v} \rangle = e^{\frac{1}{2} \langle v^2 \rangle} \tag{2.9}
\end{equation}

implies that

\begin{equation}
\langle e^{-\int d^D x \tilde{V} \Psi^\dagger \Psi} \rangle = e^{\frac{1}{2} \int d^D x \int d^D x' \Psi^\dagger(\Psi(x)\Psi(x) u(x-x') \Psi^\dagger(x'))}. \tag{2.10}
\end{equation}

If

\begin{equation}
u(x-x') = b \delta (x-x') \tag{2.11}
\end{equation}

it follows that

\begin{equation}
\langle F \rangle = \int \mathcal{D} \Psi \mathcal{D} \Psi^\dagger F e^{-S} \tag{2.12}
\end{equation}

with $S$ given by (1.1). A special case is

\begin{equation}
Z = \int \mathcal{D} \Psi \mathcal{D} \Psi^\dagger e^{-S} \tag{2.13}
\end{equation}

but according to (2.3)

\begin{equation}
Z = 1. \tag{2.14}
\end{equation}

To make the expression for $\langle F \rangle$ independent of how the measure is defined in the path integral, we can rewrite (2.12) as

\begin{equation}
\langle F \rangle = \frac{1}{Z} \int \mathcal{D} \Psi \mathcal{D} \Psi^\dagger F e^{-S}. \tag{2.15}
\end{equation}

To summarize this section, the supersymmetric action (1.1) follows from the purely bosonic action (2.1), if it is assumed that the observable bosonic fields are randomly perturbed by unobservable fields $\tilde{\Psi}_b$.

Notice that the fermionic variables $\Psi_f$ represent true degrees of freedom, and that they originate from the bosonic variables $\Psi'_b$. The coupling between the fields $\Psi_b$ and $\Psi_f$ (or $\Psi'_b$) is due to the random perturbing potential $\tilde{V}$.
3 Statistical Origin of the Bosonic Action

The bosonic action (2.1) still has a phenomenological form. Let us now turn to a microscopic statistical picture which leads to this form. The starting point is a fundamental system which consists of \( \bar{N} \) “bits”. Each bit can exist in any of \( \bar{M} \) states, with the number of bits in the \( i \)th state represented by \( n_i \). A microstate of the fundamental system is determined by specifying the state of each bit, together with a set of global parameters \( x^M \) which are interpreted as coordinates in Euclidean spacetime. A macrostate is determined by specifying only the occupancies \( n_i \) of the states, together with the parameters \( x^M \). For concreteness, suppose that the values of \( x^M \) are discrete and lie on a cubic mesh with spacing \( a \sim \ell_P \), where \( \ell_P \) is the Planck length. The density of bits in the \( i \)th state is then

\[
\rho_i = n_i / a^D. \tag{3.1}
\]

Let

\[
\phi_i^2 = \rho_i. \tag{3.2}
\]

The initial bosonic fields \( \phi_i \) are then real (but defined only up to a phase factor \( \pm 1 \)). Notice that (3.1) implies large fluctuations in \( \phi_i \) at the Planck scale. This feature, however, is already present in standard physics [17].

The entropy associated with a single point \( x \) is \( \bar{S} (x) = \log W (x) \) (in units with \( k_B = \hbar = c = 1 \)), where \( W (x) \) is the number of microstates available in a given macrostate: \( W (x) = \bar{N}! / \Pi_i n_i (x)! \) with \( \bar{N} = \sum_i n_i (x) \). Stirling’s formula gives

\[
\bar{S} (x) = (\bar{N} \log \bar{N} - \bar{N}) - \sum_i [n_i (x) \log n_i (x) - n_i (x)] + \ldots. \tag{3.3}
\]

The total number of available microstates for all points \( x \) is \( W = \Pi_x W (x) \), so the total entropy is \( \bar{S} = \sum_x \bar{S} (x) \).

A physical configuration of all the fields \( \phi_i \) corresponds to a specification of all the densities \( \rho_i (x) \). In the present picture, the probability of such a configuration is proportional to \( W = e^{\bar{S}} \). In the Euclidean path integral, the probability is proportional to \( e^{-S} \), where \( S \) is the Euclidean action. We conclude that

\[
S = -\bar{S} + \text{constant}. \tag{3.4}
\]

As in the preceding section, it will be assumed that only some of the fields \( \phi_i \) are directly observable. Let us expand \( \bar{S} \) about a point where these fields are relatively small, so that \( \bar{N} \) and \( \bar{S} \) are dominantly due to the remaining fields which are not directly observable. A systematic expansion involves the digamma function and its derivatives, but to lowest order

\[
\frac{\partial \bar{S}}{\partial n_i (x)} = \log \bar{N} - \log n_i (x) \approx \log \bar{N} \tag{3.5}
\]

\[
\frac{\partial^2 \bar{S}}{\partial n_i (x) \partial n_j (x)} = \frac{1}{\bar{N}} - \frac{\delta_{ij}}{n_i (x)} \approx - \frac{\delta_{ij}}{n_i (x)} \tag{3.6}
\]
or
\[
\frac{\partial S}{\partial \rho_i(x)} = -A \quad \text{(3.7)}
\]
\[
\frac{\partial^2 S}{\partial \rho_i(x) \partial \rho_j(x)} = B \frac{\delta_{ij}}{\rho_i(x)} \quad \text{(3.8)}
\]
where \( A = a^D \log \bar{N} \) and \( B = a^D \). If the densities are perturbed by \( \Delta \rho_i(x) \), therefore, the change in the action to second order is given by

\[
\Delta S = -A \sum_{x_i} \Delta \rho_i(x) + \frac{1}{2} B \sum_{x_i} \frac{[\Delta \rho_i(x)]^2}{\rho_i(x)} \quad \text{(3.9)}
\]

In the foregoing, the points \( x \) can be regarded as the centers of cubic cells. Let us now, however, shift our attention to another set of points — namely, the vertices at which \( 2^D \) cells intersect. Since the number of vertices is equal to the number of cells (and the two sets of points differ only by a shift of \( a/2 \) along each axis), this is clearly an equivalent choice. Let us define the value of \( S \) at a vertex to be the average over all the surrounding cells: with \( S = S_0 + \Delta S \),

\[
2^D \Delta S (\text{vertex}) = -A \sum_{x_i} \Delta \rho_i(x) + \frac{1}{2} B \sum_{x_i} \frac{[\Delta \rho_i(x)]^2}{\rho_i(x)} \quad \text{(3.10)}
\]

where \( x \) now represents the centers of surrounding cells. We have, to lowest order,

\[
\Delta \rho_i(x) = \Delta \rho_i(\text{vertex}) + \sum_M \frac{\partial \rho_i}{\partial x^M} \delta x^M \quad \text{(3.11)}
\]

\[
[\Delta \rho_i(x)]^2 = [\Delta \rho_i(\text{vertex})]^2 + 2\Delta \rho_i(\text{vertex}) \sum_M \frac{\partial \rho_i}{\partial x^M} \delta x^M + \left( \sum_M \frac{\partial \rho_i}{\partial x^M} \delta x^M \right)^2 \quad \text{(3.12)}
\]

where \( \delta x^M = \pm a/2 \). Because of the cubic symmetry, the terms that are odd in \( \delta x^M \) will cancel, leaving

\[
\Delta S (\text{vertex}) = -A \sum_i \Delta \rho_i(\text{vertex}) + \frac{1}{2} B \sum_i [\Delta \rho_i(\text{vertex})]^2 + \frac{1}{8} B a^2 \sum_{iM} \left( \frac{\partial \rho_i/\partial x^M}{\rho_i} \right)^2 \quad \text{(3.13)}
\]

