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Interference.

We have already seen examples of interference. For two counter-propagating waves of the same frequency and wavelength (on a string), there will exist fixed positions that have zero amplitude for all time.

\[ y(x,t) = A \sin(kx - wt) + A \sin(kx + wt) \]
\[ = 2A \sin(kx) \cos(wt) \]

The relative phase of the two waves at the node positions will be odd multiples of \( \pi \) (180° out of phase) whereas the antinode positions are characterized by multiples of \( 2\pi \) (0° phase shift).

Expanding this concept to two sources of waves embedded in 2 or 3D. In order to have interference the two sources of waves must be coherent, that is their phase relationship must be fixed for all time (If they are initially un-phased they will always be in phase.). One easy way to do this is to have two slits in a screen that are illuminated by a single plane wave source.

At position A we see two crests (constructive interference).

At position B we see a crest and a trough (destructive interference).
To determine the relative phase between the two rays that reach the screen at point C, we need to determine the difference between $\gamma_1$ and $\gamma_2$. From the law of cosines

$$r_1^2 = r^2 + (d/2)^2 - 2(d/2)r \cos \left[ 90 - \phi \right]$$

$$r_2^2 = r^2 + (d/2)^2 - 2(d/2)r \cos \left[ 90 + \phi \right]$$

and since $\cos \left( 90 - \phi \right) = \sin \phi$ and $\cos \left( 90 + \phi \right) = -\sin \phi$ we have that

$$r_1^2 = r^2 + (d/2)^2 - rd \sin \phi$$

$$r_2^2 = r^2 + (d/2)^2 + rd \sin \phi$$

$$(r_2^2 - r_1^2) = (r_2 - r_1)(r_2 + r_1) = 2r \ d \ sin \phi$$

For $D \gg d$ then $r_1 + r_2 \approx 2r$ and so

$$(r_2 - r_1) = d \ \sin \phi$$

Note: If we Taylor expand $r_2 - r_1$ using $d$ as the small parameter we have

$$(r_2 - r_1) = d \ \sin \phi - \frac{d^2}{8} \ \frac{\sin \phi \ \cos \phi}{r^2} + \ldots$$

$$(r_2 + r_1) = 2r + \frac{d^2}{4r} \ \frac{\sin^2 \phi}{r^2} + \ldots$$
Now that we have the path difference, how do we determine their relative phases?

If the two waves start at zero phase at the slits, then the phase each one has accumulated, phase given by the distance traveled divided by the wavelength and multiplied by 2π. i.e.

\[ \phi_1 \text{ (at screen)} = \frac{r_1}{\lambda} 2\pi \quad \phi_2 \text{ (at screen)} = \frac{r_2}{\lambda} 2\pi \]

and so \( \Delta \phi = \frac{(r_2 - r_1)}{\lambda} 2\pi = \frac{2\pi ds \sin \theta}{\lambda} \)

If \( \Delta \phi = 0, \pm 2\pi, \pm 4\pi - (m \cdot 2\pi) \) then the two waves will interfere constructively.

If \( \Delta \phi = \pm \pi, \pm 3\pi \) or \( (2m+1)\pi \) then the two waves will interfere destructively.

Constructive interference \( \Delta \phi = 2\pi m \quad m = 0, \pm 1, \pm 2 \ldots \)

\[ ds \sin \theta = m \lambda \]

Destructive interference \( \Delta \phi = \pi (2m+1) \quad m = 0, \pm 1, \pm 2 \ldots \)

\[ ds \sin \theta = (2m+1)\frac{\lambda}{2} = (m+\frac{1}{2})\lambda \]

At the positions of the constructive interference we expect a bright spot (for light) and at the positions of the destructive interference we expect a dark spot.

To see such effects note that \( d \) must be comparable to the \( \lambda \).

