Chapter 8

Electromagnetic Waves

8.1 Maxwell Equations

8.1.1 Gauss’s Law

Gauss’s law states that the net flux of the electric field on a closed volume is equal to the total charge within the volume

\[ Q_{total} = \iiint_{\text{volume}} \rho(x, y, z) \, dx \, dy \, dz = \iint_{\text{surface}} \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds \]  

(8.1)

where \( \rho(x, y, z) \) is the charge density and \( \hat{n} \) is the outward pointing normal unit vector of the surface.

Consider a differential element of volume which has a corner at \((x, y, z)\)
The contributions to the integral from the surfaces (1) and (2) perpendicular to the x-axis are

\[ \int_{\text{surface}} \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds = \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds \]  

\[ \int_{\text{(1)}} \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds + \int_{\text{(2)}} \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds \]  

Because \( \hat{n} \) is parallel to the x-axis, \( ds = dy \, dz \) and \( \mathbf{E} \cdot \hat{n} \) will be \( \pm E_x \) evaluated at \( x \) and \( x + \Delta x \)

\[ \int_{\text{(1)}} \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds + \int_{\text{(2)}} \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds \]

\[ = \int \int \varepsilon_0 E_x[x + \Delta x, y, z] \, dy \, dz - \int \int \varepsilon_0 E_x[x, y, z] \, dy \, dz \]

\[ = \varepsilon_0 (-E_x[x, y, z] + E_x[x + \Delta x, y, z]) \Delta y \Delta z \]

\[ = \varepsilon_0 \left( \frac{\partial E_x[x, y, z]}{\partial x} \right) \Delta x \Delta y \Delta z \]  

Repeating for the surfaces perpendicular to the y and z axes we have that

\[ \int_{\text{surface}} \varepsilon_0 \mathbf{E} \cdot \hat{n} \, ds = \varepsilon_0 \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \Delta y \Delta z \]  

\[ = \varepsilon_0 \nabla \cdot \mathbf{E} \Delta x \Delta y \Delta z \]  

Equating this to the total charge,

\[ \int \int \int \rho[x, y, z] \, dx \, dy \, dz = \rho[x, y, z] \Delta x \Delta y \Delta z = \varepsilon_0 \nabla \cdot \mathbf{E} \Delta x \Delta y \Delta z \]  

(8.5)

The differential form of Gauss’s law is therefore

\[ \nabla \cdot \mathbf{E} = \frac{\rho[x, y, z]}{\varepsilon_0} \]  

(8.6)

### 8.1.2 Faraday’s Law

Faraday’s law states that the line integral of the electric field around a closed loop is related to the time rate of change of the magnetic flux through the loop
Consider a loop differential element loop in the x y plane

The contributions to the integral from the line elements (1), (2), (3) and (4) are

\[
\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t} \int_{\text{surface}} \mathbf{B} \cdot \mathbf{n} \, ds
\]

Equating this to the time rate of change of the magnetic flux through the loop
\[-\frac{\partial}{\partial t} \int_B \hat{n} \cdot ds = -\frac{\partial}{\partial t} (B_z \Delta x \Delta y) = \left( -\frac{\partial E_x[x, y]}{\partial y} + \frac{\partial E_y[x, y]}{\partial x} \right) \Delta x \Delta y \]  

(8.9)

or

\[-\frac{\partial B_z}{\partial t} = \left( -\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} \right) = \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \]  

(8.10)

Repeating this for loops oriented in the \( \hat{x} \) and \( \hat{y} \) directions

\[-\frac{\partial B_x}{\partial t} = \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} \right) \]  

\[-\frac{\partial B_y}{\partial t} = \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \]  

(8.11)

These equations can be summarized as

\[-\frac{\partial B}{\partial t} = \nabla \times E \quad \text{(curl of } E) \]  

(8.12)

The curl can be evaluated by taking a determinant of a matrix formed from the unit vectors, \( \nabla \) and the vector field.

\[\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x} \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{y} \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \hat{z} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \]  

