Physics 607 Exam 2

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work, so please do not just write down answers with no explanation!

Do all your work on the blank sheets provided, writing your name clearly. (You may keep this exam.)

The variables have their usual meanings: \( E = \) energy, \( S = \) entropy, \( V = \) volume, \( N = \) number of particles, \( T = \) temperature, \( P = \) pressure, \( \mu = \) chemical potential, \( B = \) applied magnetic field, \( C_V = \) heat capacity at constant volume, \( C_P = \) heat capacity at constant pressure, \( F = \) Helmholtz free energy, \( G = \) Gibbs free energy, \( k = \) Boltzmann constant, \( h = \) Planck constant. Also, \((\cdots)\) represents an average.

\[
\lambda = e^{\mu/kT} \quad \lambda_{th} = \frac{h}{(2\pi mkT)^{1/2}} \quad n = \frac{N}{V} \quad v = \frac{V}{N} \quad g_n(\lambda) = \sum_{\ell=1}^{\infty} \frac{\lambda^\ell}{\ell^n}
\]

\[
\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \quad \kappa_S = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_S
\]

\[
\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \tanh x = \frac{\sinh x}{\cosh x} \quad \text{sech } x = \frac{1}{\cosh x}
\]

\[
\cosh^2 x - \sinh^2 x = 1 \quad \frac{d}{dx} \sinh x = \cosh x \quad \frac{d}{dx} \cosh x = \sinh x \quad \frac{d}{dx} \tanh(x) = \text{sech}^2 x
\]

1. A system at temperature \( T \) has two available states, with energies \( \pm \epsilon \).

Please express your answers in a simple form using hyperbolic functions like \( \cosh x \).

In part (c) you may use any valid method.

(a) (5) Obtain the partition function \( Z \).

(b) (5) Using the fact that \( F = -kT \ln Z \), and that \( F = E - TS \), so that \( dF = -SdT - PdV + \mu dN \), calculate the Helmholtz free energy \( F \) and the entropy \( S \).

(c) (5) Calculate the energy \( E \) and the heat capacity at constant volume \( C_V \).

(d) (5) Sketch the graph of \( C_V \) versus \( T \).
2. For an ideal gas of bosons, you are given the results

\[ \frac{P}{kT} = \frac{1}{\lambda_{th}^3} g_{5/2}(\lambda) , \quad \frac{N - N_0}{V} = \frac{1}{\lambda_{th}^3} g_{3/2}(\lambda) \]

\[ P \nu^{5/3} = \text{constant} , \quad \nu T^{3/2} = \text{constant} , \quad \frac{P}{T^{5/2}} = \text{constant} \]

with \( N_0 = 0 \) here.

(a) (10) Show that the isothermal compressibility \( \kappa_T \) is given by

\[ \kappa_T = \frac{1}{nkT} \frac{g_{n_T}(\lambda)}{g_{n'_T}(\lambda)} \]

where you will determine the constants \( n_T \) and \( n'_T \), which are not integers, while doing the calculation.

(b) (10) Show that the adiabatic compressibility \( \kappa_S \) is given by

\[ \kappa_S = \frac{3}{5} \frac{1}{nkT} \frac{g_{n_S}(\lambda)}{g_{n'_S}(\lambda)} \]

where you will determine the constants \( n_S \) and \( n'_S \), which again are not integers, while doing the calculation.

(c) (10) One can show that \( \lambda \to 0 \) as \( n \to 0 \) and that consequently one can write \( \lambda = a_n + a_2 n^2 + \ldots. \)

Use this Taylor series expansion and the equations given at the top of this problem to obtain

\[ \frac{P}{kT} = n + a \lambda_{th}^3 h^2 + \ldots \]

where you will obtain the (positive or negative) constant \( a \).
3. For an ideal gas of fermions we obtained

\[ \frac{PV}{kT} = \sum_k \ln \left( 1 + e^{-\frac{\varepsilon_k - \mu}{kT}} \right) \quad \text{and} \quad N = \sum_k n(\varepsilon_k) = \frac{1}{1 + e^{-\frac{\varepsilon_k - \mu}{kT}}} \quad \text{and} \quad n(\varepsilon_k) = \frac{1}{e^{\frac{\varepsilon_k - \mu}{kT}}} \frac{\varepsilon_k}{k \sum} \].

(a) (5) Calculate the density of states function \( \rho(\varepsilon) \) for (nonrelativistic) spin 1/2 particles (in 3 dimensions), or at least show that

\[ \rho(\varepsilon) = AV \varepsilon^p \]

where \( A \) is a constant, \( V \) is the volume, and you will determine the constant \( p \) in doing the calculation.

(b) (5) Using \( \rho(\varepsilon) \), write the equations for \( \frac{PV}{kT} \) and \( N \) as integrals over \( \varepsilon \).

**Below this point we specialize to \( T = 0 \), so that \( \mu = \varepsilon_F \), the Fermi energy.**

(c) (5) Calculate the number density \( n = N / V \) in terms of \( \varepsilon_F \) and \( A \).

(d) (5) Calculate the pressure \( P \) (at \( T = 0 \)) in terms of \( \varepsilon_F \) and \( n \).

**Hint:** Start with the expression for \( \frac{PV}{kT} \) in part (b), at a general temperature \( T \). Then perform an integration by parts to get a new integral for \( \frac{PV}{kT} \) involving \( \varepsilon \rho(\varepsilon)n(\varepsilon) \).

(e) (5) Calculate the isothermal compressibility \( \kappa_T \) (with \( N \) fixed), in terms of \( \varepsilon_F \) and \( n \) (or alternatively in terms of \( n \) and \( A \)).
4. (25) For the Rayleigh-Bénard instability, we obtained the following eigenvalue equation:

\[
\left( \frac{d^2}{dZ^2} - \alpha^2 \right)^3 V(Z) = -R\alpha^2 V(Z)
\]

where

\[
Z = \frac{z}{d}, \quad \alpha = ka
\]

\[
R = \frac{gaa \alpha d^4 \rho_0^2 c_v}{\eta K}, \quad v_z = V(Z)e^{i(k_x x + k_y y)}e^{1/2}
\]

with \( \omega \to 0^+ \) in the initial time-dependent solution, so that one has a solution that initially grows exponentially in time and then settles down to a nontrivial steady state.

(Here \( x, y, z, t \) are the coordinates, \( d \) is the spacing between the hot plate on the bottom and the cold plate on top, \( k = (k_x^2 + k_y^2)^{1/2} \), \( v_z \) is the fluid velocity in the \( z \) direction (with the \( z \) axis perpendicular to the plane of each flat plate), and the parameters in the Rayleigh number \( R \) are the acceleration of gravity, temperature gradient, thermal expansion coefficient, \( d \), unperturbed density, heat capacity per volume, shear viscosity, and thermal conductivity.

Now assume smooth boundaries at both plates, but also assume that the experiment is performed with a very thin sheet midway between the plates, which is permeable to heat etc., but forces \( v_z \) to be zero, so that the boundary conditions are now

\[
V(Z) = 0 \quad \text{for} \quad Z = \frac{1}{2} \quad \text{as well as} \quad V(Z) = 0 \quad \text{for} \quad Z = 0 \quad \text{and} \quad Z = 1.
\]

Calculate \( R_c \), the smallest value of the Rayleigh number \( R \) for which there will be a nontrivial (rolling) solution for the fluid velocity.

(You can leave your answer in terms of well-defined constants like \( \pi \).)