From (3.2), we obtain the remarkable simplification

\[
\left( \frac{\partial \rho_i/\partial x^M}{\rho_i} \right)^2 = 4 \left( \frac{\partial \phi_i/\partial x^M}{\rho_i} \right)^2 \quad \text{(3.14)}
\]

Then (3.13) can be written, to lowest order in \( \Delta (\phi^2_i) \),

\[
\Delta S (\text{vertex}) = -A \sum_i \Delta (\phi^2_i) + \frac{1}{2} B a^2 \sum_{iM} \left( \frac{\partial \phi_i}{\partial x^M} \right)^2 \quad \text{(3.15)}
\]

or, if the constant of (3.4) is chosen so that \( S = 0 \) when the \( \phi_i \) and their derivatives are zero,

\[
S (\text{vertex}) = -A \sum_i \phi^2_i + \frac{1}{2} B a^2 \sum_{iM} \left( \frac{\partial \phi_i}{\partial x^M} \right)^2 \quad \text{(3.16)}
\]
The philosophy behind the above treatment is simple: In (3.10) we have \( \langle f \rangle \) and \( \langle f^2 \rangle \), where \( f = \Delta \rho_i (x) \) and \( \langle \cdots \rangle \) represents the average over surrounding points. We essentially wish to replace \( \langle f^2 \rangle \) by \( (\partial f / \partial x)^2 \), and this can be accomplished because
\[
\langle f^2 \rangle - \langle f \rangle^2 \approx \langle (\partial f / \partial x)^2 \rangle = (\partial f / \partial x)^2 \left( a / 2 \right)^2.
\] (3.17)

The form of (3.16) also has a simple interpretation: The entropy \( \bar{S} = -S \) increases with the number of bits, but decreases when the bits are not uniformly distributed over the points \( x \).

Let \( \bar{N} = N_0 + \Delta \bar{N} \). As the number of bits in unobserved states varies randomly, so does
\[
\log \bar{N} = \log N_0 + \log \left( 1 + \Delta \bar{N}/N_0 \right) \approx \log N_0 + \Delta \bar{N}/N_0.
\] (3.18)

We can then write
\[
S_{(vertex)} = \left( -\mu + \bar{V} \right) a^D \sum_i \Phi_i^2 + (2m)^{-1} a^D \sum_i \left( \frac{\partial \Phi_i}{\partial x^M} \right)^2
\] (3.19)
where
\[
\mu = a^{-1} \log N_0 , \quad \bar{V} = -a^{-1} \Delta \bar{N}/N_0 , \quad m = a^{-1} , \quad \Phi_i = a^{1/2} \phi_i,
\] (3.20)
and \( \bar{V} \) is a Gaussian random variable whose mean is zero. The total action in the continuum approximation is obtained by summing over all vertices and letting \( \Sigma_{\text{vertices}} a^D \rightarrow \int d^D x \):
\[
S = \int d^D x \left( \frac{1}{2m} \frac{\partial \Phi}{\partial x^M} \frac{\partial \Phi}{\partial x^M} - \mu \Phi^\dagger \Phi + \bar{V} \Phi^\dagger \Phi \right).
\] (3.21)

Here \( \Phi \) is the vector with components \( \Phi_i \). If we assume that the number of these observable real fields is even, we can group them in pairs to form complex fields \( \Psi_{b,i} \). (One motivation for doing so is that complex fields can have well-defined values for physical quantities like momentum, energy, and charge. In particular, a charged bosonic field is complex.) Then we finally have
\[
S = \int d^D x \left( \frac{1}{2m} \frac{\partial \Psi_b^\dagger}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} - \mu \Psi_b^\dagger \Psi_b + \bar{V} \Psi_b^\dagger \Psi_b \right)
\] (3.22)
which is exactly the same as (2.1).

It is remarkable that this extremely simple statistical picture leads to the bosonic action (3.21), and thus to the supersymmetric action (1.1).
4 Four-Dimensional Action and Scalar Superpartners

In the present theory, the standard features of four-dimensional physics – including gauge symmetries and chiral fermions – arise from a $d$-dimensional internal space which contains an instanton. Let us begin with a picture in which this instanton occupies an unbounded volume, and then move to a physically more acceptable description in which it is confined to a finite region $r < r_0$. The finite instanton has finite action, and can be viewed as a “spinning” ball of condensate. The corresponding order parameter has a node at $r = r_0$, from which the condensate rises to become fully formed at large $r$. The region $r < r_0$ corresponds to our physical universe, and the region $r > r_0$ is unobservable.

Within this section it will be necessary to cite many equations from Ref. 1, which will be distinguished with the prefix I. Let us begin with I(5.14), the nonlinear equation of motion for the internal order parameter $\Psi_B$:

$$\left(-\frac{1}{2m}\partial^m \partial_m + V - \mu_B\right) \Psi_B = 0. \quad (4.1)$$

Although $\Psi_B$ depends primarily on the $d = D - 4$ internal coordinates $x^m$, it also has a slow parametric dependence on the 4 external coordinates $x^\mu$, since $\mu_B = \mu - \mu_A$ with $\mu_A = \frac{1}{2} m v^\mu v_\mu$.