\( d = 1 \text{nm} \quad \lambda = 5000 \text{Å} \) then the 1st non-central max. occurs at \( m=1 \)

\[ \sin \theta = \frac{1 \lambda}{d} = \frac{5 \times 10^{-7} \text{m}}{1 \times 10^{-9} \text{m}} = 0.5 \quad \theta = 30^\circ \]

or if the screen is at a distance of 1 meter, \( D = 1 \text{meter} \) then the distance between the central max. and the \( m=1 \) max is

\[ \Delta x = D \tan \theta = 0.5 \text{meters} \]
If, however, \( d = 0.5 \text{ mm} \) and \( \lambda = 5000 \text{Å} \) then the location of the

\[
\sin \theta = \frac{5 \times 10^{-2} \text{m}}{5 \times 10^{-4} \text{m}} = 10^{-3} \quad d = 10^{-3} R = 0.05^\circ
\]

and the separation from the central max becomes \( \Delta x = 1 \text{ mm} \). Consequently you can hardly distinguish the difference between the central max and the next one. (It looks like only the result of a single hole in the screen).

One can improve the analysis a bit by introducing a lens. Consider the following point source of light placed at the focal point of a lens. The light rays are refracted by the 1st lens into parallel rays.

The second lens then refocuses them to its focal point.

Although the ray \( A \) travels a larger distance in air than the ray \( B \), the ray \( B \) travels a larger distance in the two lenses. Since the propagation velocity is slower in the glass, it is conceivable that the two rays actually take the same time in going from \( F_1 \) to \( F_2 \). This is a statement of Fermat’s Principle.

Light traveling from one point to another will follow a path such that when compared to other paths the time required to traverse the path will either be a minimum, a maximum or remain the same.

In the above example, all the rays shown take the same amount of time to traverse their path.

If all the waves (represented by each ray) leave \( F_1 \) in phase, they will consequently arrive at \( F_2 \) in phase. This allows us to use a lens to simplify the analysis of the 2-slits. By placing a lens in front of the
slits with the screen at the focal plane of the lens, then we need only consider rays that leave the slits that are parallel since they will be focused at the same point on the screen. Moreover, from our previous example, we see that the two rays will accumulate the same phase from a point perpendicular to their rays. Thus the rays accumulate the same phase from A and A' to the focal point B. The overall phase difference is therefore in the path length from the slit to the point A' which is just $d \sin \theta$.

Intensity of Double Slit Pattern.

To determine the variation of the intensity we must consider how the amplitude of the individual waves varies as they leave each slit. For definiteness, let us choose the electric field of each wave to be in the z-axis (perpendicular to the paper). Then at the point in the screen the total electric field will be

$$E = E_1 + E_2 = (E_0 \cos (k r_1 - \omega t) + E_0 \cos (k r_2 - \omega t))^2$$

indicating the each wave has traveled along paths of different lengths. Adding the two waves we get

$$E = 2E_0 \cos \left( \frac{k (r_1 + r_2)}{2} - \omega t \right) \cos \left( \frac{k (r_2 - r_1)}{2} \right)$$
The intensity can then be found from the time-average magnitude of the Poynting vector $\mathbf{S} = \frac{1}{\mu_0 c} \mathbf{E} \times \mathbf{B}$, where

$$\mathbf{B} = \frac{2E_0}{c} \cos \left[ \frac{k(r_1 + r_2) - \omega t}{2} \right] \cos \left[ \frac{k(r_1 - r_2)}{2} \right]$$

and so

$$|\mathbf{S}| = \frac{4E_0^2}{\mu_0 c} \cos^2 \left[ \frac{k(r_1 + r_2) - \omega t}{2} \right] \cos^2 \left[ \frac{k(r_1 - r_2)}{2} \right]$$

Taking the time average eliminates the $\cos \left[ \frac{k(r_1 + r_2) - \omega t}{2} \right] = \frac{1}{2}$ and so

$$I = \frac{|\mathbf{S}|}{\mu_0 c} = \frac{2E_0^2}{\mu_0 c} \cos^2 \left[ \frac{k(r_1 - r_2)}{2} \right]$$

but $r_2 - r_1 = d \sin \theta$ and $k = \frac{2\pi}{\lambda}$

$$I = \frac{2E_0^2}{\mu_0 c} \cos^2 \left[ \frac{\pi d \sin \theta}{\lambda} \right]$$

