Physics 607 Exam 2

Please show all significant steps clearly in all problems.

1. Consider spin waves at the surface of a ferromagnetic solid, with a single polarization and the dispersion relation $\omega = c k^2$, where $c$ is a constant. Since a surface is 2-dimensional, you may consider this to be a 2-dimensional system with area $A$.

   (a) (5) Show that the number of wavevectors with magnitude in the interval $dk$ at $k$ is given by
   
   \[ \rho(k) \, dk = \text{constant} \times A \, k \, dk \]  
   
   and determine the constant.

   (b) (15) Show that the heat capacity at low temperature $T$ is given by
   
   \[ C_V = \text{constant} \times T^\alpha. \]  

   At the same time, determine the constant prefactor and the value of $\alpha$. 
2. In three dimensions, the number of wavevectors whose magnitude lies in the interval $dk$ at $k$ is given by

$$\rho(k)\,dk = \frac{4\pi k^2\,dk}{(2\pi)^3/V}. \tag{3}$$

(a) (10) Starting with this expression, show that the energy density for a gas of photons is given by

$$\frac{E}{V} = \int_0^\infty \rho(\omega, T)\,d\omega \tag{4}$$

where

$$\rho(\omega, T) = \text{constant} \times \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1}, \tag{5}$$

and determine the constant.

(b) (10) If $\omega_{\text{max}}$ is the frequency at which $\rho(\omega, T)$ is a maximum, show that

$$\frac{\hbar\omega_{\text{max}}}{k_B T} = 3 \left[1 - \exp(-\hbar\omega/k_B T)\right] \tag{6}$$

and obtain an approximate value for $\hbar\omega_{\text{max}}/k_B T$. Try to get two significant figures.
3. Consider a classical one-dimensional gas with \( N \) noninteracting particles. Each particle has a kinetic energy \( \frac{p^2}{2m} \) and a potential energy

\[
V(x) = a|x|^n
\]  

(7)

where \( a \) is a constant and \( n \) is an integer. (For a harmonic oscillator, \( n = 2 \) and \( a = K/2 \), where \( K \) is the force constant.)

(a) (10) Calculate the partition function \( Z \), and show that it has the form

\[
Z = \text{constant} \times (k_B T)^\nu
\]  

(8)

(At the same time, of course, you will determine the values of \( \nu \) and the other constant.) Give your answer in terms of \( \Gamma(1/n) \), with

\[
\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}dt.
\]  

(9)

(b) (10) Show that the energy per particle is given by

\[
\frac{E}{N} = \text{constant} \times k_B T
\]  

(10)

and determine the constant.
4. The density of states of an ideal gas of $N$ bosons is given by

\begin{align}
g(\varepsilon) &= b N \varepsilon^2 \quad \text{if } \varepsilon > 0 \\
&= 0 \quad \text{if } \varepsilon < 0
\end{align}

where $b$ is a constant and $\varepsilon$ is the single-particle energy.

(a) (6) Starting with the Bose-Einstein distribution function at temperature $T$ (and with chemical potential $\mu$), write down the expression for the number density

\[ \rho \equiv \frac{N}{V} \]  \hspace{1cm} (13)

as an integral over the energy $\varepsilon$.

(b) (6) Using the fact that $(1 - x)^{-1} = 1 + x + x^2 + \ldots$, and integrating term-by-term, write $\rho$ in the form

\[ \rho = (\text{function of } T) \times g_n(\lambda) + \rho_0 \]  \hspace{1cm} (14)

where

\[ g_n(\lambda) = \sum_{\ell=1}^{\infty} \frac{\lambda^\ell}{\ell^n} \]  \hspace{1cm} (15)

and $\rho_0 = \pi_0/V$ is the number density in the condensate. (In obtaining this form, you will also obtain the value of $n$.) Recall that $\lambda \equiv \exp (\mu/k_B T)$.

(c) (6) Calculate the transition temperature $T_c$ in terms of the constants $b$, $k_B$, and $g_n(1)$.

(d) (2) Since $g(\varepsilon)$ has the dimensions of inverse energy, $b^{1/3}$ also has the dimensions of inverse energy. If

\[ \left( k_B b^{1/3} \right)^{-1} = 1 \text{ } K \]  \hspace{1cm} (16)

estimate the value of $T_c$ in $K$ (i.e. degrees Kelvin). Try to obtain two significant figures.
5. Let

\[ I \equiv \int_{-\infty}^{\infty} f(\varepsilon) h(\varepsilon) \, d\varepsilon \]  

(17)

and

\[ H(\varepsilon) \equiv \int_{-\infty}^{\varepsilon} h(\varepsilon') \, d\varepsilon' \]  

(18)

where \( f(\varepsilon) \) is the Fermi function (or Fermi-Dirac distribution function), which gives the probability that a fermion state is occupied at temperature \( T \) with a chemical potential \( \mu(T) \).

We showed that

\[ I \equiv H(\mu) + \frac{1}{6} \pi^2 H''(\mu) (k_B T)^2 + \ldots \]  

(19)

where

\[ H''(\varepsilon) = \frac{d^2 H(\varepsilon)}{d\varepsilon^2} = \frac{dh(\varepsilon)}{d\varepsilon} = h'(\varepsilon) \]  

(20)

and you can assume this result in doing the problem. Also, let

\[ \varepsilon_F \equiv \mu(0) \]  

(21)

(a) (10) For an ideal gas of fermions with density of states \( \rho(\varepsilon) \), at a temperature which is low compared to \( T_F \equiv \varepsilon_F/k_B \), calculate the chemical potential \( \mu(T) \) in terms of \( \varepsilon_F, \rho(\varepsilon_F), \rho'(\varepsilon_F), k_B, \) and \( T \).

(b) (10) Using the result of (a), calculate the heat capacity at constant volume \( C_V \) in terms of the same quantities.