Physics 607 Final Exam

Please be well-organized, and show all significant steps clearly in all problems. You are graded on your work, so please do not just write down answers with no explanation! And show your work on separate pages, not on these pages with the problem statements.

The variables have their usual meanings: \( E = \) energy, \( S = \) entropy, \( V = \) volume, \( N = \) number of particles, \( T = \) temperature, \( P = \) pressure, \( \mu = \) chemical potential, \( B = \) applied magnetic field, \( C_V = \) heat capacity at constant volume, \( F = \) Helmholtz free energy, \( k = \) Boltzmann constant. Also, \( \langle \cdots \rangle \) represents an average.

1. (20) Fun with the compressibility of a Fermi gas. The velocity of sound is obtained from the adiabatic compressibility:

\[
\kappa_S \equiv - \frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_S .
\]

Calculate \( \kappa_S \) for an ideal (quantum) Fermi gas, in terms of the functions

\[
f_n(z) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} z^\ell}{\ell^n} = z - \frac{2}{3^n} + \frac{3}{3^n} - \ldots
\]

where \( z \) is the fugacity (in our textbook’s notation). You may start with the equation:

\[
P_v^{5/3} = \text{constant} \quad , \quad v \equiv V / N .
\]

Give your answer in terms of the number density \( n \equiv N / V \), the Boltzmann constant \( k \), the temperature \( T \), \( f_{3/2}(z) \), and \( f_{5/2}(z) \).

You may also use

\[
P \quad kT \quad \lambda_{th}^{3} = \frac{1}{\lambda_{th}^{3}} f_{5/2}(z) \quad , \quad \frac{N}{V} = \frac{1}{\lambda_{th}^{3}} f_{3/2}(z) \quad , \quad \frac{\partial f_n(z)}{\partial z} = \frac{1}{z} f_{n-1}(z) \quad , \quad \text{with} \quad \lambda_{th} = \frac{h}{(2\pi mkT)^{1/2}}
\]

although you do not need all of these equations.

*Not the isothermal compressibility \( \kappa_T \), since there is ordinarily not enough time during one period of oscillation to achieve thermal equilibrium. The velocity of sound is given by

\[
v_{\text{sound}} = \sqrt{\left( \frac{\partial P}{\partial \rho_m} \right)_S} = \sqrt{\frac{\kappa_S^{-1}}{\rho_m}} = \sqrt{\frac{\text{stiffness}}{\text{mass density}}} .
\]

**This equation is also true for a classical ideal gas of particles with no internal excitations, but for the quantum ideal gas considered here \( \gamma \equiv \frac{C_P}{C_V} = \frac{5}{3} \frac{f_{5/2}(z)f_{1/2}(z)}{\left[ f_{3/2}(z) \right]^2} \neq \frac{5}{3} \). I.e., for a quantum ideal gas, it is no longer true that \( P_v \gamma \) = constant during an adiabatic process.
2. **Fun with the virial theorem.** Consider the average of the quantity \( \frac{dG}{dt} \) over a time \( \tau \), with \( \tau \to \infty \) and

\[
G \equiv \sum_i q_i p_i .
\]

The \( q_i \) are a set of coordinates, and the \( p_i \) are the corresponding momenta. We assume a classical system in which the magnitude of \( G \) is bounded.

(a) (10) Show that

\[
\sum_i \langle u_ip_i \rangle = - \sum_i \langle q_i F_i \rangle
\]

where \( u_i \) is the velocity and \( F_i \) is the force corresponding to the coordinate \( q_i \).

(b) (10) Suppose that the particles are confined to a volume \( V \) by a pressure \( P \). From kinetic arguments, one finds that

\[
- \sum_i \langle q_i F_i \rangle = - \sum_i \langle q_i F'_i \rangle + 3PV
\]

where the forces \( F'_i \) are due to interactions among the particles.*

Also suppose that

\[
F'_i = - \frac{\partial U}{\partial q_i}
\]

and that the potential energy \( U \) is a homogeneous function of order \( n \) of all the particle coordinates:

\[
U(\lambda q_1, \lambda q_2, ...) = \lambda^n U(q_1, q_2, ...).
\]

For nonrelativistic particles, show that

\[
\langle K \rangle = \text{constant} \times \langle U \rangle + \text{different constant} \times PV
\]

while at the same time obtaining the constants. Here \( \langle K \rangle \) is the mean value of the total kinetic energy and \( \langle U \rangle \) is the mean value of the total potential energy.**

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*If there is no confinement to a fixed volume (as for dark matter particles in a galactic halo, which are just bound by gravity), \( P = 0 \) and \( F'_i = F_i \).

