1. Atomic transitions due to time-dependent electric field.

Consider a hydrogen atom which is in its ground state for \( t < 0 \). For \( t > 0 \) it is subjected to a spatially uniform electric field
\[
E_0 e^{-t/\tau}
\]
which is directed along the \( z \) axis. Using first-order time-dependent perturbation theory, let us calculate the probability of finding the atom in the excited state with \( n = 2, \ell = 1, m = 0 \) after a time \( t \) has elapsed.

First, however, let us obtain the expression which gives the probability. The eigenstates of the unperturbed Hamiltonian are \( \ket{n} : H_0 \ket{n} = \epsilon_n \ket{n} \). The full Hamiltonian in the original Schrödinger picture is
\[
H = H_0 + V_t,
\]
but recall that the time dependence of the state \( \ket{\psi(t)} \) in the interaction picture arises only from
\[
V(t) \equiv e^{iH_0 t/\hbar} V_t e^{-iH_0 t/\hbar}.
\]
\[
i \hbar \frac{d}{dt} \ket{\psi(t)} = V(t) \ket{\psi(t)}.
\]

(a) (3) Show that the solution of this equation to first order is
\[
\ket{\psi(t)} = \ket{\psi(t_0)} + \frac{1}{i \hbar} \int_{t_0}^{t} dt' V(t') \ket{\psi(t_0)}.
\]

(b) (3) If the system is in an initial state \( \ket{i} \) at time \( t_0 = 0 \), show that the probability that it is in a different state \( \ket{n} \) at time \( t \) is
\[
P_{i \rightarrow n}(t) = \left| \frac{1}{i \hbar} \int_{t_0}^{t} dt' e^{i(\epsilon_n - \epsilon_i)t'/\hbar} \langle n|V_t|i \rangle \right|^2.
\]

(c) (3) Write down the perturbing Hamiltonian \( V_t \) for the electron in a hydrogen atom subjected to an electric field with \( E_z = E_0 e^{-t/\tau} \) and \( E_x = E_y = 0 \). Recall that force = charge × electric field and that \( z = r \cos \theta \).

(d) (15) Calculate the matrix element \( \langle n|V_t|i \rangle \) for \( \ket{i} = \ket{100} \) and \( \ket{n} = \ket{210} \), with the states labeled as usual by the quantum numbers \( \ket{n \ell m} \). The wavefunctions are given by
\[
R_{10} = \frac{2}{a_0^{3/2}} e^{-r/a_0} \quad , \quad Y_{oo} = \frac{1}{\sqrt{4\pi}} \quad , \quad R_{21} = \frac{1}{(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \quad , \quad Y_{10} = \frac{\sqrt{3}}{\sqrt{4\pi}} \cos \theta.
\]

Note: One way to evaluate an integral of the form \( \int du \; u^N \; e^{au} \) is to rewrite it as \( \frac{d^N}{da^N} \int du \; e^{au} \) and take the derivative after evaluating \( \int du \; e^{au} \). (Another way is to repeatedly integrate by parts.)
(e) (8) Calculate the probability $P_{100 \rightarrow 210}(t)$ that an electron initially in the ground state $|100\rangle$ will be found in the excited state $|210\rangle$ after a time $t$ has elapsed. Give your answer in terms of $t$, $\tau$, $E_o$, $e$, $h$, and

$$\omega \equiv \frac{E_2 - E_1}{h}.$$ 

(f) (3) Calculate the probability after a long time has elapsed, $P_{100 \rightarrow 210}(t \rightarrow \infty)$. 
2. (20) **Field emission of electrons from a metal.**

In this problem we have a time-*independent* electric field which we can treat using the WKB approximation, in which the transmission probability is given by:

\[ T = e^{-2 \int_a^b \kappa(x') \, dx'} \]

where \( a \) and \( b \) are the classical turning points and

\[ \kappa(x) = \frac{1}{\hbar} \sqrt{2m(V(x) - E)} \]

Let us adopt a model in which

\[ V(x) = -e \varepsilon_x \quad \text{for} \quad x > 0 \quad (\text{region outside metal}) \]

and

\[ V(x) = -V_0 \quad \text{for} \quad x < 0 \quad (\text{region inside metal}) \]

where \( V_0 \) is positive and \(-V_0 < E < 0\).

(This approximates a more realistic picture in which the potential energy inside the metal rises from an average internal value of \(-V_0\) to the vacuum level of 0 over a distance of a few Angstroms, as the surface is approached.)

Assuming that the WKB approximation can be used with \( x = 0 \) as one turning point, calculate the transmission probability \( T \) in terms of the various constants.

[Hint: you may want to define \( \varepsilon_c = \frac{4 \sqrt{2m}}{3 e\hbar} \left| E \right|^{3/2} \).]
3. Electron in magnetic field is (mathematically) equivalent to harmonic oscillator.

Let

$$\Pi_x = p_x - \frac{q}{c} A_x, \quad \Pi_y = p_y - \frac{q}{c} A_y$$

where \( q = -e \) is the charge on an electron, \( p \) is the momentum operator, and the vector potential \( A(r) \) corresponds to a magnetic field in the z direction: \( B = B \hat{z} \).

(a) (10) Evaluate the commutator of these operators in the coordinate representation and show that

$$[\Pi_x, \Pi_y] = iB \times \text{constant}$$

(also determining the constant, of course).

(b) (15) Writing the Hamiltonian for an electron in this magnetic field in terms of \( \Pi_x \) and \( \Pi_y \), and comparing the above commutation relation with the one for a harmonic oscillator, show that the energy eigenvalues can be written as

$$E = \frac{\hbar^2 k^2}{2m} + \text{constant}' \times \left( n + \frac{1}{2} \right)$$

(also determining \( \text{constant}' \) in terms of \( e, B, \hbar, m, \) and \( c \), of course).

You may just quote results for the harmonic oscillator, which has the Hamiltonian

$$H = \frac{1}{2m} p^2 + \frac{1}{2m} (m \omega x)^2$$.
4. **Operator form of Ehrenfest’s theorem.**

Consider the usual 1-dimensional Hamiltonian

\[ H = \frac{1}{2m} p^2 + V(x). \]

(a) (5) The operators in the Heisenberg picture are \( x(t) \), \( p(t) \), etc. Starting with the usual commutation relation for the operators \( x \) and \( p \) in the Schrödinger picture, obtain the commutation relation for \( x(t) \) and \( p(t) \).

(b) (5) Using the Taylor series expansion

\[ V(x(t)) = \sum_n V_n x(t)^n \]

show that

\[ \left[ p(t), V(x(t)) \right] = -i\hbar \frac{\partial V(x(t))}{\partial x(t)}. \]

(b) (10) Using the result of part (a) and the Heisenberg equations of motion for \( x(t) \) and \( p(t) \), obtain the operator form of Ehrenfest’s theorem:

\[ m \frac{d^2 x(t)}{dt^2} = F(t) \quad , \quad F(t) = -\frac{\partial V(x(t))}{\partial x(t)}. \]