(8.13)

8.1.3 Ampère’s Law (with modifications by Maxwell)

Ampère’s law states that the line integral of the magnetic field around a closed loop is equal to the total enclosed current. Maxwell indicated that in addition to currents, another source of magnetic fields are time dependent electric fields

\[\oint B \cdot dl = \mu_0 \int \mathbf{J} \cdot \hat{n} \, ds + \epsilon_0 \mu_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot \hat{n} \, ds \]  

(8.14)

Mathematically, this equation has the same form as Faraday’s law. Following this example, we can immediately state the differential form of Ampère’s law

\[\int \mathbf{E} \cdot dl = -\frac{\partial}{\partial t} \int_B \hat{n} \, ds \quad \rightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  

(8.15)
\[ \int \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\text{surface}} \mathbf{J} \cdot \hat{n} \, ds + \epsilon_0 \mu_0 \int_{\text{surface}} \frac{\partial \mathbf{E}}{\partial t} \cdot \hat{n} \, ds \quad \rightarrow \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \] (8.16)

### 8.1.4 Maxwell's Equations

To complete the differential forms of Maxwell's equations, we need to specify the divergence of the magnetic field. At this time, there is no experimental evidence for a magnetic charge (or magnetic monopole). Consequently the divergence of the magnetic field is identically equal to zero.

\[
\nabla \cdot \mathbf{E} = \rho(x, y, z) \\
\nabla \cdot \mathbf{B} = 0 \\
\n\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
\n\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}
\] (8.17)

### 8.1.5 Wave Equation

By combining the above the equations in free space (no electrical charges or currents), one obtains an equation involving only the electric or magnetic field. For example, consider the magnetic field in the z direction. From \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \), we have that

\[
\frac{\partial B_z}{\partial t} = \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) 
\] (8.18)

Taking the time derivative of the entire equation, we have that

\[
\frac{\partial^2 B_z}{\partial t^2} = \left( \frac{\partial}{\partial y} \left( \frac{\partial E_x}{\partial t} \right) - \frac{\partial}{\partial x} \left( \frac{\partial E_y}{\partial t} \right) \right) 
\] (8.19)

Replacing the time derivatives of the electric field in terms of the magnetic field using \( \nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \)

\[
\frac{\partial^2 B_z}{\partial t^2} = \frac{1}{\epsilon_0 \mu_0} \left( \frac{\partial}{\partial y} \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_x}{\partial z} \right) - \frac{\partial}{\partial x} \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \right)
\] (8.20)
\[
\frac{\partial^2 B_z}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \left( \frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2} \right) = \frac{1}{\varepsilon_0 \mu_0} \nabla^2 B_z
\]  

(8.21)

where \( \nabla^2 = \nabla \cdot \nabla \). Similarly, one can show that each component of the magnetic field satisfies this equation

\[
\frac{\partial^2 B}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \nabla^2 B
\]  

(8.22)

From the symmetry of Maxwell’s equation, one can also show that the electric field also satisfies this equation.

\[
\frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \nabla^2 E
\]  

(8.23)

Therefore the electric and magnetic fields satisfy the wave equation. The speed of propagation can then be read from the equation

\[
c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}} = \sqrt{\frac{1}{8.85 \times 10^{-12} \text{ Farad m}^{-1} \cdot 4\pi \times 10^{-7} \text{ Webers Amp}^{-1} \text{ m}^{-1}}} = \sqrt{\frac{A \text{ m}^2}{\text{ Farad Weber}}} = \frac{2.99863 \times 10^8 \text{ m}}{s}
\]  

(8.24)

8.1.6 Energy Conservation

Starting with the wave equation, we were able to show that the energy of a section of the string can change with time if the power at the ends of the string is nonzero. A similar treatment can be done now for electromagnetic waves or pulses by starting with one of Maxwell’s equations.

\[
\varepsilon_0 \frac{\partial E}{\partial t} = \frac{\mu_0}{\partial t} \nabla \times B
\]  

(8.25)