At $\theta = 0$ we have a maximum intensity $I(\theta = 0) = \frac{2E_0^2}{\mu_0 c}$ and so the above expression can be rewritten as

$$I(\theta) = I(\theta = 0) \cos^2 \left[ \frac{\pi d \sin \theta}{\lambda} \right]$$
In some problems the interfering rays travel through media of different indices of refraction. Consider the two rays shown. Over the length \( l \) they travel through media of indices of refraction \( n_1 \) and \( n_2 \).

![Diagram of interfering rays]  

Although the two rays start in phase after travelling the distance \( l \) they have a phase difference given by the relation:

\[
\Delta \phi = \frac{2\pi l}{\lambda_1} - \frac{2\pi l}{\lambda_2} \quad \text{where} \quad \lambda_1 = \frac{v_1}{\gamma_1} = \frac{c}{n_1 v} = \frac{\lambda_{vac}}{n_1} \\
\lambda_2 = \frac{v_2}{\gamma_2} = \frac{c}{n_2 v} = \frac{\lambda_{vac}}{n_2}
\]

\[
\Delta \phi = \frac{2\pi}{\lambda_{vac}} \left[ \ell n_1 - \ell n_2 \right] \quad \text{The product of the physical distance} \ l
\]

and the corresponding index of refraction is called the optical path length.

If \( \Delta \phi = m \, 2\pi \) (\( m = 0, \pm 1, \pm 2 \)) then two waves will interfere constructively, i.e. \( (\ell n_1 - \ell n_2) = m \lambda_{vac} \).

If \( \Delta \phi = (2M+1)\pi \) (\( M = 0, \pm 1, \pm 2 \)) the two waves will interfere destructively and

\[
(\ell n_1 - \ell n_2) = (2M+1) \frac{\lambda_{vac}}{2}
\]

Consider the interference between two rays reflecting from the front and rear surfaces of a thin film.
The incident ray comes in a normal incidence. (the rays shown are drawn with non-normal incidence for clarity). At the first surface the incident ray is both reflected and refracted. The transmitted ray is also partially reflected at the 2nd surface.

A portion of this ray will be refracted through the 1st surface where it can combine with the originally reflected ray A. Ray B will no longer be in phase with A since it has traveled a different distance. The phase difference should be given by

$$\Delta \phi = \frac{2\pi}{\lambda} \left[ 2dn - 0 \right]$$

since Ray B has a distance 2d in the medium with an index of refraction n, whereas Ray A does not.

If we were to believe the above calculation, we would find that as d → 0

$$\Delta \phi \rightarrow 0$$

is no phase shift. Rays A & B add constructively. Consequently one would always expect a reflection from very thin films such as soap bubbles. Experimentally however soap bubble become non-reflecting as they become thinner.

What we have forgotten is that there can be a discrete phase change associated with the reflection. Remembering back to our analysis of a pulse on a string.

When a reflection occurs from an interface for which the propagation velocity decreases the reflected wave is shifted by $\pi$ or 180°. When the propagation velocity increases there is no phase shift for the reflected wave. In either case the transmitted wave has no phase shift at the interface.
We see that ray A now picks up an additional $\pi$ shift upon reflecting from the 1st interface. The velocity of light decreases from its value in air ($n=1$) to its value in the medium with an index of refraction greater than 1.

Ray B does not pick up a phase shift upon reflection because now the velocity of light goes from slow in fact at the 2nd interface. The overall phase shift between $A \leftrightarrow B$ is then

$$\Delta \phi = \left[ \frac{2\pi}{\lambda_{\text{vac}}} \right] - \pi = \frac{4\pi d n}{\lambda_{\text{vac}}} - \pi$$

Constructive interference $\Delta \phi = m(2\pi) = \frac{4\pi d n}{\lambda_{\text{vac}}} - \pi$

Destructive interference $\Delta \phi = (2m+1)\pi = \frac{4\pi d n}{\lambda_{\text{vac}}} - \pi$

Therefore as $d \to 0$ the phase shift $\Delta \phi \to -\pi$ and the two rays will destructively interfere and no reflection is possible. This explains the "black" spots that appear on a soap bubble just before it bursts.