Just as I(3.7) leads to I(3.20), (4.1) leads to the internal Bernoulli equation

$$-\frac{1}{2m} n_B^{-1/2} \partial^m \partial_m n_B^{1/2} + \frac{1}{2} m \eta_B^\dagger v^m v_m \eta_B + b n_A n_B - \mu_B = 0 \quad (4.2)$$

where $\Psi_B = n_B^{1/2} U_B \eta_B$ and $m v_m = -i U_B^{-1} \partial_m U_B$. As mentioned above, the internal space is assumed to contain an instanton with the symmetry of a $(d-1)$-sphere:

$$\eta_B^\dagger v^m v_m \eta_B = (\bar{a}/mr)^2 \quad (4.3)$$

$$\partial^m \partial_m n_B^{1/2} = r^{-d'} \frac{d}{dr} \left(r^{d'} \frac{d}{dr} n_B^{1/2}\right), \quad d' = d - 1. \quad (4.4)$$

Then (4.2) can be rewritten as

$$-\frac{1}{\rho^{d'}} \frac{d}{d\rho} \left(\rho^{d'} \frac{df}{d\rho}\right) + \frac{\bar{a}^2}{\rho^2} f + f^3 - f = 0 \quad (4.5)$$

where $\rho = r/\xi_B$ and $f = n_B^{1/2}/\bar{n}_B^{1/2}$, with $\xi_B = (2m\mu_B)^{-1/2}$ and $\bar{n}_B = \mu_B/b n_A$. The asymptotic solutions to (4.5) are

$$f \propto \rho^n \quad \text{as } \rho \to 0 \quad (4.6)$$

$$f = 1 - \frac{\bar{a}^2}{2\rho^2} \quad \text{as } \rho \to \infty \quad (4.7)$$

where

$$n = \frac{1}{2} \left[\sqrt{(d-2)^2 + 4\bar{a}^2} - (d - 2)\right] \quad (4.8)$$

so that

$$n = 1 \quad \text{if } \bar{a}^2 = d - 1. \quad (4.9)$$
It is easy to show that (4.9) holds for a minimal vortex in two dimensions or a minimal SU(2) instanton in four dimensions.

Since the volume element is proportional to $\rho^{d-1}d\rho$ and $1 - f^2$ is proportional to $\rho^{-2}$ as $\rho \to \infty$, the above solution has infinite action. However, we can obtain a solution with finite action by requiring that

$$
\Psi_B = R(r) \bar{n}_B^{1/2} U_B \eta_B , \quad \rho < \rho_0 \tag{4.10}
$$

$$
\Psi_B = 0 , \quad \rho = \rho_0 \tag{4.11}
$$

$$
\Psi_B = \bar{R}(r) \eta_B , \quad \rho > \rho_0 \tag{4.12}
$$

so that the instanton is confined to the region inside a radius $\rho_0$ which is determined by the boundary conditions below. Then (4.5) is replaced by

$$
-\frac{1}{\rho^d} \frac{d}{d\rho} \left( \rho^d \frac{dR}{d\rho} \right) + \frac{\tilde{a}^2}{\rho^2} R + R^3 - R = 0 , \quad \rho < \rho_0 \tag{4.13}
$$

$$
-\frac{1}{2m} \frac{1}{r^d} \frac{d}{dr} \left( r^d \frac{d\bar{R}}{dr} \right) + bn_AR^3 - \mu \bar{R} = 0 , \quad \rho > \rho_0. \tag{4.14}
$$

$R$ is required to satisfy (4.13) with the boundary condition $R \to 0^+$ as $\rho \to 0$. $\bar{R}$ is required to satisfy (4.14) with the boundary condition $\bar{R} \to -(\mu/bn_A)^{1/2}$ as $r \to \infty$ (and with $\partial \Psi_B/\partial r$ continuous at $\rho = \rho_0$). In the following, we will be concerned only with the physical region $\rho < \rho_0$, and the integrals are over only this region; e.g.,

$$
V_B = \int d^d x = \int_{\rho<\rho_0} d^d x. \tag{4.15}
$$

The internal basis functions $\Psi_r^B$ satisfy the linear equation I(5.20) with $\varepsilon_r = 0$:

$$
\left(-\frac{1}{2m} \partial^m \partial_m + V - \mu_B\right) \Psi_r^B = 0. \tag{4.16}
$$

Let

$$
\Psi_b = \Psi_s + \Phi_b \tag{4.17}
$$

as in I(3.1), where $\Psi_s$ is the order parameter. Both the fermionic excitations $\Psi_f$ and the bosonic excitations $\Phi_b$ can be represented as in I(4.2) and I(5.18):

$$
\Psi_f = U \sum_r \psi_r(x_A) \psi_r^B(x_B) \tag{4.18}
$$

$$
\Phi_b = U \sum_r \Phi_r(x_A) \psi_r^B(x_B) \tag{4.19}
$$

where

$$
\psi_r^B = \chi(r) \bar{\psi}_r^B \tag{4.20}
$$

with the normalization

$$
\bar{\psi}_r^B \psi_r^B = \delta_{rs} \tag{4.21}
$$

$$
\int d^d x \chi^2 = 1. \tag{4.22}
$$
Since $\Psi_r^B = U_B \psi_r^B$, according to I(5.19), (4.16) implies that

$$-\frac{1}{\rho^d} \frac{d}{d\rho} \left( \rho^d \frac{d\chi}{d\rho} \right) + \frac{\bar{a}^2}{\rho^2} \chi + R^2 \chi - \chi = 0. \quad (4.23)$$

This equation resembles (4.13) but is linear, so a general solution is

$$\chi = A R \quad (4.24)$$

where $A$ is a constant.

One expects that $V_B \sim \xi^d$, where $\xi = (2m\mu)^{-1/2}$ is the coherence length, and that $\chi^2 \sim \xi^{-d}$ in an average sense, so that

$$I \equiv \int d^d x \chi^4 \sim \xi^{-d}. \quad (4.25)$$

With these results, let us now turn to the interactions I(2.14) involving fermions and fundamental scalar bosons:

$$S_{\text{int}} = S_{bb} + S_{bf} + S_{ff} \quad (4.26)$$

$$S_{bb} = \frac{1}{2} b \int d^d x \left( \psi_b^\dagger \psi_b \right)^2 \quad (4.27)$$

$$S_{bf} = b \int d^d x \left( \psi_b^\dagger \psi_b \right) \left( \psi_f^\dagger \psi_f \right) \quad (4.28)$$

$$S_{ff} = \frac{1}{2} b \int d^d x \left( \psi_f^\dagger \psi_f \right)^2 \quad (4.29)$$

The order parameter can be written in the form

$$\Psi_s = U \Phi_0 (x_A) \chi (r) \eta_B \quad (4.30)$$

as in (4.19) and (4.20). Suppose that we ignore the scalar bosons corresponding to direct excitations of the order parameter, which presumably are GUT-scale Higgs bosons having masses comparable to $m_{GUT} \sim 10^{-2} m_P$ (where $m_P$ is the Planck mass of I(2.8)). This means that $\Psi_s^\dagger \Phi_b = 0$ and

$$\Psi_b^\dagger \Psi_b = \Phi_b^\dagger \Phi_b + \Psi_s^\dagger \Psi_s. \quad (4.31)$$