Multiply both sides by \( E \)
\[ \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \epsilon_0 = \frac{1}{\mu_0} \mathbf{E} \cdot \nabla \times \mathbf{B} \]  \hspace{1cm} (8.26)

Using the identity \( \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \) we have that

\[ \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \epsilon_0 = \frac{1}{\mu_0} \left( \mathbf{B} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \right) \]  \hspace{1cm} (8.27)

Substituting \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) and bringing the magnetic field term to the left side of the equation

\[ \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) \]  \hspace{1cm} (8.28)

or

\[ \frac{\partial \left( \frac{1}{2} \epsilon_0 \mathbf{E}^2 + \frac{1}{2} \frac{1}{\mu_0} \mathbf{B}^2 \right)}{\partial t} = -\frac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) \]  \hspace{1cm} (8.29)

If one now integrates over a finite volume

\[ \frac{\partial}{\partial t} \int_{\text{volume}} \left( \frac{1}{2} \epsilon_0 \mathbf{E}^2 + \frac{1}{2} \frac{1}{\mu_0} \mathbf{B}^2 \right) d^3 \mathbf{r} = -\frac{1}{\mu_0} \int_{\text{volume}} \nabla \cdot (\mathbf{E} \times \mathbf{B}) d^3 \mathbf{r} \]  \hspace{1cm} (8.30)

We previously showed that a volume integral of a divergence can be converted to a surface integral, i.e.

\[ \int_{\text{volume}} \nabla \cdot \mathbf{A} d^3 \mathbf{r} = \int_{\text{surface}} \mathbf{A} \cdot \hat{n} d^2 \mathbf{r} \]  \hspace{1cm} (8.31)

Applying this to the right side of the equation we obtain

\[ \frac{\partial}{\partial t} \int_{\text{volume}} \left( \frac{1}{2} \epsilon_0 \mathbf{E}^2 + \frac{1}{2} \frac{1}{\mu_0} \mathbf{B}^2 \right) d^3 \mathbf{r} = -\frac{1}{\mu_0} \int_{\text{surface}} (\mathbf{E} \times \mathbf{B}) \cdot \hat{n} d^2 \mathbf{r} \]  \hspace{1cm} (8.32)

To interpret the above equation, first consider a capacitor with area \( A \), thickness \( d \) and surface charge \( \sigma = \frac{Q}{A} \).
From Gauss’s law, the electric field in the capacitor is given by \( \mathbf{E} = \frac{\sigma}{\varepsilon_0} \). The energy of the capacitor is given by \( \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C} \) where the capacitance is given by \( C = \frac{A \varepsilon_0}{d} \). Expressing the energy in terms of the electric field within the capacitor we have that

\[
E = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \frac{A^2 \sigma^2}{\varepsilon_0 d} = \frac{1}{2} \frac{A d \sigma^2}{\varepsilon_0} = \frac{1}{2} \varepsilon_0 E^2 (A d)
\]

(8.33)

Therefore the energy of the capacitor is just \( \frac{1}{2} \varepsilon_0 \mathbf{E}^2 \) times the volume of the capacitor. Consequently, \( \frac{1}{2} \varepsilon_0 \mathbf{E}^2 \) is the energy density associated with an electric field.

Now consider an inductor.

\[
\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 i_{\text{enclosed}} \quad \Rightarrow \quad B \ell = \mu_0 I n \ell \quad \Rightarrow \quad B = \mu_0 I n
\]

(8.34)

The inductance of a coil is defined in terms of the EMF that is associated with the changing current, i.e. \( \text{emf} = -L \frac{dI}{dt} \). For the above coil, each loop contributes an \( \text{emf}_i = -\frac{d\Phi}{dt} \) where the flux through one loop is \( \Phi = AB \). For the entire coil, then the

\[
\text{emf} = -n \ell \frac{\partial \Phi}{\partial t} = -n \ell A \frac{\partial B}{\partial t}
\]

(8.35)

\[
= -n \ell A \frac{\partial (\mu_0 I n)}{\partial t} = -n^2 \mu_0 \ell A \frac{\partial I}{\partial t}
\]

