3. This problem involves a quantum ideal gas of fermions, for which \( a = 1 \) below, or bosons, for which \( a = -1 \).

You are given the following:

\[
\frac{PV}{kT} = \frac{1}{a} \sum_k \ln \left( 1 + a e^{-\left(\varepsilon_k - \mu\right)/kT} \right)
\]

\[
\langle n(\varepsilon_k) \rangle = \frac{1}{e^{\left(\varepsilon_k - \mu\right)/kT} + a}
\]

where \( \langle n(\varepsilon_k) \rangle \) is the average number of particles in the single-particle state with energy \( \varepsilon_k \).

(a) (4) Demonstrate that the above equations lead to the equation of state for a classical ideal gas in the limit \( a \to 0 \). (As always, show each significant step.)

In the remainder of the problem, return to a (spinless) quantum ideal gas, and assume that the single-particle energy is a function of only the magnitude \( p \) of the momentum: \( \varepsilon = \varepsilon(p) \).

(b) (4) Convert the sum above to an integral of the form \( \int_0^\infty f(p)dp \) by using the density of states \( \rho(p) \) in momentum space, which is obtained as usual from \( \rho(p)dp = \frac{4\pi p^2dp}{h^3/V} \).

(c) (4) Now perform an integration by parts to express the pressure \( P \) as an integral involving \( \langle n(p) \rangle \equiv \langle n(\varepsilon(p)) \rangle \) and the particle velocity \( u(p) = \frac{d\varepsilon(p)}{dp} \), as well as \( p \) and constants.

(d) (4) Write the number of particles \( N \) as an integral over \( p \) with a similar form.

(e) (4) Use your results of parts (c) and (d) to show that

\[
P = \text{constant} \times n \langle pu \rangle \quad n \equiv \frac{N}{V}
\]

while at the same time determining the constant.

(f) (4) For particles with a dispersion relation \( \varepsilon(p) = ap^s \), where \( a \) is a constant, obtain the relation between \( P \) and the energy density \( E/V \).

(g) (4) Apply the result of part (f) to the cases of (i) nonrelativistic particles and (ii) ultrarelativistic particles. I.e., obtain the relation between the pressure \( P \) and the energy density \( E/V \) in each case.
4. Let $\Phi(V)$ be the static energy of a solid with volume $V$, approximately given by

$$\Phi(V) = \frac{(V - V_0)^2}{2\kappa_0 V_0}$$

where $V_0$ and $\kappa_0$ are constants, with $\kappa_0 C_V T \ll V_0$. Then the total energy (in the harmonic approximation) is

$$U(T) = \Phi(V) + E_{vib}(T)$$

where $E_{vib}(T)$ is the energy of the vibrational modes with (angular) frequencies $\omega_i$.

Assume that each mode has the same value for the Grüneisen constant $\gamma$:

$$-\frac{\partial \ln \omega_i}{\partial \ln V} = \gamma.$$

(a) (4) Calculate the canonical partition function $Z_{vib}(T)$ as a product over the modes $i$ (with each mode treated as a harmonic oscillator). Recall that this means summing over all the possible quantum numbers $n_i$ for each mode.

(b) (4) From the result of part (a), calculate the Helmholtz free energy $F_{vib}(T)$ (again as a summation over $i$).

(c) (4) Now switch to the equivalent description involving the Bose-Einstein distribution function $\langle n_i \rangle$ for the phonons in each mode, and obtain $E_{vib}(T)$ as a summation over the modes $i$.

(d) (4) Show that this same expression for $E_{vib}(T)$ results when one calculates it from the result for $Z_{vib}(T)$ in part (a).

(e) (4) From the result of part (c) or (d), calculate $C_V$ (again as a summation over $i$).

(f) (4) Show that the pressure is given by

$$P = -\frac{\partial \Phi(V)}{\partial V} + \gamma \frac{E_{vib}(T)}{V}.$$

(g) (4) Show that the coefficient of thermal expansion at constant pressure, defined by

$$\alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_{N,P}$$

and with $P \approx 0$, $V \approx V_0$, is approximately given by

$$\alpha = \gamma \frac{\kappa_0}{V_0} C_V.$$

(h) (4) Starting with the well-known relation (which we derived in the first week of class)

$$C_P - C_V \approx TV \frac{\alpha^2}{\kappa_T},$$

where the isothermal compressibility is defined by

$$\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_{N,T},$$

show that $\kappa_T \approx \kappa_0$, and then obtain $C_P - C_V$ in terms of $\gamma$, $\kappa_0$, $C_V$, $V_0$, and $T$. 