** With the appropriate reinterpretation of averages, this result can be generalized to a quantum system.
3. Fun with the density matrix. The density operator is
\[ \rho = \sum_\alpha w_\alpha \, |\alpha \rangle \langle \alpha | \]
where the \(|\alpha\rangle\) are time-dependent states available to the system, each with a weight (or probability) \(w_\alpha\). The density matrix in a particular representation with basis states \(|i\rangle\) is
\[ \rho_{ij} = \langle i | \rho | j \rangle = \sum_\alpha w_\alpha \langle i | \alpha \rangle \langle \alpha | j \rangle. \]

In Part (c) we will specialize to (i) \(|\alpha\rangle = \text{stationary state with Hamiltonian operator } \hat{H} \), so that \(|\alpha, t\rangle = e^{-iE_\alpha t/\hbar} |\alpha, 0\rangle\), (ii) thermal equilibrium at temperature \(T\), so that \(w_\alpha = \frac{e^{-E_\alpha/kT}}{Z}\), (iii) the coordinate representation, so that \(\langle i | \alpha \rangle = \langle \vec{r} | \alpha \rangle \equiv \psi_\alpha (\vec{r}, t)\), and (iv) the case when \(\psi_\alpha (\vec{r}, t)\) is a one-component function:
\[ \rho (\vec{r}, \vec{r}', t) = \sum_\alpha \frac{e^{-E_\alpha/kT}}{Z} \psi_\alpha (\vec{r}, t) \psi_\alpha^* (\vec{r}', t) \]
with \(Z\) determined by
\[ \int d^3r \rho (\vec{r}, \vec{r}', t) = \sum_\alpha \frac{e^{-E_\alpha/kT}}{Z} \int d^3r \psi_\alpha^* (\vec{r}, t) \psi_\alpha (\vec{r}, t) = \frac{1}{Z} \sum_\alpha e^{-E_\alpha/kT} = 1. \]

(a) (7) Starting with the time-dependent Schrödinger equation
\[ i\hbar \frac{d}{dt} |\alpha\rangle = \hat{H} |\alpha\rangle \]
show that the equation of motion for the density operator is
\[ \frac{d}{dt} \hat{\rho} = \text{constant} \times [\hat{\rho}, \hat{H}]_- \]
while at the same time obtaining the constant. Here \([\hat{A}, \hat{B}]_-\) is the commutator of the operators \(\hat{A}\) and \(\hat{B}\).

(b) (3) The equation of motion in the Heisenberg picture for an operator \(\hat{O}(t)\) representing a physical quantity is
\[ \frac{d}{dt} \hat{O} = \frac{1}{i\hbar} [\hat{O}, \hat{H}]_- \]
Is your result consistent with this equation? Explain.

(c) (7) For a single nonrelativistic free particle in 3 dimensions, set up the integral that you have to evaluate to obtain \(\rho (\vec{r}, \vec{r}', t)\). Your expression should have the form
\[ \frac{1}{Z} \int \frac{d^3k}{(2\pi)^3} \left( \text{function of } \vec{k}, \vec{r}, \vec{r}' \right). \]
Recall that, for a free particle, the Hamiltonian and wavefunctions in the coordinate representation are respectively \(-\frac{\hbar^2}{2m} \nabla^2\) and \(\frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{r} - i\omega t}\).

(d) (3) Does your expression in Part (c) for \(\rho (\vec{r}, \vec{r}', t)\) have a dependence left on the time \(t\)? Why or why not?
4. Fun with superconductivity. Let us obtain the BCS scaling prediction

\[
\frac{2\Delta(0)}{kT_c} = 3.53
\]

starting with the BCS gap equation

\[
1 = N(0)V_0 \int_{0}^{\hbar \omega_D} d\varepsilon \frac{\tanh \left( \frac{E}{2kT_c} \right)}{E}, \quad E = \left( e^2 + \Delta(T)^2 \right)^{1/2}.
\]

Here \( \Delta(0) \) is the gap in the electronic spectrum at \( T = 0 \), which can be measured in microwave absorption or tunneling experiments. \( T_c \) is the transition temperature, below which the resistivity falls to zero. Also, \( N(0) \) is the electronic density of states (for one spin), \( V_0 \) is the effective phonon-mediated attraction, and \( \hbar \omega_D \) is the maximum (Debye) phonon energy. Using \( \tanh(\infty) = 1 \), you will need the integrals

\[
\int_{0}^{a} \frac{\tanh x}{x} \, dx = \ln(2.268 \, a) \quad \text{for} \quad a \gg 1
\]

\[
\int_{0}^{b} \frac{1}{\left(1 + x^2\right)^{1/2}} \, dx = \sinh^{-1} b \quad \text{with} \quad \sinh y = \frac{1}{2} e^y \quad \text{for} \quad y \gg 1.
\]