Comparing this with the definition of the inductance, we have that \( L = \mu_0 n^2 \ell A \) The energy of an inductor is given by \( E = \frac{1}{2} LI^2 \). Substituting the expression for the inductance and expressing the current in terms of the magnetic field.

\[
\frac{1}{2} LI^2 = \frac{1}{2} \left( n^2 \ell A \mu_0 \right) \frac{B^2}{n^2 \mu_0^2} = \frac{B^2}{2 \mu_0} A \ell
\]

(8.36)

The energy of the inductor is \( \frac{1}{2} \mu_0 B^2 \) times the volume of the inductor. Consequently \( \frac{1}{2} \mu_0 B^2 \) is the energy density associate with a magnetic field.

With these identifications, the left hand side of Eq. 8.32 is the time rate of change of the electromagnetic energy within the volume. Therefore the right hand side of the equation must be
related to the rate at which energy is flowing through the surface of the volume. Because \( \hat{n} \) is the outward pointing normal of the surface of the volume, \( \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \) must be the energy flux or the rate at which energy flows per unit area.

\[
\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})
\]  

(8.37)

is called the Poynting vector. The direction of the Poynting vector indicates the direction of the energy flow and its magnitude represents the energy that passes through a unit area per unit time.

8.1.7 Radiation Pressure

Consider two parallel planes which have opposing currents flowing through them. The current in each plane generates a uniform magnetic field in space. In the space between the planes, the magnetic fields point in the same direction and therefore add. Outside of this volume, the magnetic fields from each plane point in opposite directions and the net magnetic field is zero.

If one suddenly turns off the current, the magnetic field goes to zero. However because the information that the current has vanished takes time to propagate outwards. Therefore the region in which the magnetic field is zero propagates outwards at the speed of light. Because the planes are displaced from each other, there will be two regions in which the magnetic field from one plane will be zero but the magnetic field from the more distant plane will be nonzero. These regions of non zero field also propagate outwards at the speed of light.

From this example, one sees that one can have a block of magnetic field that moves through space. Associated with this moving block of magnetic field is a block of electric field.
As the block of magnetic field crosses side 1 of the loop, the magnetic flux though the loop increases. From Faraday’s law, this generates an electric field along the loop. From the principal of locality, the electric field can only appear along the side 1 that is within the magnetic field. Consequently, superposed upon the magnet field, there is a copropagating electric field block with field strength, $E = c\, B$. The direction of $E \times B$ points in the direction of motion. The energy density of fields is given by $\mathcal{E} = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 B^2$

Imagine now that this block of magnetic and electric fields, reflects from a mirror. To understand the force on the wall, one can make an analogy to a mass bouncing off a wall.

The momentum of the mass changes from $mv$ to $-mv$. From momentum conservation, the momentum transferred to the wall is $2\, m\, v$. The force of the wall is given by the impulse or the time rate of change of the momentum.

For a block of electromagnetic energy, the total energy within the block is given by $\mathcal{E} A L$. 
The amount of energy that passes a surface perpendicular to the propagation direction in a time \( \Delta t \) is \( A c \Delta t E \). Therefore the Poynting vector or the energy flux (the amount of energy that passes this surface per unit time per unit area) is just \( |S| = cE \).

Using Einstein’s relation, \( E = mc^2 \), we can define a mass to the entire energy block, \( m_{\text{eff}} = \frac{EAL}{c^2} \). If the entire block is reflected by a surface, the total change in the momentum of the surface is

\[
\Delta p = 2v m_{\text{eff}} = 2\left(\frac{ALE}{c^2}\right)c = \frac{2ALE}{c}
\]

where the velocity of the block is \( c \). This change in momentum occurs over a time \( \Delta t = L/c \), the time it takes for the entire block to bounce off the surface. The force on the surface is

\[
\frac{\Delta p}{\Delta t} = \frac{2(EAL)}{c} = \frac{L}{c} = 2AE
\]

and consequently the pressure on the surface is \( P = F/A = 2E = 2 \left(\frac{|S|}{c}\right) \). Note that because \( S \) is
the energy flux and the energy-momentum relationship for photons is $\varepsilon = c \rho$, \( \frac{|S|}{c} \) is also the momentum flux. Remembering that any flux is equal to a velocity times a density, we can interpret \( \frac{S}{c^2} = \frac{1}{c^2 \mu_0} \mathbf{E} \times \mathbf{B} = \varepsilon_0 \mathbf{E} \times \mathbf{B} \) as the momentum density.

