1. Fun with hydrogen

For an electron in the hydrogen atom, the normalized ground state is $|100\rangle$, with wavefunction

$$\psi_{100} = \frac{2}{a_0^{3/2}} e^{-r/a_0} \frac{1}{\sqrt{4\pi}}$$

which satisfies the time-independent Schrödinger equation

$$H_0 \psi_0 = E_0 \psi_0 \quad , \quad E_0 = -\frac{e^2}{2a_0}.$$  \hspace{1cm} (1)

(a) (5) Calculate $\langle \frac{1}{r^2} \rangle$, where $\langle \cdots \rangle$ indicates an expectation value in the ground state. Give your answer in terms of $a_0$, both here and in part (b).

(b) (5) Calculate $\langle \frac{1}{r} \rangle$. (One way involves an integration by parts.)

(c) (15) In special relativity the kinetic energy is

$$T = \sqrt{c^2 \frac{-\bar{p}^2}{2m} + m^2 c^4} - mc^2$$

$$= \frac{-\bar{p}^2}{2m} - \frac{1}{2mc^2} \left( \frac{-\bar{p}^2}{2m} \right)^2.$$ 

Using first-order perturbation theory, calculate the relativistic correction to the ground-state energy for an electron in a hydrogen atom.

The perturbation in the Hamiltonian is, of course, the last term above, with $\bar{p}$ being the momentum operator. Notice that

$$\frac{-\bar{p}^2}{2m} = H_0 + \frac{e^2}{r}$$

where $H_0$ is the unperturbed (i.e. nonrelativistic) Hamiltonian.

The easiest way is to use this equation, together with the results of parts (a) and (b), and Eq. (1) above. Give your answer in terms of $m$, $c$, and $a_0$.

(d) (5) Using $e^2 / 2a_0 = 13.6$ eV and $mc^2 = 511$ keV, estimate this relativistic correction in eV.
2. Fun with the variational principle

Let $H$ be the Hamiltonian of some system.

Suppose that the first excited state $|1\rangle$, with wavefunction $\psi_1(x) = \langle x | 1 \rangle$ and energy $E_1$, is known to be orthogonal to the ground state $|0\rangle$: $\langle 1 | 0 \rangle = 0$.

Then let a trial state $|\psi\rangle$, with wavefunction $\psi(x) = \langle x | \psi \rangle$, be some approximation to $|1\rangle$ which is chosen to be orthogonal to $|0\rangle$:

\[ \langle \psi | 0 \rangle = 0. \]

(a) (10) Show that

\[ \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \geq E_1. \]

(b) (10) Apply the result of part (a) in estimating the energy of the first excited state of an anharmonic oscillator, with Hamiltonian

\[ H = \frac{p^2}{2m} + ax^4. \]

For your normalized trial wavefunction, use

\[ \psi(x) = \left( \frac{4\alpha^3}{\pi} \right)^{1/4} x \ e^{-\alpha x^2/2}. \]

Compare your final answer with the exact numerical result of

\[ 3.80 \left( \frac{\hbar}{2m} \right)^{2/3} a^{1/3}. \]

You may use

\[ \int_{-\infty}^{\infty} dx \ e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \]
\[ \int_{-\infty}^{\infty} dx \ x^2 \ e^{-\alpha x^2} = \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}} \]
\[ \int_{-\infty}^{\infty} dx \ x^4 \ e^{-\alpha x^2} = \frac{3}{4\alpha^2} \sqrt{\frac{\pi}{\alpha}} \]
\[ \int_{-\infty}^{\infty} dx \ x^6 \ e^{-\alpha x^2} = \frac{15}{8\alpha^3} \sqrt{\frac{\pi}{\alpha}}. \]
3. In class we obtained, for scattering off a potential \( V(r) \),
\[
\psi_k(r) = e^{i k \cdot r} + \frac{e^{i k r}}{r} f(k', k)
\]
\[
f(k', k) = -\frac{m}{2\pi \hbar^2} \int d^3 r' e^{-i k \cdot r'} V(r') \psi_k(r')
\]

In the case of a spherically symmetric potential, and in the first Born approximation, we then obtained
\[
f(\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty dr \ r \sin qr \ V(r)
\]
where \( hq \) is the magnitude of the momentum transfer. You may use these results below in this problem.

(a) (10) In the first Born approximation, calculate the differential cross section \( \frac{d\sigma}{d\Omega} \) for the potential
\[
V(r) = \begin{cases} 
V_0, & r \leq R \\
0, & r > R 
\end{cases}
\]

(b) (10) Show that \( \frac{d\sigma}{d\Omega} \) is a constant (independent of both \( k \) and \( \theta \)) in the limit of low momentum transfer \( q \).

(c) (5) Now consider the 3-dimensional delta-function potential
\[
V(r) = A \delta(r).
\]
Using the first Born approximation again, calculate \( \frac{d\sigma}{d\Omega} \). Determine the constant \( A \) which gives the same result as was found in part (b).
4. A hydrogen atom is placed in a static electric field $\vec{E}_0 = E_0 \hat{z}$, where $\hat{z}$ is the unit vector in the $z$ direction. You are given that $\langle 200|z|210\rangle = -3a_0$, where $a_0$ is the Bohr radius and $|nlm\rangle$ designates a state with the usual quantum numbers $n$, $l$, $m$.

(a) (15) Using degenerate perturbation theory, calculate the 4 first-order shifts in the energy levels of the $n = 2$ states due to this electric field. (Give your answers in terms of $e$, $E_0$, $a_0$, etc.)

(b) (10) Calculate the 4 energy eigenstates as superpositions of the original unperturbed states $|200\rangle$, $|210\rangle$, $|211\rangle$, and $|21,-1\rangle$. (I.e., if the new states are labeled $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$, you might obtain solutions of the form $|1\rangle = a|200\rangle + b|210\rangle$, etc.)