We then have

$$S_{bb} = \int d^4 x \frac{1}{2} b \left[ \left( \phi_s^\dagger \phi_s \right)^2 + 2 \left( \phi_s^\dagger \phi_s \right) \left( \phi_0^\dagger \phi_0 \right) + \left( \phi_0^\dagger \phi_0 \right)^2 \right] \quad (4.32)$$

$$S_{bf} = \int d^4 x \ m_b^{-1} \left[ \left( \phi_s^\dagger \phi_s \right) \left( \psi^\dagger \psi \right) + \left( \phi_0^\dagger \phi_0 \right) \left( \psi^\dagger \psi \right) \right] \quad (4.33)$$

$$S_{ff} = \int d^4 x \left[ \frac{1}{2} m_f^{-2} \left( \psi^\dagger \psi \right)^2 \right] \quad (4.34)$$

where $\psi$ and $\phi$ are respectively the vectors with components $\psi_r$ and

$$\phi_r = (2m)^{-1/2} \Phi_r \quad (4.35)$$

and where

$$\bar{b} = (2m)^2 bI \quad , \quad m_b^{-1} = 2mbI \quad , \quad m_f^{-2} = bI. \quad (4.36)$$
Since $\xi^{-d} b \sim m_P^{-2}$ and $m \sim m_P$, according to I(7.28) and I(7.34), (4.25) implies that

$$\bar{b} \sim 1 \ , \ m_b \sim m_f \sim m_P.$$  \hspace{1cm} (4.37)

It follows that the four-fermion, dimension-six contribution of (4.34) can ordinarily be neglected, and that the same is true of the first term in (4.33). Also, the terms involving the order parameter $\phi_0$ are implicitly included in the treatment below, which is based on I(8.32) and I(6.4). The only remaining interaction term is

$$S_{bb} = \int d^4x \left[ \frac{1}{2} \bar{b} \left( \phi^\dagger \phi \right)^2 \right].$$  \hspace{1cm} (4.38)

Notice that the present theory contains interactions, like $m^{-1} b \left( \phi^\dagger \phi \right) \left( \psi^\dagger \psi \right)$, which are nonrenormalizable but are also negligible at energies far below the Planck scale. This is precisely what one might expect of a fundamental theory [18-20]. On the other hand, interactions like $\bar{m}^{-1} GUT \psi^\dagger \ell \phi^\dagger GUT \phi h \psi$ (see (4.58) and (4.59)) are effectively renormalizable at ordinary energies, because $\phi_{GUT}$ acquires a vacuum expectation value at the GUT scale. It follows that the present theory is effectively renormalizable up to $m_{GUT} \sim 10^{-2} m_P$.

In the present paper, a factor $U = U_A U_B$ is included in the definitions (4.18) and (4.19) for both the fermionic fields $\psi$ and bosonic fields $\phi$. The physical motivation for this is that these fields should be defined in the frame of reference of the condensate, which has a nonzero “superfluid velocity” $v_M$. I.e., after this GUT-scale condensate forms in the very early universe, with an order parameter $\Psi_s$ which rotates as a function of position and time, it defines a frame of reference for the bosonic and fermionic fields which are subsequently born into it. These fields can be viewed as “moving with the condensate”, like a hypothetical observer moving with the flow of an ordinary superfluid.

In Ref. 1, on the other hand, the factor $U_A$ was omitted in the definition of $\phi$. This means that I(8.32) becomes, in the present notation and in Lorentzian spacetime,

$$L_b = - \left[ \eta^{\mu \nu} D_\mu \tilde{\phi}^\dagger D_\nu \tilde{\phi} - \tilde{\mu}^2 \tilde{\phi}^\dagger \tilde{\phi} + \frac{1}{2} \bar{b} \left( \phi^\dagger \phi \right)^2 \right]$$  \hspace{1cm} (4.39)

where $\eta^{\mu \nu} = \text{diag}(-1, 1, 1, 1)$,

$$\tilde{\phi} = U_A \phi,$$  \hspace{1cm} (4.40)

and $\tilde{\mu}^2 = m^2 \eta^{\mu \nu} v_{\mu \alpha} v_{\nu \alpha}$. (The last two terms are unchanged because $\tilde{\phi}^\dagger \tilde{\phi} = \phi^\dagger \phi$.) I(8.24), I(3.13), and I(3.23) imply that

$$D_\mu \tilde{\phi} = (\partial_\mu + i A_\mu) U_A \phi = U_A (\partial_\mu + i A_\mu + im v_\mu) \phi$$  \hspace{1cm} (4.41)

where

$$A_\mu = A^i_\mu t_i$$  \hspace{1cm} (4.43)

corresponds to an $SO(10)$ grand-unified theory and

$$v_\mu = v_\mu^\alpha \sigma_\alpha$$  \hspace{1cm} (4.44)
is the four-dimensional “superfield velocity” for the GUT Higgs field $\phi_0$ (or $\Psi_s$). These ideas are discussed further in Ref. 1, where
\[ e^{\mu}_\alpha = v^{\mu}_\alpha = \delta^{\mu\nu}v_{\nu\alpha} \] (4.45)
is interpreted as the contravariant vierbein:
\[ g^{\mu\nu} = \eta^{\alpha\beta}e^{\mu}_\alpha e^{\nu}_\beta. \] (4.46)

In the cosmological model of Ref. 1, it is assumed that
\[ e^{\mu}_\alpha = \tilde{e}^{\mu}_\alpha \equiv \lambda\delta^{\mu}_\alpha \] (4.47)
in regions of spacetime where the local, inhomogeneous gravitational field is weak. It follows that
\[ \eta^{\mu\nu}D_\mu\tilde{\phi}^\dagger D_\nu\tilde{\phi} - \tilde{\mu}^2\tilde{\phi}^\dagger\phi = \frac{1}{2} \left[ \lambda^{-2}\tilde{g}^{\mu\nu}D_\mu\phi^\dagger D_\nu\phi - 2i\tilde{m}\phi^\dagger e^{\mu}_\alpha\sigma^\alpha D_\mu\phi \right] + \text{conj} \] (4.48)
where
\[ \tilde{g}^{\mu\nu} \equiv \eta^{\alpha\beta}\tilde{e}^{\mu}_\alpha \tilde{e}^{\nu}_\beta = \lambda^2\eta^{\mu\nu}. \] (4.49)

As in Ref. 1, “conj” represents a term which has the form of the Hermitian conjugate, but it is understood that $\phi$ and $\phi^\dagger$ vary independently in the path integral, as do $\psi$ and $\psi^\dagger$. Then (4.39) becomes
\[ \mathcal{L}_b = -\frac{1}{2} \left[ \lambda^{-2}\tilde{g}^{\mu\nu}D_\mu\phi^\dagger D_\nu\phi - 2i\tilde{m}\phi^\dagger e^{\mu}_\alpha\sigma^\alpha D_\mu\phi + \frac{1}{2} \tilde{b} \left( \phi^\dagger\phi \right)^2 \right] + \text{conj}. \] (4.50)