Consider $\lambda = 5520 \text{Å}, n = 1.33$ (water).

Constructive interference: $\frac{4\pi d n}{\lambda_{\text{vac}}} = (2m+1)\pi$ or $d = \frac{(2m+1) \lambda_{\text{vac}}}{4\pi}$

Bright bands for $m=0$ $d_0 = \frac{\lambda_{\text{vac}}}{4\pi} = 1.000 \text{Å}$

$m=1$ $d_1 = \frac{3\lambda_{\text{vac}}}{4\pi} = 3.000 \text{Å}$

Destructive interference $\frac{4\pi d n}{\lambda_{\text{vac}}} = (2m+2)\pi$ or $d = \frac{(m+1) \lambda_{\text{vac}}}{2\pi}$

Dark bands for $m=-1$ $d_{-1} = 0$

$m=0$ $d_0 = \frac{\lambda_{\text{vac}}}{2\pi} = 2.700 \text{Å}$
When looking at a soap bubble with white light, each wavelength will have its own bright and dark bands occurring at different thickness. Consequently, color will appear on the film.

Since the first "bright" band is given by:

\[ d = \frac{2m+1}{4\pi} \frac{\lambda_{\text{vac}}}{n} \]

as the film thickens from the \( d = 0 \) dark band the first color one should see should correspond to the smallest wavelength in the blue. Further thickening will cause the blue to be in a dark band while red will have a bright band giving the film a red color.

Non-Reflecting Coating:

- \( n = 1 \) Air
- \( n = 1.38 \) MgF₂
- \( n = 1.50 \) Glass

The overall phase shift then is:

\[ \Delta \phi = \frac{4\pi d n}{\lambda_{\text{vac}}} \]

Open reflection ray A has a \( \pi \) phase shift. (\( \nu_1 = \frac{\nu}{2} \) to \( \nu_2 = \frac{\nu}{1.38} \)) But ray B will also have a \( \pi \) phase shift upon reflection because the speed of propagation also decreases as the interface (\( \nu_2 = \frac{\nu}{1.38} \) to \( \nu_3 = \frac{\nu}{1.50} \)).
To get no reflection $\Delta \phi = (2m+1)\pi$ or $\frac{4\pi d \eta}{\lambda_{\text{vac}}} = (2m+1)\pi$ are

$$d = (2m+1)\frac{\lambda_{\text{vac}}}{4\eta}$$

Therefore the thinnest coating will have $\eta = 0$ or $d_{\text{smallest}} = \frac{\lambda_{\text{vac}}}{4\eta}$
Generalization to multiple slits.

\[ E_i = R_e \left[ E_0 e^{i(kr - wt)} \right] = E_0 \cos(kr - wt) \]
\[ E_2 = R_e \left[ E_0 e^{i(kr + \Delta x - wt)} \right] \]
\[ E_3 = R_e \left[ E_0 e^{i(kr + 2\Delta x - wt)} \right] \]
\[ \vdots \]
\[ E_n = R_e \left[ E_0 e^{i(kr + (n-1)\Delta x - wt)} \right] \]

where \( \Delta x = d \sin \varphi \).

