Physics 607 Exam 2

Please be well-organized, and show all significant steps clearly in all problems. You are graded on your work, so please do not just write down answers with no explanation! And show your work on separate pages, not on these pages with the problem statements.

The variables have their usual meanings: $E =$ energy, $S =$ entropy, $V =$ volume, $N =$ number of particles, $T =$ temperature, $P =$ pressure, $\mu =$ chemical potential, $B =$ applied magnetic field, $C_V =$ heat capacity at constant volume, $F =$ Helmholtz free energy, $k =$ Boltzmann constant. Also, $\langle \cdots \rangle$ represents an average.

$$\langle n_k \rangle = \frac{1}{e^{|\varepsilon_k - \mu|/kT} \pm 1}$$

1. (a) (5) For any system, recall that $F \equiv E - TS$, and obtain $dF$ in terms of $dT$, $dV$, and $dN$. Then obtain the relation between $S$ and $\left( \frac{\partial F}{\partial T} \right)_{V,N}$.

(b) (5) Use your result from Part (a) to obtain the relation between $C_V$ and $\left( \frac{\partial^2 F}{\partial T^2} \right)_{V,N}$.

Now specialize to a system with free energy having the form

$$F = -a|\psi|^2 + \frac{1}{2}b|\psi|^4, \quad a = a_0(T_c - T), \quad b > 0 \text{ and } a_0 > 0.$$

(c) (5) Calculate the equilibrium value of $|\psi|^2$ for $T < T_c$ and $T > T_c$. Be sure to show that your solution is the stable one.

(d) (5) Calculate the heat capacity $C_V$ for $T < T_c$.

(e) (4) Sketch $C_V$ as a function of $T$. What kinds of system show this sort of behavior?
2. Recall that for a solid

\[ E_{\text{vib}}(T) = \sum_i \frac{1}{2} \hbar \omega_i + \sum_i \frac{\hbar \omega_i}{e^{\hbar \omega_i / kT}} - 1. \]

In your work on the parts below, you can assume this expression, but you should obtain your results from only this expression, being especially careful about the behavior at \( T \to \infty \).

(a) (12) For \( \frac{\hbar \omega_i}{kT} \ll 1 \), obtain \( \frac{\hbar \omega_i}{e^{\hbar \omega_i / kT}} \) to first order in \( \frac{\hbar \omega_i}{kT} \) -- i.e., neglecting only terms of order \( \left( \frac{\hbar \omega_i}{kT} \right)^2 \).

(b) (12) With \( C_V(\infty) = 3Nk \), show that

\[ \int_0^\infty [C_V(\infty) - C_V(T)]dT \]

is exactly equal to the zero-point energy of the solid.
3. Consider an ideal quantum gas of bosons in $D$ dimensions, with a relation between energy and momentum of the form $\varepsilon = ap^s$. We wish to determine the relation between $D$ and $s$ if Bose-Einstein condensation is to occur.

Recall that the density of states in momentum space is given by

$$\rho(p) dp = A p^{D-1} dp, \quad A = \frac{2\pi^{N/2}}{\Gamma(D/2)} \cdot \frac{1}{h^D/V},$$

where $V$ is the $D$-dimensional volume in which the particles are confined.

(a) (4) At temperature $T$, write down the equation for the number of particles $N$ in term of (i) the number of particles $N_0(T)$ in the state with $p = 0$ and (ii) an integral over $p$ (i.e., over all the excited states with $p > 0$).

(b) (4) Derive the value of the chemical potential $\mu$ below the transition temperature if Bose-Einstein condensation is to occur. (Please be clear in your argument.)

(c) (4) Assuming the value of the chemical potential $\mu$ obtained in Part (b), rewrite the integral of Part (a) as an integral over $\varepsilon$.

(d) (4) Using the result of part (c), obtain an equation of the form

$$N = \text{constant} \times T^X + N_0$$

while at the same time obtaining the constant prefactor and the other constant $X$, in terms of $A$, $D$, $s$, etc.

(e) (4) Obtain the condition on $D$ and $s$ if Bose-Einstein condensation is to occur.

(f) (4) For those cases where Bose-Einstein condensation does occur at some temperature $T_c \neq 0$, obtain $\frac{N_0}{N}$ as a function of $\frac{T}{T_c}$ for $T \leq T_c$. 
4. Let us calculate the speed of sound \( u \) in an ideal quantum gas of fermions at \( T = 0 \) using
\[
u^2 = \left( \frac{\partial P}{\partial \rho} \right)_{T=0}
\]
where \( \rho = m n \), \( m \) is the mass of one particle, and \( n = \frac{N}{V} \) is the number density. (See the bottom of this page for the general expression for \( u \).) These are nonrelativistic spin 1/2 fermions in 3 dimensions, with energy \( \epsilon = \frac{p^2}{2m} \).

(a) (4) Using the fact that the density of states in momentum space is given by
\[
\rho(p)dp = \frac{4\pi p^2dp}{h^3/V}
\]
calculate the value of the Fermi momentum \( p_F \) and the Fermi energy \( \epsilon_F = \frac{p_F^2}{2m} \) (in terms of the various constants).

(b) (4) Again using this density of states, and the result for \( p_F \), show that the energy of the system is given by
\[
\frac{E}{V} = \text{constant} \times n^{5/3}
\]
while at the same time determining the constant.

(c) (4) For a nonrelativistic ideal gas, recall that \( P = \frac{2E}{3V} \). Calculate \( u \) at \( T = 0 \), giving your answer in terms of \( n \) and \( m \).

(d) (4) Show that \( u = \text{constant} \times u_F \), while at the same time obtaining the constant. Here \( u_F \) is the Fermi velocity: \( \epsilon_F = \frac{1}{2} m u_F^2 \).

(e) (4) Using the standard expression for \( dE \) in terms of \( dS \), \( dV \), and \( dN \), and the Euler relation for \( E \) in terms of \( TS \), \( PV \), and \( \mu N \), obtain the Gibbs-Duhem relation involving \( s dT \), \( v dP \), and \( d\mu \). Here \( s = \frac{S}{N} \) and \( v = \frac{V}{N} \).

(f) (4) Use the Gibbs-Duhem relation to obtain the relation between \( \left( \frac{\partial P}{\partial n} \right)_{T} \) and \( \left( \frac{\partial \mu}{\partial n} \right)_{T} \). Then obtain the relation between \( \left( \frac{\partial P}{\partial \rho} \right)_{T} \) and \( \left( \frac{\partial \mu}{\partial n} \right)_{T} \).

(g) (4) Finally, using the above relation and the fact that \( \mu = \epsilon_F \) at \( T = 0 \), again calculate \( u \) at \( T = 0 \), giving your answer in terms of \( n \) and \( m \). Do you get the same answer as in Part (c)?

\[
u = \sqrt{\frac{1}{\rho \kappa_S}} \], \( \kappa_S \equiv -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_S \) = adiabatic compressibility, with \( T = 0 \) the same as \( S = 0 \).