Physics 607 Final Exam

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work, so please do not just write down answers with no explanation!

To state the obvious: If you are asked for an answer or explanation, it must be clear, precise, and unambiguous. **Fuzzy answer, argument, or explanation => zero credit.**

Do all your work on the blank sheets provided, writing your name clearly. (You may keep this exam.)

The variables have their usual meanings: $E =$ energy, $S =$ entropy, $V =$ volume, $N =$ number of particles, $T =$ temperature, $P =$ pressure, $\mu =$ chemical potential, $B =$ applied magnetic field, $C_V =$ heat capacity at constant volume, $C_P =$ heat capacity at constant pressure, $F =$ Helmholtz free energy, $G =$ Gibbs free energy, $k =$ Boltzmann constant, $h =$ Planck constant. Also, $\langle \cdots \rangle$ represents an average.

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \quad \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi}
\]

\[
\langle E \rangle = kT^2 \frac{\partial}{\partial T} \ln Z \\
Z = \sum_j e^{-\beta E_j} \\
Z = \sum_{N_j} e^{-\beta E_{N_j}} e^{\gamma N}
\]

\[
\left\langle x_i \frac{\partial H}{\partial x_j} \right\rangle = \delta_{ij} kT
\]
1. The Gibbs formula for the entropy has the very general form

\[ S = -k \sum_r p_r \ln p_r \].

(a) (12) In the canonical ensemble, with the constraint

\[ \sum_j p_j E_j = \bar{E} = \text{constant} \]

show that maximization of \( S \) gives for the probability of a state \( j \)

\[ p_j = \frac{e^{-\beta E_j}}{Z} \]

where \( Z \) is the canonical partition function and \( \beta \) is a Lagrange multiplier.

(b) (12) In the grand canonical ensemble, with the constraints

\[ \sum_{N_j} p_{N_j} E_{N_j} = \bar{E} = \text{constant} \quad \text{and} \quad \sum_{N_j} p_{N_j} N = \bar{N} = \text{constant} \]

show that maximization of \( S \) gives for the probability of a state \( j \) with \( N \) particles

\[ p_{N_j} = \frac{e^{-\beta E_{N_j}} e^{\gamma N}}{Z} \]

where \( Z \) is the grand partition function and \( \beta \) and \( \gamma \) are Lagrange multipliers.
2. It is possible to create a monolayer of noble gas atoms on a smooth graphite surface.

Let us therefore consider a 2-dimensional ideal gas of (spinless nonrelativistic) bosons in an area $A$.

(a) (4) Using the fact that the area in momentum space per allowed 2-dimensional momentum $\vec{p}$ is $\hbar^2 / A$, obtain the density of states in momentum space $\rho(p)$ and then the usual density of states $\rho(\varepsilon)$, where $\varepsilon$ is the single-particle energy.

(b) (4) Let us first treat this system as a classical ideal gas (as is valid at high enough temperatures). Write the classical (canonical) partition function $Z$ as an integral over $p_x$ and $p_y$, or over $p$, or over $\varepsilon$ (you may choose), and then evaluate it to get $Z$ as a function of $A$, $\hbar$, $k$, $T$, and the particle mass $m$.

(c) (4) Using the partition function, calculate the average energy per particle $\langle \varepsilon \rangle$.

(d) (4) Then obtain $\langle \varepsilon \rangle$ from the equipartition theorem in the form given on the first page of this exam, and see if the results agree.

(e) (4) If the system is cooled to low temperatures, it must be treated as a quantum ideal gas. The pressure and number of particles are respectively given by

$$\frac{PA}{kT} = - \int_0^\infty d\varepsilon \rho(\varepsilon) \ln \left( 1 - e^{-(\varepsilon - \mu)/kT} \right) \quad \text{and} \quad N - N_0 = \int_0^\infty d\varepsilon \rho(\varepsilon) \frac{1}{e^{(\varepsilon - \mu)/kT} - 1}.$$ 

Use an integration by parts to relate $P$ to the energy density $\frac{\langle E \rangle}{A}$ when $N_0 = 0$.

(f) (4) The integral for $N - N_0$ can be performed. Use your result for this integral to demonstrate whether there is or is not Bose-Einstein condensation in this 2-dimensional ideal gas of bosons at nonzero temperature. (Assume that it remains a liquid down to $T = 0$.)

(g) (4) Now assume that, like the isotopes of helium, this system becomes a crystalline solid under pressure, with vibrations that are described by a 2-dimensional Debye model. The mean-square displacements are given by

$$\langle x^2 \rangle = c \int_0^{\omega_D} d\omega \bar{\rho}(\omega) \frac{kT}{\omega^2}$$

where $c$ is a constant. Obtain the density of states function $\bar{\rho}(\omega)$ for 2-dimensional vibrational modes and determine whether $\langle x^2 \rangle$ is finite at nonzero temperatures. Assume two modes with the same velocity.