\[ E_{\text{total}} = \sum E_i = R_e \left( E_0 e^{i(kr - wt)} \right) \frac{1 + e^{i\kappa a} + e^{2i\kappa a} + \cdots + e^{(N-1)i\kappa a}}{1 - e^{i\kappa a}} \]

\[ \text{Ans.} \quad S = 1 + x + x^2 + x^3 + \cdots + x^{n-1} \]
\[ xS = x + x^2 + x^3 + \cdots + x^{n-1} + x^n \]
\[ (1-x)S = 1 - x^n \quad \text{or} \quad S = \frac{1-x^n}{1-x} \]

\[ E_{\text{total}} = R_e \left\{ E_0 e^{i(kr - wt)} \frac{1 - e^{i\kappa a}}{1 - e^{i\kappa a/2}} \right\} \]
\[ = R_e \left\{ E_0 e^{i(kr - wt + (N-1)\kappa a)} \frac{e^{-i\kappa a/2} - e^{i\kappa a/2}}{e^{-i\kappa a/2} - e^{i\kappa a/2}} \right\} \]
\[ E_{\text{total}} = R_e \left\{ E_0 e^{i(kr - wt + (N-1)\kappa a)} \frac{\sin \left[ \frac{\kappa a}{2} \right]}{\sin \left[ \frac{\kappa a}{2} \right]} \right\} \]
\[ E_{\text{total}} = E_0 \cos \left[ kr - wt + \left( \frac{N-1}{2} \right) \kappa a \right] \frac{\sin \left[ \frac{\kappa a}{2} \right]}{\sin \left[ \frac{\kappa a}{2} \right]} \]
The total magnetic field is therefore
\[ B = \frac{E_0}{c} \cos \left[ k (r - w t) - \frac{(N-1) \pi d}{2} \right] \frac{\sin \frac{\pi d}{k a}}{\sin \frac{\pi a}{k}} \]
and therefore the intensity \( I = |S| \) becomes
\[ I = \frac{E_0^2}{2 \epsilon_0} \cos^2 \left[ k (r - w t) - \frac{(N-1) \pi d}{2} \right] \frac{\sin^2 \frac{\pi d}{k a}}{\sin^2 \frac{\pi a}{k}} \]

The average of the \( \cos^2 \) term is just \( \frac{1}{2} \), so that
\[ I(\Omega) = \frac{E_0^2}{2 \epsilon_0} \frac{\sin^2 \frac{\pi d}{k a}}{\sin^2 \frac{\pi a}{k}} = \frac{E_0^2}{2 \epsilon_0} \frac{\sin^2 \frac{\pi d \sin \Omega}{\lambda}}{\sin^2 \frac{\pi \sin \Omega}{\lambda}} \]
(since \( k = \frac{2\pi}{\lambda} \) and \( d = \frac{\pi d \sin \Omega}{\lambda} \))

This expression can be simplified by looking at \( I(\Omega = 0) \) limit.

If \( \Omega \to 0 \) both numerator and denominator vanish. Consequently one need to use L'Hopital's Rule or the Taylor expansion. For small \( \frac{\pi d \sin \Omega}{\lambda} \) will be small and so
\[ \frac{\sin \frac{\pi d \sin \Omega}{\lambda}}{\lambda} \approx \frac{\pi d \sin \Omega}{\lambda} \frac{\pi d \sin \Omega}{\lambda} \]
and
\[ I(\Omega = 0) = \frac{E_0^2}{2 \epsilon_0} \] Consequently,
\[ I(\Omega) = I(\Omega = 0) \frac{\sin^2 \frac{\pi \lambda}{\lambda}}{\sin^2 \frac{\pi \lambda}{\lambda}} \frac{\pi d \sin \Omega}{\lambda} \]

\[ \lambda = \frac{\pi d \sin \Omega}{\lambda} \]
Special Case \( N=2 \)

\[
I(\theta)_{N=2} = \frac{I(\theta=0) \sin^2 2\theta}{4} \quad \text{but} \quad \sin 2\theta = 2\sin \theta \cos \theta
\]

and

\[
I(\theta)_{N=2} = I(\theta=0) \cos^2 \left( \frac{\pi \sin \theta}{\lambda} \right) \quad \text{which is just our previous result from our discussion of the double slit}
\]

\( N=4 \).

\[
I(\theta) = \frac{I(\theta=0) \sin^2 4\theta}{16} \quad \frac{\sin^2 4\theta}{\sin^2 \theta}
\]

Note: at \( \theta=0 \) \( \alpha=0 \) one obtains a maximum as before with \( I(\theta=0) \neq I(\theta=0) \).