Assume weak coupling,

\[N(0)V_0 \ll 1,\]

which according to the results below also implies that

\[
\frac{kT_c}{\hbar \omega_D} \ll 1 \quad \text{and} \quad \frac{\Delta(0)}{\hbar \omega_D} \ll 1.
\]

(a) (9) By using the gap equation at \( T_c \), show that \( kT_c = \text{prefactor} \times e^{-1/N(0)V_0} \), while at the same time determining the prefactor.

(b) (9) By using the gap equation at \( T = 0 \), show that \( \Delta(0) = \text{different prefactor} \times e^{-1/N(0)V_0} \), while at the same time determining this different prefactor.

(c) (2) Using your results of Parts (a) and (b), obtain the BCS scaling prediction (see above) for \( \frac{\Delta(0)}{T_c} \).
5. Fun with neutron stars. This problem closely parallels our calculation for a white dwarf. Let us make the rough approximation that a neutron star is entirely supported by the degeneracy pressure of the neutrons (so that nuclear forces are ignored). We also make the approximations that the density is uniform and that the neutrons are highly relativistic. You may assume that the central gravitational pressure is given by

\[ P_{\text{grav}} = \frac{3}{8\pi} \frac{GM^2}{R^4}, \quad V = \frac{4}{3} \pi R^3, \quad M = Nm_n \]

where \( G = 6.67 \times 10^{-11} \) N m\(^2\) / kg\(^2\) is the gravitational constant, \( m_n = 1.67 \times 10^{-27} \) kg is the mass of a neutron, and \( N \) is the number of neutrons in the neutron star, which has mass \( M \), radius \( R \), and volume \( V \).

The relativistic expression for the energy \( \epsilon \) of a neutron is

\[ \epsilon^2 = p^2 c^2 + m_n^2 c^4 \quad \text{with} \quad c = 3.00 \times 10^8 \text{ m/s}. \]

In the following, treat the \( N \) neutrons as simply a quantum ideal gas of spin ½ fermions confined to a volume \( V \). According to a general argument for a quantum ideal gas

\[ P = \frac{1}{3} n \left\langle p u \right\rangle, \quad n = \frac{N}{V}, \quad u = \frac{\partial \epsilon}{\partial p} \]

where \( P \) is the pressure due to the gas and \( u \) is the particle velocity.

In the following, assume that \( pc \gg m_n c^2 \), but keep the leading 2 terms in the energy and pressure, as indicated below.

(a) (2) Obtain \( \epsilon \) in the form

\[ \epsilon \approx pc + \frac{ac}{p} \quad \text{(where you will determine the constant} \ a). \]

(b) (2) Obtain the pressure \( P \) in the form

\[ P = \frac{1}{3} nc \left\langle p + \frac{b}{p} \right\rangle \quad \text{(where you will determine the constant} \ b). \]

(c) (2) Obtain the density of states in momentum space, in the form \( A p^2 \) (where you will determine the constant \( A \)).

(d) (3) Calculate the Fermi momentum \( p_F \) as a function of \( n = N/V \).

(e) (3) Calculate the degeneracy pressure \( P \) as a function of \( n \).

(f) (3) Set \( P \) equal to \( P_{\text{grav}} \) and show that there is a limiting mass \( M_0 \) for the neutron star, requiring \( M \leq M_0 \). Express \( M_0 \) in terms of the various constants.

(g) (3) Calculate (or carefully estimate) the value of \( M_0 \) in solar masses \( M_\odot \), with \( M_\odot = 1.99 \times 10^{30} \) kg.

Show all the steps in SI units, but you may use the fact that \( \left( \frac{\hbar c}{G} \right)^{1/2} = \text{Planck mass} = 2.18 \times 10^{-8} \) kg.

(h) (2) For a neutron star with mass \( M_0/2 \), estimate the radius \( R \) in kilometers. (It is possible that you may want to use the fact that Compton wavelength of neutron \( \equiv \frac{\hbar}{m_n c} = 1.32 \times 10^{-15} \) m.)
6. **Just fun.** (5 points extra credit as a reward for those who attended the talks)

Please discuss 3 main points in **any ONE talk** that was given last Monday evening, **other than your own talk!**

Happy Holidays, Merry Christmas, and have a successful New Year!