### 8.1.8 Oscillatory Solutions

The wave equation also has oscillatory solutions. Because the electric and magnetic fields are vectors, the solutions must include a unit vector indicating the direction of the fields. Designating the direction of the electric field to be $\hat{\mathbf{e}}$, the wave corresponding to the electric field has the form

$$
\mathbf{E} = \hat{\mathbf{e}} E_0 e^{i(k \cdot \mathbf{r} - \omega t)}
$$

(8.40)

where $k = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$ is the wavevector and $\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$. Differentiating with respect to $t$,

$$
\frac{\partial^2 \mathbf{E}}{\partial t^2} = -e^{i(k \cdot \mathbf{r} - \omega t)} \omega^2 \hat{\mathbf{e}} E_0 = -\omega^2 \mathbf{E}
$$

(8.41)

Taking the Laplacian

$$
\frac{1}{\varepsilon_0 \mu_0} \nabla^2 \mathbf{E} = c^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) e^{i(-\omega x k_x + \omega y k_y + \omega z k_z)} \hat{\mathbf{e}} E_0
$$

$$
= -c^2(k_x^2 + k_y^2 + k_z^2) \hat{\mathbf{e}} E_0 e^{i(-\omega x k_x + \omega y k_y + \omega z k_z)} = -c^2(k_x^2 + k_y^2 + k_z^2) \mathbf{E}
$$

(8.42)

$\mathbf{E} = \hat{\mathbf{e}} E_0 e^{i(k \cdot \mathbf{r} - \omega t)}$ is therefore a solution if $\omega^2 = (k_x^2 + k_y^2 + k_z^2) c^2$ or $\omega = c k = c \sqrt{k_x^2 + k_y^2 + k_z^2}$. Because there are no charges or currents, the electric field must also satisfy, $\nabla \cdot \mathbf{E} = 0$ or

$$
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -i e^{i(k \cdot \mathbf{r} - \omega t)} (k_x \varepsilon_x + k_y \varepsilon_y + k_z \varepsilon_z) E_0 = 0
$$

(8.43)

and therefore $\hat{\mathbf{e}} \cdot \mathbf{k} = 0$ or that $\hat{\mathbf{e}}$ is perpendicular to $\mathbf{k}$. Associated with this electric field, there is a magnetic field that can be calculated from Faraday’s law

$$
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}
$$

$$
-\hat{x} \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \hat{y} \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) - \hat{z} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)
$$

(8.44)

$$
= -i \hat{x} \left( k_y \varepsilon_z - k_z \varepsilon_y \right) - i \hat{y} \left( k_z \varepsilon_x - k_x \varepsilon_z \right) - i \hat{z} \left( k_x \varepsilon_y - k_y \varepsilon_x \right) e^{i(k \cdot \mathbf{r} - \omega t)} E_0
$$

$$
= -i (k \times \hat{\mathbf{e}}) E_0 e^{i(k \cdot \mathbf{r} - \omega t)}
$$

Integrating with respect to time
\[ \mathbf{B} = \frac{(k \times \hat{\mathbf{e}})}{\omega} E_0 e^{i(k \cdot r - \omega t)} = \frac{k \mathbf{E} \cdot (\mathbf{k} \times \hat{\mathbf{e}})}{\omega} = \frac{\mathbf{E} \cdot (\mathbf{k} \times \hat{\mathbf{e}})}{c} \] (8.45)

Therefore the corresponding magnetic field is perpendicular to the electric field and has a magnitude that is the magnitude of the electric field divided by \( c \), \( B_0 = \frac{E_0}{c} \).