As \( \theta \) increases, \( \alpha \) increases, however the argument of the sine in the numerator increases at a faster rate. Consequently, the numerator first vanishes at \( \alpha = \frac{\pi}{4} \) with the denominator having a non-zero value.

The numerator will also vanish at \( \frac{\pi}{2} \) and \( \frac{3\pi}{4} \). However, when \( \alpha = \frac{\pi}{4} \) both the numerator and denominator vanish. Here we use L'Hopital's Rule. Differentiate both numerator & denominator:

\[
\lim_{\alpha \to \pi} \frac{\sin 4\alpha}{\sin \alpha} = \lim_{\alpha \to \pi} \frac{4 \cos 4\alpha}{\cos \alpha} = \lim_{\alpha \to \pi} \frac{4 \cos(4\alpha)}{\cos(\alpha)} = -4
\]

Consequently \( I(\theta=\pi) = I(\theta=0) \) again a possible maximum.

As a function of \( \theta \) then we see that we have zeros at \( \theta = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}, \pm \frac{5\pi}{4}, \pm \frac{7\pi}{4} \) etc and maxima at \( \theta = 0, \pm \frac{\pi}{2}, \pm \frac{3\pi}{2} \) etc.
Given this information we see that there will be 2 additional maxima between $2\pi/3$ and $\pi$. However, they will not be as large as the principal maxima because although $\sin 4\pi = 1$ we still have a factor of $1/16$ overall. Although $\sin \xi$ will be less than one, it will not generally be small enough to compensate for the $1/16$ factor.

$$\frac{\sin^2 \left( 4 \frac{3\pi/\theta}{\lambda} \right)}{16 \sin^2 \left( \frac{3\pi/\theta}{\lambda} \right)} \Rightarrow 0.073.$$  

\[ \text{Example } \quad d = 4\lambda. \]

\[ \alpha = \pi d \text{ Sin} [\theta] / \lambda \]

Note that the principal maxima occur at the same position as in the double slit problem, $d \sin \theta = m\lambda$. In general, all $N$-slit systems have their principal maxima in the same position. As $N$ increases the principal maxima become sharper as more secondary maxima are introduced.
What are the widths of the principle peaks. The location of the $n$th order peak is $\alpha = m\pi$ or $Nd = Nm\pi$. The next maximum occurs when $N\pi$ increments by $\pi$ is $Nd = Nm\pi + \pi$. Writing out $\alpha = N\pi \sin \theta / \lambda$ and solving for $\sin \theta$ we have that

$$Nd \sin \theta_p / \lambda = Nm\pi \quad \text{and} \quad Nd \sin \theta_m / \lambda = Nm\pi + \pi$$

$$\sin \theta_{peak} = \frac{m\lambda}{d} \quad \ldots \quad \sin \theta_{min} = \frac{m\lambda}{d} + \frac{\lambda}{Nd}$$

For $N$ large we expect that $\theta_{min} = \theta_{peak} + \Delta \theta$ where $\Delta \theta$ is small, and so

$$\sin \theta_{min} = \sin (\theta_{peak} + \Delta \theta) = \sin \theta_{peak} \cos \Delta \theta + \sin \Delta \theta \cos \theta_{peak}$$

$$= \frac{m\lambda}{d} 1 + \Delta \theta \cos \theta_{peak} = \frac{m\lambda}{d} + \frac{\lambda}{Nd}$$

and so $\Delta \theta = \frac{\lambda}{Nd \cos \theta_{peak}}$. 
\[ \Delta \theta = \frac{\lambda}{\text{max} \cdot \text{cos} \theta_{\text{peak}}} \]

shows that the half-width decreases linearly with \( N \), so that the principle maxima are more well-defined for large \( N \) gratings. Usually such gratings are used instead of prisms to disperse light. One can determine the angular separation of two principle maxima generated by two wavelengths from the expression

\[ d \sin \theta = m \lambda \quad \text{and noting that} \quad \Delta \theta = \left( \frac{d \theta}{d \lambda} \right) \Delta \lambda \quad \text{and} \]

\[ \Delta \theta = \frac{m \lambda}{\text{max} \cdot \text{cos} \theta_{\text{peak}}} \]

Therefore, the dispersion of the grating increases with the order \( m \). Of course, higher \( m \) maxima will be diminished.

