1. In this problem, start with the following two equations for the scattering of a plane wave, which we derived in class:

\[ \psi_k(\vec{r}) = e^{ik\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\vec{k}', \vec{k}) \]

\[ f(\vec{k}', \vec{k}) = -\frac{m}{2\pi\hbar^2} \int d^3r' e^{-i\vec{k}\cdot\vec{r}'} V(\vec{r}') \psi_k(\vec{r}') \]

(a) (10) In the first Born approximation, and for a central potential (i.e. \( V(\vec{r}) = V(r) \)), obtain a simple integral expression for \( f \) in the form

\[ f(\theta) = \text{constant} \times \int_0^\infty dr \ (\text{function of } q \text{ and } r) \ V(r) \]

where \( q = |\vec{q}| \), \( \vec{q} = \vec{k}' - \vec{k} \), and \( \theta \) is the angle between \( \vec{k} \) and \( \vec{k}' \).

(b) (10) Let us model a target by

\[ V(r) = -V_0 \text{ for } r < R \]
\[ V(r) = 0 \text{ for } r > R \]

with \( V_0 > 0 \). Calculate \( f(\theta) \) as a function of \( qR \) (writing your final answer in a simple form).

(c) (i) (3) Now obtain the differential cross section \( \frac{d\sigma}{d\Omega} \) as a function of \( qR \).

(ii) (3) Solve for \( q \) in terms of \( k \) and \( \sin(\theta/2) \). Then roughly sketch a graph of \( \frac{d\sigma}{d\Omega} \) as a function of \( \sin(\theta/2) \) for fixed \( k \) and \( R \).

(iii) (2) Roughly sketch a graph of \( \frac{d\sigma}{d\Omega} \) as a function of \( \theta \) (again for fixed \( k \)), labeling \( \theta = \pi/2 \), \( \theta = \pi \), etc.

(iv) (2) Explain qualitatively how you could use this graph of \( \frac{d\sigma}{d\Omega} \) as a function of \( \theta \) to estimate the size \( R \) of the target.
2. (20) The proton in a hydrogen atom is not really a point charge. To roughly estimate what might be the effect of its finite size, let us model it by a hollow shell of charge of radius

\[ R = 10^{-5} a_0 \]

where \( a_0 \) is the Bohr radius. Then the potential energy of an electron is

\[
V(r) = -k \frac{e^2}{R} \quad \text{for} \quad r < R
\]

\[
V(r) = -k \frac{e^2}{r} \quad \text{for} \quad r > R.
\]

(In our CGS units, \( k = 1 \).)

For the 1s state (ground state) of hydrogen, with wavefunction

\[
\frac{1}{\sqrt{4\pi}} \frac{2}{a_0^{3/2}} e^{-r/a_0},
\]

calculate the change \( \Delta E_{1s} \) in the energy due to the finite size of the proton in this model.

You may make the approximation \( R \ll a_0 \) where appropriate.

Give your answer in terms of \( a_0 \) and \( e \).

Also give a roughly approximate numerical answer, in eV.
3. In this problem, start with the following equation which we derived in time-dependent perturbation theory:

\[ P_{0\rightarrow n}(t) = \frac{1}{i\hbar} \left[ dt' e^{i(\epsilon_n - \epsilon_0)t'/\hbar} \langle n| V_t | 0 \rangle \right]^2 \]

where \( V_t \) is the time-dependent perturbation in the Schrödinger picture.

Let \( V_t = -eE x \) for \( t > 0 \) and \( V_t = 0 \) for \( t < 0 \). For a harmonic oscillator, recall that

\[
\begin{align*}
    a &= \left( \frac{m\omega}{2\hbar} \right)^{1/2} x + \frac{i}{(2m\hbar\omega)^{1/2}} p \\
    a' &= \left( \frac{m\omega}{2\hbar} \right)^{1/2} x - \frac{i}{(2m\hbar\omega)^{1/2}} p
\end{align*}
\]

(a) (10) By integrating, find a simple expression for \( P_{0\rightarrow n}(t) \) which holds for very short times, \( t \ll \frac{\hbar}{\epsilon_n - \epsilon_0} \).

(b) (5) Calculate the short-time transition rate from the ground state \( n = 0 \) to the second excited state \( n = 2 \).

Here and in the next part (c), give the transition rate in terms of the various constants above \( (m, \omega, \hbar, e, E) \).

(c) (10) Calculate the short-time transition rate from the ground state \( n = 0 \) to the first excited state \( n = 1 \).
4. (a) (10) Calculate the probability for the electron in the ground state of a hydrogen atom to be found outside its classically allowed region near the nucleus. Recall again that the wavefunction is
\[ \frac{1}{\sqrt{4\pi}} \frac{2}{a_0^{3/2}} e^{-r/a_0}. \]

(b) (10) At time \( t = 0 \) the unnormalized wavefunction for an electron in a hydrogen atom is
\[ \psi = 2\psi_{100} + \psi_{211} + \psi_{210} + \psi_{21-1} \]
where the eigenstates are labeled as usual by the quantum numbers: \( \psi_{n,m} \).

Calculate the expectation value of the energy, in terms of the ground state energy \( \varepsilon_0 \), at the later time
\[ t = \frac{\hbar}{\varepsilon_0}. \]
(Neglect spontaneous emission here and in the rest of this problem.)

(c) At a still later time, the angular momentum \( L \) and its \( z \) component are measured, and the result of the measurement is \( L = \sqrt{2}\hbar, L_z = \hbar \).

Write down the state \( |\psi\rangle \) of the electron after the measurement, using the notation \( |n,m\rangle \).