To finish the analysis, one must also calculate the Poynting vector, \( \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \). Because the Poynting is the product of two fields, one must use the real part of both vectors.

\[
\mathbf{S} = \frac{(\mathbf{E} \times \mathbf{B})}{\mu_0} = \frac{1}{\mu_0} \left( (\hat{\mathbf{e}} E_0 \cos[\omega t - \mathbf{k} \cdot \mathbf{r}]) \times (B_0 (\mathbf{k} \times \hat{\mathbf{e}}) \cos[\omega t - \mathbf{k} \cdot \mathbf{r}]) \right) \\
= \frac{E_0^2 (\hat{\mathbf{e}} \times (\mathbf{k} \times \hat{\mathbf{e}}))}{c \mu_0} \cos[\omega t - \mathbf{k} \cdot \mathbf{r}]^2 \tag{8.46}
\]

Using the vector identity, \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \) and that \( \hat{\mathbf{e}} \) is perpendicular to \( \mathbf{k} \), the Poynting vector can be simplified

\[
\mathbf{S} = \frac{E_0^2}{c \mu_0} \cos[\omega t - \mathbf{k} \cdot \mathbf{r}]^2 \left( \hat{\mathbf{e}} \cdot \hat{\mathbf{e}} \mathbf{k} - \hat{\mathbf{e}} \cdot \mathbf{k} \hat{\mathbf{e}} \right) \\
= \frac{E_0^2}{c \mu_0} \cos[\omega t - \mathbf{k} \cdot \mathbf{r}]^2 \mathbf{k} \tag{8.47}
\]

For this wave, the energy flows in the direction of \( \mathbf{k} \). The vectors, \( \mathbf{E}, \mathbf{B} \) and \( \mathbf{k} \) form a triad of perpendicular vectors.
The average of the Poynting vector is the intensity.

\[
\langle S \rangle = \frac{E_0^2}{c \mu_0} \cos[\omega t - \hat{k} \cdot \mathbf{r}] \hat{k} = \frac{E_0^2}{2 c \mu_0} \hat{k} = \frac{1}{2} c \varepsilon_0 E_0^2 \hat{k}
\]  
(8.48)

The physical manifestation of these wave depends on the frequency or wavelength.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\lambda$</th>
<th>Physical Manifestation</th>
<th>Sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>$60 \text{ Hz}$</td>
<td>5000 km</td>
<td>Grid Power</td>
<td></td>
</tr>
<tr>
<td>$10^6 \text{ Hz (MHz)}$</td>
<td>300 m</td>
<td>Radio Waves (AM band)</td>
<td>Galaxies</td>
</tr>
<tr>
<td>$10^8 \text{ Hz}$</td>
<td>3 m</td>
<td>Radio Waves (FM band)</td>
<td></td>
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<tr>
<td>$10^{10} \text{ Hz (10 GHz)}$</td>
<td>0.03 m</td>
<td>Microwaves</td>
<td>Radar, Microwave Ovens</td>
</tr>
<tr>
<td>$10^{12} \text{ Hz}$</td>
<td>300 $\mu$m</td>
<td>Infrared Light</td>
<td>Hot Objects</td>
</tr>
<tr>
<td>$10^{15} \text{ Hz}$</td>
<td>3000 Å</td>
<td>Visible Light</td>
<td>Atoms, Molecules</td>
</tr>
<tr>
<td>$10^{16} \text{ Hz}$</td>
<td>300 Å</td>
<td>Ultraviolet Light</td>
<td>Atoms (Hydrogen)</td>
</tr>
<tr>
<td>$10^{18} \text{ Hz}$</td>
<td>3 Å</td>
<td>$X - \text{Rays}$</td>
<td>Atoms</td>
</tr>
<tr>
<td>$10^{22} \text{ Hz}$</td>
<td>30 fm</td>
<td>Gamma Rays</td>
<td>Nuclei</td>
</tr>
</tbody>
</table>