These last two parts are examples of the Mermin-Wagner theorem regarding long-range order in 2 dimensions. Two of the latest winners of the Nobel Prize in physics found a way around this theorem involving topological excitations – namely, vortices.
3. Statistical mechanics spills over into many other areas, including the modeling of how societal collapses occur, with many examples throughout history and in cultures outside history. Suppose that we include, under societal collapse, the many cases where a society undergoes a phase transition into an authoritarian repressive regime, or at least one with a dominant political orientation.

One of the simplest imaginable models is that the probability of a citizen being on the left or right (in political orientation) is proportional to \( e^{c \Delta N_\leftarrow / NI} \) or \( e^{c \Delta N_\rightarrow / NI} \), where \( \Delta N_\leftarrow = N_\leftarrow - N_\rightarrow \) or \( \Delta N_\rightarrow = N_\rightarrow - N_\leftarrow \) is the net number of citizens who have shifted to that state, \( c \) is the strength of the coercive peer pressure, and \( I \) is the average intellectual integrity or independence of the citizens, so that

\[
\frac{N_\leftarrow}{N} = \frac{e^{c \Delta N_\leftarrow / NI}}{e^{c \Delta N_\leftarrow / NI} + e^{c \Delta N_\rightarrow / NI}} \quad \text{and} \quad \frac{N_\rightarrow}{N} = \frac{e^{c \Delta N_\rightarrow / NI}}{e^{c \Delta N_\leftarrow / NI} + e^{c \Delta N_\rightarrow / NI}}.
\]

The polarization is then

\[
\frac{\Delta N_\leftarrow}{N} = \frac{e^{c \Delta N_\leftarrow / NI}}{e^{c \Delta N_\leftarrow / NI} + e^{c \Delta N_\rightarrow / NI}} - \frac{e^{c \Delta N_\rightarrow / NI}}{e^{c \Delta N_\leftarrow / NI} + e^{c \Delta N_\rightarrow / NI}} \quad \text{or alternatively} \quad \frac{\Delta N_\rightarrow}{N} = \frac{e^{c \Delta N_\rightarrow / NI}}{e^{c \Delta N_\leftarrow / NI} + e^{c \Delta N_\rightarrow / NI}} - \frac{e^{c \Delta N_\leftarrow / NI}}{e^{c \Delta N_\leftarrow / NI} + e^{c \Delta N_\rightarrow / NI}}
\]

or, with either choice,

\[
\frac{\Delta N}{N} = \frac{e^{c \Delta N / NI} - e^{-c \Delta N / NI}}{e^{c \Delta N / NI} + e^{-c \Delta N / NI}} = \tanh \left( \frac{c}{I} \frac{\Delta N}{N} \right)
\]

(a) (9) Sketch a rough plot of \( \tanh \left( \frac{c}{I} \frac{\Delta N}{N} \right) \) as a function of \( \frac{\Delta N}{N} \) for \( \frac{c}{I} = 1/2, 1, 2 \). (This is just \( \tanh \left( \frac{1}{2} x \right) \), \( \tanh(x) \), \( \tanh(2x) \) as a function of \( x \). Consider small and large values of \( x \) to pin down the end points of your sketches.)

(b) (9) For what value of \( \frac{c}{I} \) is there a phase transition – i.e., a solution for which the polarization \( \Delta N \) is nonzero?

[Common sense says that this is the stable solution. The above is, of course, the Weiss mean-field model in a different context, and it may be most appropriate to the formation of fully-aligned political domains analogous to local ferromagnetic domains in the original context.]
4. Here we wish to compare two systems, each of which has a fixed volume $V$ and an energy $E$ which fluctuates. Let $\langle E \rangle$ be the average energy, with $(\Delta E)^2 = \left( \langle E - \langle E \rangle \rangle \right)^2 = \langle E^2 \rangle - \langle E \rangle^2$ being the variance – i.e., the usual measure of the size of the fluctuations.

Notice that $\langle E \rangle$ and $\langle N \rangle$ are to be interpreted as the usual thermodynamic variables (internal energy and number of particles), but we use $\langle \cdot \rangle$ here to avoid confusion. But the canonical $N$ and the grand canonical $\langle N \rangle$ can be regarded as equivalent and interchangeable when treated as thermodynamic variables.

In the first system, the number of particles $N$ is fixed, so this system can be described with the canonical ensemble and the canonical partition function $Z$. We will represent the quantities in this case by $\langle E \rangle_C$ and $(\Delta E)^2_C$.