With the scaling $\phi' = \lambda\phi$, this can be written
\[ \mathcal{L}_b = -\frac{1}{2} \tilde{g} \left[ \tilde{g}^{\mu\nu}D_\mu\phi'^\dagger D_\nu\phi' - i\tilde{m}\phi'^\dagger e^{\mu}_\alpha\sigma^\alpha D_\mu\phi' + \frac{1}{2} \tilde{b} \left( \phi'^\dagger\phi' \right)^2 \right] + \text{conj} \] (4.51)
where $\tilde{g} = (-\det \tilde{g}_{\mu\nu})^{1/2} = \lambda^{-4}$ and $\tilde{m} = 2\lambda^2m$. Expressions (4.50) and (4.51) are essentially the same as I(8.32) and I(8.37). The only differences are that (i) $\phi$ and $\phi'$ of Ref. 1 become $\tilde{\phi} = U_A\phi$ and $\tilde{\phi}' = U_A\phi'$ in the notation of the present paper, (ii) $\mathcal{L}_b$ is the Lagrangian density in Lorentzian rather than Euclidean spacetime, and (iii) we have used a tilde in $\tilde{g}^{\mu\nu}$ and $\tilde{g}$ as a reminder that these are not dynamical quantities.

Let $\Phi = (2m)^{1/2}\phi$ be the vector with components $\Phi_r$. Then (4.50) can also be written
\[ \mathcal{L}_b = -\frac{1}{2} \tilde{g} \left[ \tilde{g}^{\mu\nu}D_\mu\Phi^\dagger D_\nu\Phi - i\Phi^\dagger e^{\mu}_\alpha\sigma^\alpha D_\mu\Phi + \frac{1}{2} \tilde{b}' \left( \Phi^\dagger\Phi \right)^2 \right] + \text{conj} \] (4.52)
where $\tilde{b}' = (2m)^{-2}\tilde{b}$. Because of the symmetry between fundamental bosons and fermions, it immediately follows that
\[ \mathcal{L}_f = -\frac{1}{2} \tilde{g} \left[ \tilde{g}^{\mu\nu}D_\mu\psi^\dagger D_\nu\psi - i\psi^\dagger e^{\mu}_\alpha\sigma^\alpha D_\mu\psi + \frac{1}{2} \tilde{b}' \left( \psi^\dagger\psi \right)^2 \right] + \text{conj} \] (4.53)
in the corresponding Lagrangian density for fermions. (Alternatively, one can obtain (4.53) directly from I(2.13) and I(2.14), if the approximations below I(4.1), I(4.4), and I(6.7) are not made. Then the symmetry between fermions and fundamental bosons leads back to (4.52) and (4.50).) With the scaling $\psi' = \lambda^2 \psi$, this becomes

$$\mathcal{L}_f = -\frac{1}{2g} \left[ \tilde{m}^{-1} \tilde{g}^{\mu\nu} D_\mu \psi'^\dagger D_\nu \psi' - i \tilde{m} \lambda^2 \psi'^\dagger e^\mu_\alpha \sigma^\alpha D_\mu \psi' + \frac{1}{2} b' \left( \psi'^\dagger \psi' \right)^2 \right] + \text{con.} \quad (4.54)$$

As in other grand-unified theories, the initial fermion fields $\psi_r$ all have the same chirality. (In the cosmological model described by (4.47), they are all right-handed.) One then obtains fields of the opposite chirality by charge conjugation. For the present $SO(10)$ theory, the result is 8 left-handed and 8 right-handed two-component spinors (per generation), with the Lagrangian

$$\mathcal{L}_f = -\frac{1}{2g} \left[ \tilde{m}^{-1} \tilde{g}^{\mu\nu} D_\mu \psi'^\dagger R D_\nu \psi_R - i \tilde{m} \lambda^2 \psi'^\dagger e^\mu_\alpha \sigma^\alpha D_\mu \psi_R + \tilde{m}^{-1} \tilde{g}^{\mu\nu} D_\mu \psi'^\dagger L D_\nu \psi_L - i \tilde{m} \lambda^2 \psi'^\dagger e^\mu_\alpha \sigma^\alpha D_\mu \psi_L \right] + \text{con.} \quad (4.55)$$

where $\tilde{\sigma}^0 = \sigma^0$, $\tilde{\sigma}^k = -\sigma^k$, and the four-fermion interaction of (4.53) has been neglected. Because of the symmetry between fermions and bosons, (4.51) can be rewritten in the same form as (4.55), with $\psi_R, \psi_L \rightarrow \tilde{m}^{1/2} \phi_R, \tilde{m}^{-1/2} \phi_L$ and the four-boson interaction retained. Like their fermionic partners, $\phi_R$ and $\phi_L$ each consist of 8 two-component complex fields. There are important differences, however: The bosonic fields consist of ordinary numbers, rather than anticommuting Grassmann numbers, and they transform as scalars rather than spinors.

$\mathcal{L}_b$ does not contain mass terms, and $\mathcal{L}_f$ does not contain Yukawa interactions, so it is necessary to assume that these contributions come from radiative corrections involving gauge interactions together with the four-particle interactions of (4.32)-(4.34) (after Fierz rearrangements). For concreteness, let us focus on just the electroweak Higgs doublet $\phi_h$, the left-handed lepton doublet $\psi_L$, and the right-handed electron singlet $\psi_e$, so that

$$\phi_h = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}. \quad (4.56)$$

If a negative mass term is added, (4.51) gives

$$\mathcal{L}_h = \frac{1}{2g} \left[ - \tilde{g}^{\mu\nu} D_\mu \phi^\dagger_h D_\nu \phi_h + i \tilde{m} \lambda^2 \phi^\dagger_h e^\mu_\alpha \sigma^\alpha D_\mu \phi_h + \mu^2 \phi^\dagger_h \phi_h - \frac{1}{2} \tilde{h} \left( \phi^\dagger_h \phi_h \right)^2 \right] + \text{con.} \quad (4.57)$$

and if a Yukawa coupling is added (4.55) gives

$$\mathcal{L}_f = \frac{1}{2g} \left[ - \tilde{m}^{-1} \tilde{g}^{\mu\nu} D_\mu \psi^\dagger_L D_\nu \psi_L + i \tilde{m} \lambda^2 \psi^\dagger_L e^\mu_\alpha \sigma^\alpha D_\mu \psi_L - \tilde{m}^{-1} \tilde{g}^{\mu\nu} D_\mu \psi^\dagger_e D_\nu \psi_e + i \tilde{m} \lambda^2 \psi^\dagger_L e^\mu_\alpha \sigma^\alpha D_\mu \psi_e - \lambda_{\text{eff}} \psi^\dagger_L \phi_h \psi_e \right] + \text{con.} \quad (4.58)$$