One criterion to be able to resolve two neighboring peaks associated with \( 2 \lambda \)s. We need a separation between the two peaks which is at least as large as the separation between one peak and its first minimum.

\[ \Delta \lambda = \frac{\lambda}{\text{max} \cdot \text{cos} \theta_{\text{peak}}} \]

\[ \Delta \lambda > \Delta \theta_{\text{min-max}} \]

\[ \Delta \theta_{\text{min-max}} = \frac{\lambda}{N \cdot \text{cos} \theta_{\text{peak}}} \]

\[ \frac{\text{max} \cdot \text{cos} \theta_{\text{peak}}}{\text{cos} \theta_{\text{peak}}} \geq \frac{\lambda}{N \cdot \text{cos} \theta_{\text{peak}}} \]

\[ \lambda > \frac{1}{N \cdot m} \]

The \( N \cdot m \) factor is sometimes called the resolving power of the grating.
Single Slit.

To treat a single slit we can artificially separate the single slit into \( N \) slits and ask how the rays from different sections of the single slit interfere with each other.

The distance between the centers of each of the \( N \) slits is \( a/N \).

From our analysis of the \( N \)-slit problem we have that

\[
I(\psi) \propto \frac{E_0^2}{2\varepsilon_0} \sin^2 \left( \frac{kV a}{2} \right) \sin^2 \left( \frac{k a \sin \psi}{2} \right)
\]

where now \( aL = \frac{a}{N} \sin \psi \). Note also we probably should replace \( E_0 \) with \( E_0/N \) since we are effectively dividing up the incident field into \( N \) different components.

\[
I(\psi) \propto \frac{(E_0/N)^2}{2\varepsilon_0} \sin^2 \left( \frac{\pi a \sin \psi}{N} \right) \sin^2 \left( \frac{\pi a \sin \psi}{N} \right)
\]

We now take \( N \to \infty \) and we obtain

\[
I(\psi) \propto \frac{(E_0/N)^2}{2\varepsilon_0} \sin^2 \left( \frac{\pi a \sin \psi}{N} \right) = \frac{E_0^2}{2\varepsilon_0} \sin^2 \left( \frac{\pi a \sin \psi}{N} \right)
\]

where \( \beta = \pi a \sin \psi / N \). As before we can simplify the form by noting that as \( \psi \to 0 \) \( \beta \to 0 \) and

\[
I(\psi \to 0) = \frac{E_0^2}{2\varepsilon_0} \quad \text{and so}
\]

\[
I(\psi) = I(\psi = 0) \frac{\sin^2 \beta}{\beta^2}
\]
For $\theta = 0$ we obtain the central maximum. The first minimum on either side occurs for $\beta = \pi$ or

$$\frac{\pi a \sin \theta}{\lambda} = \sin \beta_{\text{min}} = \pm \frac{\lambda}{a}$$

Subsequent minima all occur at

$$\sin \beta = \pm m \lambda$$

(Note: the similarity with the $N$-slit problem. The difference is that this locates the minima, not maxima.)