In the second system, the chemical potential is fixed, so this system can be described with the grand canonical ensemble and the grand partition function $Z$. We will represent the quantities in this case by $\langle E \rangle_G$ and $(\Delta E)^2_G$.

In the following, it will be convenient to work with $\beta = \frac{1}{kT}$ and $\gamma = \frac{\mu}{kT}$ as well as $T$ and $\mu$.

You will want to recall the first law of thermodynamics, $d\langle E \rangle = TdS - PdV + \mu d\langle N \rangle$.

(a) (2) At a given temperature $T$, which system should have the larger value of $(\Delta E)^2$? Why? (Give a clear physical reason.)

(b) (2) Obtain an expression for $\langle E \rangle_C$ in terms of $\left( \frac{\partial \ln Z}{\partial \beta} \right)_N$.

c) (2) Obtain an expression for $(\Delta E)^2_C$ in terms of $T$ and $\left( \frac{\partial \langle E \rangle_C}{\partial T} \right)_N$.

d) (2) Obtain an expression for $\langle E \rangle_G$ in terms of $\left( \frac{\partial \ln Z}{\partial \beta} \right)_\gamma$.

e) (2) Obtain an expression for $(\Delta E)^2_G$ in terms of $T$ and $\left( \frac{\partial \langle E \rangle_G}{\partial T} \right)_\gamma$.

(f) (2) Obtain an expression for $\langle N \rangle_G$ in terms of $\left( \frac{\partial \ln Z}{\partial \gamma} \right)_T$.

g) (2) Obtain an expression for $(\Delta N)^2_G$ in terms of $\left( \frac{\partial \langle N \rangle_G}{\partial \gamma} \right)_T$. 

Regarding $\langle E \rangle$ as a function of $T$ and $\langle N \rangle$, obtain the obvious expression for $d\langle E \rangle$ in terms of $dT$, $d\langle N \rangle$, and the partial derivatives of $\langle E \rangle$ with respect to $T$ and $\langle N \rangle$.

Then obtain

$$\left( \frac{\partial \langle E \rangle}{\partial T} \right)_T \left( \frac{\partial \langle E \rangle}{\partial T} \right)_N + \left( \frac{\partial \langle E \rangle}{\partial \langle N \rangle} \right)_T \left( \frac{\partial \langle N \rangle}{\partial T} \right)_T .$$

Finally obtain

$$\left( \frac{\partial \langle E \rangle}{\partial T} \right)_N \left( \frac{\partial \langle N \rangle}{\partial T} \right)_N + \left( \frac{\partial \langle E \rangle}{\partial \langle N \rangle} \right)_T \left( \frac{\partial \langle N \rangle}{\partial T} \right)_T .$$

Now, recalling that the grand canonical potential is defined by $\Omega = \langle E \rangle - TS - \mu \langle N \rangle$, obtain $d\Omega$ in terms of $dT$ and $d\mu$ when $V$ is fixed.

Show that $d\Omega = -(S + k\gamma N)dT - NTKd\gamma$.

Then use the result of part (j) to obtain a relation between $\left( \frac{\partial (S + k\gamma N)}{\partial \gamma} \right)_T$ and $N + T \left( \frac{\partial N}{\partial T} \right)_T$.

Next, use the first law of thermodynamics ($d\langle E \rangle = TdS - PdV + \mu d\langle N \rangle$ with $V$ fixed) to obtain $d(S + k\gamma N)$ in terms of $d\langle E \rangle$ and $d\gamma$, and from this get $\left( \frac{\partial (S + k\gamma N)}{\partial \gamma} \right)_T$ in terms of $\langle N \rangle, T$, and $\left( \frac{\partial \langle E \rangle}{\partial \gamma} \right)_T$.

Combine the results of parts (k) and (l) to obtain a relation between $\left( \frac{\partial N}{\partial T} \right)_\gamma$ and $\left( \frac{\partial \langle E \rangle}{\partial \gamma} \right)_T$.

Use $\left( \frac{\partial \langle E \rangle}{\partial \gamma} \right)_T = \left( \frac{\partial \langle E \rangle}{\partial \langle N \rangle} \right)_T \left( \frac{\partial \langle N \rangle}{\partial \gamma} \right)_T$ and the results of parts (g) and (m) to obtain

$$kT^2 \left( \frac{\partial \langle N \rangle}{\partial T} \right)_\gamma = \left( \frac{\partial \langle E \rangle}{\partial \langle N \rangle} \right)_T (\Delta N)_G^2 .$$

Then use the results of parts (n) and (h) to write $(\Delta E)_G^2 - (\Delta E)_C^2$ in terms of $\left( \frac{\partial \langle E \rangle}{\partial \langle N \rangle} \right)_T$ and $(\Delta N)_G^2$.

Is this result in accordance with your answer to part (a)? Explain.

Please have a good summer!