In the present theory, however, the effective Yukawa coupling must have the form

$$\lambda_{\text{eff}} = \frac{\lambda_0 \hat{\psi}_L^\dagger \psi_e}{\tilde{G}_U T / M} \quad (4.59)$$
where $\lambda_0$ is dimensionless and $\langle \phi^\dagger_{\text{GUT}} \phi_{\text{GUT}} \rangle \sim M^2 \sim m^2_{\text{GUT}}$. There are several reasons for this: (i) The fundamental Lagrangian does not contain any three-particle interactions involving scalar bosons. Instead it contains only four-particle interactions of the form (4.32)-(4.34). (ii) In the present theory, each component of the doublet $\phi_h$ consists of two complex fields rather than one. It is therefore necessary to contract $\phi_h$ with another scalar field in forming the Lagrangian. (iii) Fundamental bosons have the same coupling to gauge fields as fermions, and the same quantum numbers (for example, lepton number). This implies that $\phi_h$ must be contracted with another scalar field which has compensating quantum numbers.

In the simplest version of the present theory, $\phi_h$ is the superpartner of $\psi^c_\ell$. Then $\phi_h$ has a lepton number of -1, and an R-parity of -1:

$$R = (-1)^{3(B-L)+2s} = -1$$  \hspace{1cm} (4.60)

where $B, L,$ and $s$ are the baryon number, lepton number, and spin. Similarly, all fundamental scalar bosons have an R-parity of -1, and are true superpartners of the fermions. Since these are the only superpartners (in the simplest version of the theory), the most natural candidate for cold dark matter is then a scalar boson which couples only through the weak interaction. This prediction should be testable in spin-dependent dark matter searches.

There are two unconventional features of the symmetry-breaking that results from (4.57). (1) Since $\phi_h$ has a lepton number of -1, the electroweak Higgs condensate has net lepton number. This kind of feature, however, is already present in the Standard Model [21]: Since $\phi_h$ has a hypercharge of +1, the electroweak condensate has net hypercharge. In fact, these are special cases of the familiar result that the vacuum in standard physics has nonzero values for many physical quantities, including energy and charge. One can often eliminate such vacuum contributions through the cheat of imposing normal ordering [22], but in any fundamental physical theory they are actually present, and in most cases they do not lead to observable effects. (An exception is the vacuum energy, which is revealed through gravitational and Casimir effects, and which will be treated in Section 5.) For example, the vacuum lepton number does not affect conservation of lepton number any more than the vacuum energy affects conservation of energy. Lepton number violation occurs only via the exchange of superheavy vector bosons, as in proton decay and neutrinoless double beta decay [2-8, 21], so lepton number, baryon number, and R-parity are conserved to a very good approximation in the present theory. (2) There is an extra first-order term in (4.57). This term will influence the equation of motion of fundamental scalar particles, as discussed below, but it will not change the results of the Standard Model for, e.g., the masses of W bosons: In a locally inertial coordinate system, with $\partial_\mu \phi_h = 0$, the relevant terms are

$$\phi_h^\dagger A^\mu \lambda \phi_h - \bar{m} \phi_h^\dagger \sigma^\mu A^\mu \phi_h = A^\mu A^\mu_{\phi_h} \phi_h^\dagger t_i \phi_h - A^\mu \bar{m} \phi_h^\dagger \sigma^\mu t_i \phi_h$$  \hspace{1cm} (4.61)

$$= a_{ij} A^\mu b^\mu_{t_i} - A^\mu_{\phi_h} \phi_h^\dagger t_i \phi_h + \text{constant}$$  \hspace{1cm} (4.62)

$$= a_{ij} \left( A^\mu_{t_i} - c^\mu t_i \right) - a_{ij} c^\mu c^\mu_{t_i}$$  \hspace{1cm} (4.63)

$$= \phi_h^\dagger \tilde{A}^\mu \phi_h + \text{constant}$$  \hspace{1cm} (4.64)

with obvious definitions for $a_{ij}, b^\mu, c^\mu,$ and $\tilde{A}_\mu$. We thus regain standard physics, except for unobservable constant shifts in the gauge fields $A_\mu = A^i_\mu t_i$ and in the action or energy.
In a weak gravitational field and a locally inertial coordinate system, the Lagrangian (4.58) yields the equation of motion

\[
\left( \bar{m}^{-1}\eta^{\mu\nu}D_{\mu}D_{\nu} + i\sigma^{\mu}D_{\mu} \right) \psi^f_R - m_f\psi^f_L = 0 \tag{4.66}
\]

\[
\left( \bar{m}^{-1}\eta^{\mu\nu}D_{\mu}D_{\nu} + i\bar{\sigma}^{\mu}D_{\mu} \right) \psi^f_L - m_f\psi^f_R = 0 \tag{4.67}
\]

where \( \eta^{\mu\nu} \) is the Minkowskian metric tensor, \( \psi^f_R \) and \( \psi^f_L \) are the right- and left-handed fields for a specific fermion, and \( m_f \) is determined by \( \langle \phi^\dagger_{\text{GUT}}\phi_h \rangle / M \). For energies that are small compared to \( \bar{m} \), the first term can be neglected in each expression, and we obtain the Dirac equation.

The corresponding equation of motion for a fundamental scalar boson follows from (4.57) with a positive mass term:

\[
-\eta^{\mu\nu}D_{\mu}D_{\nu}\phi_s - i\bar{m}\sigma^{\mu}D_{\mu}\phi_s + m^2_s\phi_s = 0. \tag{4.68}
\]

The second term implies an unconventional equation of motion for the Higgs boson. It also implies unconventional dynamics for dark matter WIMPs, with an effect on density profiles which will be discussed elsewhere.

Notice, however, that these deviations from standard physics are predicted only for (i) fermions at extremely high energy and (ii) fundamental scalar bosons which have not yet been observed.

5 Canonical Quantization and Vacuum Energy

The preceding sections were implicitly based on the path-integral approach to quantization, with commuting and anticommuting classical fields \( \phi' \) and \( \psi' \). In this section, let us switch to the canonical approach, and determine whether the present theory permits consistent extensions of standard field theory [18, 22]. (This is not a trivial issue because the Lagrangian (1.1) is quite unconventional. For example, it is not Lorentz invariant at high energy, although it retains many of the features of Lorentz invariance, including rotational invariance, CPT invariance, and the requirement that \( p^2 = 0 \) for massless particles, and it appears to be in agreement with even the most sensitive experimental tests of Lorentz invariance [23, 24].)