As the slit size becomes smaller than $\sin \theta_{\text{min}} \rightarrow 0$, the central maximum becomes broader and broader, and will eventually illuminate the screen uniformly. This was our assumption when we derived all of a $N$-slit formula. If $a$ is not too much smaller than $\lambda$, then the single slit pattern will mimic that of the $N$-slit pattern. Consequently, some of the $N$ slit principle maxima may be missing unless they occur at the minima of the single slit pattern.
In the above example the principle maxima of the N-slit pattern occur at
\[ \sin \theta_{\text{max}} = \frac{m \lambda}{d} = 0, \pm \frac{1}{4}, \pm \frac{1}{2}, \pm \frac{3}{4}, \pm 1 \quad 9 \text{ maxima} \]
argue that there are only 7 since the \( \sin \theta_{\text{max}} = 1 \) are not perfectly developed. Their maxima of the single slit occur at
\[ \sin \theta_{\text{min}} = \frac{m \lambda}{a} = \pm \frac{1}{2}, \pm 1 \quad \text{and so eliminate 4 principle maxima.} \]
Note if \( a \) and \( d \) are chosen arbitrarily the pattern will be far more complex.
What is the width of the central maximum of the single slit pattern? Taking the width at half maximum we want $\delta$ such that

$$\left( \frac{\sin \beta}{\beta} \right)^2 = \frac{1}{2} \quad \Rightarrow \quad \sin \beta = \frac{\beta}{\sqrt{2}}$$

This is a transcendental equation and must be solved numerically. Note that $\sin \beta = \beta$, $\frac{\beta}{\sqrt{3}}$, ... and so

$$\beta = \frac{\beta}{\sqrt{2}} \quad \Rightarrow \quad \beta = 6 \left[ 1 - \frac{1}{12} \right]^{\frac{1}{2}} \quad \Rightarrow \quad \beta = 6 \left[ \frac{2 - 12}{2} \right]$$

or $\beta = \sqrt{3(2 - 12)} = 1.335$ (The actual answer is 1.3915).

(Initial on your calculator $\beta = \sqrt{2 \sin \beta}$).

And so $\pi \sin \theta = 1.3915 \quad \Rightarrow \quad \sin \theta_{\text{width}} = 0.443 \frac{\lambda}{\delta}$.

For a circular slit, one can similarly show that the minimum next to the central maximum is given by:

$$\sin \theta_{\text{min}} = 1.22 \frac{\lambda}{D} \quad \text{where } D \text{ is the diameter of the aperture.}$$

If we use the same criterion as before then the image of two point sources can be resolved if their images have an angular separation greater than $1.22 \frac{\lambda}{D}$.

Point Sources

$$\Delta \phi_{\text{sep}} = \frac{\Delta x}{D_0} \quad \Rightarrow \quad \Delta \phi = 0 \Delta \phi_{\text{sep}} = \frac{\Delta y}{D_i}$$

$$\frac{\Delta x}{D_0} > 1.22 \frac{\lambda}{D} \quad \text{in order for the two images to be resolved as two separate images. This gives the ultimate resolution.}$$
Note the dependence of $\lambda$. For microscopes one can improve one's resolution by using shorter wavelengths.

For telescopes it is much harder to change the wavelength and so you make your telescope larger. For radio telescopes there is the VLA (Very Large Array) in New Mexico and now it is possible to link telescopes across the world to form a baseline with length of 5000 km.

For a human pupil $D = 0.5 \text{ cm}$. Sunlight is refracted by the eye which is filled with a fluid of index of refraction $n = 1.33$ then we should use the $\lambda$ inside the eye to calculate.

\[
\text{v_{sep}} = \frac{1.22\lambda}{D} \quad \text{if} \quad \lambda = 5000 \text{Å} \quad n = 1.33.
\]

\[
= \frac{1.22 \times 5000 \text{Å}}{2.2 \times 10^{-4} \text{ cm}} = 2.2 \times 10^{-4} \text{ cm}.
\]

Head lights on a car are about 1.4 meters apart and so

\[
\text{v_{sep}} = \frac{\Delta x}{D_0} \quad \Rightarrow \quad D_0 = \frac{1.4 \text{ meters}}{2.2 \times 10^{-4}} = 6104 \text{ meters}.
\]

or about 20,000 feet. So in principle one could almost distinguish a pair of head lights from a plane at 20,000 ft. (Nominal cruising altitude = 36,000 feet).

Similarly one can ask how large must a lens be in order for a spy satellite to be able to observe a missile or truck on the ground.