Let us also change notation by letting \( \phi \) and \( \psi \) represent 2-component, complex, massless bosonic and fermionic fields with Lagrangians of the form (4.51) and (4.54). In a locally inertial coordinate system, and with interactions neglected, (4.51) gives

\[
\mathcal{L}_\phi = -\eta^{\mu\nu}\partial_{\mu}\phi^\dagger\partial_{\nu}\phi + \frac{1}{2}\left( i\bar{m}\phi^\dagger\sigma^\mu\partial_{\mu}\phi + \text{conj} \right) \tag{5.1}
\]

\[
\mathcal{L}_\phi = \phi^\ddagger\phi - \partial^k\phi^\dagger\partial_k\phi + \frac{1}{2}\left( i\bar{m}\phi^\dagger\phi + i\bar{m}\phi^\dagger\sigma^k\partial_k\phi + \text{conj} \right) \tag{5.2}
\]
where \( \dot{\phi} = \partial_0 \phi \). The canonical momenta are (in a slightly unconventional notation)

\[
\pi^\dagger_\phi = \frac{\partial L_\phi}{\partial \dot{\phi}} = \dot{\phi}^\dagger + \frac{1}{2} \vec{m} \phi^\dagger \\
\pi_\phi = \frac{\partial L_\phi}{\partial \dot{\phi}^\dagger} = \dot{\phi} - \frac{1}{2} \vec{m} \phi
\]

(5.3)

(5.4)

and the Hamiltonian density is

\[
\mathcal{H}_\phi = \pi^\dagger_\phi \dot{\phi} + \phi^\dagger \pi_\phi - L_\phi \\
= \dot{\phi}^\dagger \phi + \partial^k \phi^\dagger \partial_k \phi - \frac{1}{2} \left( \vec{m} \phi^\dagger \sigma^k \partial_k \phi + \text{conj} \right).
\]

(5.5)

(5.6)

From (5.1) we obtain the equation of motion

\[
\eta^{\mu\nu} \partial_\mu \partial_\nu \phi + i \vec{m} \sigma^\mu \partial_\mu \phi = 0.
\]

(5.7)

Let \( \phi_n \) be a solution to this equation, and let \( \phi_n^\dagger \) be a solution to the equation that one similarly obtains for \( \phi^\dagger \). (Since \( \phi \) and \( \phi^\dagger \) vary independently, \( \phi_n^\dagger \) is not necessarily the Hermitian conjugate of \( \phi_n \).) Then we can write

\[
\phi = \sum_n a_n \phi_n, \quad \phi^\dagger = \sum_n a_n^\dagger \phi_n^\dagger.
\]

(5.8)

For each 3-momentum \( \vec{p} \), there are four solutions to (5.7):

\[
\phi_{p1} = A_{p1} u_p e^{i \vec{p} \cdot \vec{x} - i \omega_{p1} x^0}, \quad \omega_{p1} = |\vec{p}|
\]

(5.9)

\[
\phi_{p2} = A_{p2} u_p e^{i \vec{p} \cdot \vec{x} - i \omega_{p2} x^0}, \quad \omega_{p2} = -\vec{m} - |\vec{p}|
\]

(5.10)

\[
\phi_{p3} = A_{p3} v_p e^{i \vec{p} \cdot \vec{x} - i \omega_{p3} x^0}, \quad \omega_{p3} = -|\vec{p}|
\]

(5.11)

\[
\phi_{p4} = A_{p4} v_p e^{i \vec{p} \cdot \vec{x} - i \omega_{p4} x^0}, \quad \omega_{p4} = -\vec{m} + |\vec{p}|
\]

(5.12)

where

\[
\vec{\sigma} \cdot \vec{p} u_p = + |\vec{p}| u_p
\]

(5.13)

\[
\vec{\sigma} \cdot \vec{p} v_p = - |\vec{p}| v_p
\]

(5.14)

\[
n \leftrightarrow \vec{p}, \lambda \quad \text{with} \quad \lambda = 1, 2, 3, 4
\]

(5.15)

and the \( A_{p\lambda} \) are normalization constants specified below. We can choose

\[
u_p^\dagger u_p = v_p^\dagger v_p = 1, \quad u_p^\dagger v_p = v_p^\dagger u_p = 0
\]

(5.16)

\[
u_p u_p^\dagger + v_p v_p^\dagger = 1
\]

(5.17)

where \( 1 \) is the \( 2 \times 2 \) identity matrix. The \( \phi_n^\dagger \) are obtained by taking the Hermitian conjugates of (5.9)-(5.12), except that the coefficients \( A_n^\dagger \) are not necessarily the Hermitian conjugates of the \( A_n \). We then have

\[
\dot{\phi}_n = -i \omega_n \phi_n, \quad \phi_n^\dagger = i \omega_n \phi_n^\dagger
\]

(5.18)
and (5.3)-(5.4) give

\[ \pi_\phi^\dagger = \frac{1}{2} i \sum_n (2\omega_n + \bar{m}) a_n^\dagger \phi_n \]  
(5.19)

\[ \pi_\phi = -\frac{1}{2} i \sum_n (2\omega_n + \bar{m}) a_n \phi_n. \]  
(5.20)

We quantize by interpreting \( \phi \) and \( \pi_\phi^\dagger \) as operators, and requiring that

\[ [\phi (\vec{x}, x^0), \pi_\phi^\dagger (\vec{x}', x^0)]_- = i\delta (\vec{x} - \vec{x}') \mathbf{1} \]  
(5.21)

or more explicitly

\[ [\phi_\alpha (\vec{x}, x^0), \pi_{\phi,\beta}^\dagger (\vec{x}', x^0)]_- = i\delta (\vec{x} - \vec{x}') \delta_{\alpha\beta} \]  
(5.22)

where \( \alpha \) and \( \beta \) label the two components of \( \phi \) and \( \pi_\phi^\dagger \), with \( [X,Y]_\pm = XY \pm YX \). This requirement will be satisfied if

\[ [a_n, a_m^\dagger]_- = \delta_{nm} \omega_n / |\omega_n| \]  
(5.23)

\[ [a_n, a_m]_- = [a_n^\dagger, a_m^\dagger]_- = 0 \]  
(5.24)

\[ A_n^\dagger A_n = A_n A_n^\dagger = V^{-1} (2\omega_n + \bar{m})^{-1} \omega_n / |\omega_n| \]  
(5.25)

where \( V \) is the normalization volume, since this last equation implies that

\[
\frac{1}{2} \sum_n (2\omega_n + \bar{m}) \phi_n (\vec{x}, x^0) \phi_n^\dagger (\vec{x}', x^0) \frac{\omega_n}{|\omega_n|} = \frac{1}{2} \sum_{\rho,\lambda=1,2} (2\omega_{\rho\lambda} + \bar{m}) \frac{\omega_n}{|\omega_n|} A_{\rho\lambda} A_{\rho\lambda}^\dagger u_{\rho} u_{\rho}^\dagger e^{i\vec{p}^\prime \cdot (\vec{x} - \vec{x}')}
\]

\[ + \frac{1}{2} \sum_{\rho,\lambda=3,4} (2\omega_{\rho\lambda} + \bar{m}) \frac{\omega_n}{|\omega_n|} A_{\rho\lambda} A_{\rho\lambda}^\dagger v_{\rho} v_{\rho}^\dagger e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \]

\[ = V^{-1} \sum_{\vec{p}^\prime} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} (u_{\rho} u_{\rho}^\dagger + v_{\rho} v_{\rho}^\dagger) \]  
(5.26)

\[ = \delta (\vec{x} - \vec{x}') \mathbf{1}. \]  
(5.27)

From (5.9)-(5.12), (5.16), and (5.25), it follows that

\[ \phi_n^\dagger (\vec{x}, x^0) \phi_n (\vec{x}', x^0) = V^{-1} (2\omega_n + \bar{m})^{-1} \omega_n / |\omega_n|. \]  
(5.28)

Since \( \phi \) satisfies (5.7), and \( \phi^\dagger \) satisfies its conjugate equation of motion, (5.1) implies that

\[ \mathcal{L}_\phi = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu (\phi^\dagger \phi) \]  
(5.29)

so the Hamiltonian density of (5.5) is

\[ \mathcal{H}_\phi = \pi_\phi^\dagger \dot{\phi} + \dot{\pi_\phi} \phi + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu (\phi^\dagger \phi). \]  
(5.30)

(These last two equations hold only when \( \phi \) and \( \phi^\dagger \) satisfy their equations of motion.) The term involving \( \eta^{\mu\nu} \partial_\mu \partial_\nu (\phi^\dagger \phi) \) can be ignored, since it does not contribute to the integrals for
the action and total energy. In any state with a well-defined number of particles, we then have

\[ \langle H \phi \rangle = \int d^3 x \langle H \phi \rangle \]
\[ = \sum_n \omega_n (2 \omega_n + \bar{m}) \langle a_n^\dagger a_n \rangle \int d^3 x \phi_n^\dagger \phi_n \]
\[ = \sum_n \langle a_n^\dagger a_n \rangle |\omega_n| . \]

As usual, let us define

\[ N_n^b = \langle a_n^\dagger a_n \rangle , \omega_n > 0 \] (5.34)
\[ N_n^b = \langle b_n^\dagger b_n \rangle , \omega_n < 0 \] (5.35)

where

\[ b_n^\dagger = a_n \quad b_n = a_n^\dagger \] (5.36)

so that

\[ \langle H \phi \rangle = \sum_{n, \omega_n > 0} \langle a_n^\dagger a_n \rangle |\omega_n| + \sum_{n, \omega_n < 0} \langle b_n^\dagger b_n \rangle |\omega_n| \]
\[ = \sum_n N_n^b |\omega_n| + \sum_{n, \omega_n < 0} |\omega_n| \]

since

\[ [b_n, b_m^\dagger]_+ = -[a_n, a_m^\dagger]_+ = +\delta_{nm} , \omega_n < 0 \] (5.39)

according to (5.23).

The above treatment can be repeated for fermions, with

\[ \phi \rightarrow \psi \quad a_n \rightarrow c_n \quad A_n \rightarrow B_n \quad b_n \rightarrow d_n , \]

and with (5.21), (5.23)–(5.25), and (5.37)–(5.39) replaced by

\[ [\psi(\vec{x}, x^0), \pi_\psi^\dagger(\vec{x}', x^0)]_+ = i\delta(\vec{x} - \vec{x}') \mathbf{1} \] (5.41)
\[ [c_n, c_m^\dagger]_+ = \delta_{nm} \] (5.42)
\[ [c_n, c_m]_+ = [c_n^\dagger, c_m^\dagger]_+ = 0 \] (5.43)
\[ B_n^\dagger B_n = V^{-1}(2\omega_n + \bar{m})^{-1} \] (5.44)

\[ \langle H \psi \rangle = \sum_{n, \omega_n > 0} \langle c_n^\dagger c_n \rangle |\omega_n| - \sum_{n, \omega_n < 0} \langle d_n d_n^\dagger \rangle |\omega_n| \]
\[ = \sum_n N_n^f |\omega_n| - \sum_{n, \omega_n < 0} |\omega_n| \] (5.46)
\[
\left[ d_n, d_m^\dagger \right]_+ = \left[ c_n, c_m^\dagger \right]_+ = \delta_{nm}.
\] (5.47)

(In the present section, a factor of \( \bar{m}^{-1/2} \) has been absorbed in both \( \psi \) and \( \phi \).) The total energy is then
\[
\langle H \rangle = \sum_n N_n^b |\omega_n| + \sum_n N_n^f |\omega_n|.
\] (5.48)

(This result is not as trivial as it may seem, because the Lagrangian (5.1) violates Lorentz invariance and the \( \omega_n \) are given by (5.9)-(5.12).) In particular, the vacuum energy is
\[
\langle 0 | H | 0 \rangle = 0.
\] (5.49)

Before the initial supersymmetry of the present theory is broken, there is thus a cancellation of the bosonic and fermionic contributions to the vacuum energy, just as in standard supersymmetry [25, 26].

6 Conclusion

In Sections 2 and 3, it was shown that the action (1.1) results from a remarkably simple picture: a single fundamental system which consists of “bits” varying randomly over all possible states.

Some physical implications of (1.1) have already been discussed in Ref. 1. In Section 4 of the present paper, a number of points were clarified and considered in more detail. In addition, it was found that the simplest form of the present theory leads to several new predictions: (1) The only superpartners are scalar bosons, which may be observable at accelerators in the near future. (2) It follows that the natural candidate for cold dark matter is a spin-zero WIMP. (3) The Higgs boson has an R-parity of -1. This means that the Higgs cannot be produced individually, but only as one member of a pair of superpartners. (4) The Higgs, dark matter WIMPs, and other scalar superpartners all satisfy an unconventional equation of motion.

Finally, in Section 5, it was shown that a consistent field theory can be formulated for the unconventional Lagrangian (5.1) and its fermionic counterpart, and that the vacuum energy is zero before supersymmetry is broken.
Acknowledgement

This work was supported by the Robert A. Welch Foundation.

Appendix: Corrections to Ref. 1

The “cosmological constant” of the abstract and last section is actually just the contribution from condensed Higgs fields, as defined in (9.22). Also, since (3.24) does not hold for a pure state, it should be discarded; (3.26) then holds only in the approximation (4.7). Finally, the approximation $\tilde{\mu} = 0$ in the paragraph containing (8.40) should be discarded.
References


