Chapter 7

Sound

7.1 “Bulk” Springs

For a single spring, of length $L$, the force generated by a small change in the length is

$$F = -k \Delta L$$

If two such springs are connected in parallel, then for a given change in length $\Delta L$, each spring contributes the same force. The net force is therefore doubled.

$$F = -2k \Delta L$$

If however, the two springs are connected in series, then for an overall change in length $\Delta L$, each spring is only stretched by $\Delta L/2$. Therefore the net force is only $-k \Delta L/2$

$$F = -\frac{k \Delta L}{2}$$

We can do the same for rods. For a single rod of length $L_0$ and cross sectional area $A_0$, the force generated by a small change in length, $\Delta L$, is

$$F = -k \Delta L$$

If we now combine several of these rods in parallel, each rod will contribute the same force, $F_i = -k \Delta L$
where \( N_p = \frac{A}{A_0} \) is the number of rods that in parallel. If we now combine two stacks in series, then the force is reduced by the number of rods in series, \( N_s = \frac{L}{L_0} \), i.e.

\[
F = -N_p k \Delta L = -\frac{A k \Delta L}{A_0}
\]

\[A = A_0 N_p\]

The effective spring constant that characterizes the force for a change in length of a rod whose area is \( A \) and length \( L \) is therefore

\[
F = -A \left( \frac{k L_0}{A_0} \right) \Delta L = -Y \frac{A}{L} \Delta L
\]

(7.1)

where \( Y \) is the Youngs modulus of the material. It has dimension of

\[
Y = k \frac{L_0}{A_0} = \left( \frac{\text{Force}}{\text{Length}} \right) \frac{\text{Length}}{\text{Area}} = \frac{\text{Force}}{\text{Area}} = \text{Pressure}
\]

(7.2)

### 7.2 Derivation of the Wave Equation for Longitudinal Waves in a Rod

Consider the forces on a small section of a rod induced by the neighboring sections.
The force $F_1$ on right side of element (2) is given by the change in length of element (3)

$$F_1 = -F_{on_3} = Y \frac{A \Delta L}{L} = Y A \frac{y[x + \Delta x] - y[x + \Delta x/2]}{\Delta x/2}$$

$$\left. Y \frac{\partial y}{\partial x} \right|_{x + \Delta x/2}$$

where we have used the fact that

$$y[x + \Delta x] = y[x + \Delta x/2] + \left. \left( \frac{\partial y}{\partial x} \right) \right|_{x + \Delta x/2} \left( \frac{\Delta x}{2} \right)$$

Similarly the force $F_2$ on left side of element (2) is given by the change in length of element (1)

$$F_2 = F_{on_1} = -Y \frac{A \Delta L}{L} = -Y A \frac{y[x - \Delta x/2] - y[x - \Delta x]}{\Delta x/2}$$

$$\left. Y \frac{\partial y}{\partial x} \right|_{x - \Delta x/2}$$

where we have used the fact that

$$y[x - \Delta x] = y[x - \Delta x/2] - \left. \left( \frac{\partial y}{\partial x} \right) \right|_{x - \Delta x/2} \left( \frac{\Delta x}{2} \right)$$

The total force on element (2) is equal to its mass, $\rho A \Delta x$, times the acceleration, $\left( \frac{\partial^2 y}{\partial x^2} \right)$

$$A \Delta x \rho \frac{\partial^2 y}{\partial t^2} = F_1 + F_2 = A Y \left( \left. \left( \frac{\partial y}{\partial x} \right) \right|_{x + \Delta x/2} - \left. \left( \frac{\partial y}{\partial x} \right) \right|_{x - \Delta x/2} \right) = A Y \Delta x \frac{\partial^2 y}{\partial x^2}$$

or
\[
\frac{\partial^2 y}{\partial t^2} - \frac{1}{\rho} \frac{\partial^2 y}{\partial x^2} = \frac{Y}{\rho} \frac{\partial^2 y}{\partial x^2}
\]  
(7.8)

Again we obtain the wave equation. We can immediately say that the propagation velocity of displacements in the material in the direction of propagation is
\[
c = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{\text{Pressure}}{\text{Mass} / \text{Volume}}}
\]  
(7.9)

These are longitudinal waves whereas the displacements on the string were perpendicular to the propagation direction or transverse waves.

We can immediately take what we have learned about strings and apply them to these waves.

For example, the force at a point along the rod is
\[
F = -A Y \frac{\partial y}{\partial x}
\]  
(7.10)

If one has a fixed boundary condition, \(y[x, t] = 0\), there is no motion of the material. Alternatively, at an open boundary condition, \(\frac{\partial y}{\partial x}[x, t] = 0\), and the force is zero.

Consider the demonstration of the singing aluminum rod. The length of the rod is 60 inches. The density of aluminum is \(\rho_{\text{Al}} = 2.7 \text{ g/cm}^3\) and the speed of the longitudinal sound is
\[
c = \frac{5100 \text{ m/sec}}{\text{sec}} = \frac{16700 \text{ ft/ sec}}{\text{sec}} = \frac{200400 \text{ inches/ sec}}{\text{sec}}
\]  
(7.11)

and the Young's modulus is
\[
Y = c^2 \rho_{\text{Al}} = \frac{2.7 \text{ g}}{\text{cm}^3 \text{ sec}^2} \left(\frac{5100 \text{ m}}{\text{sec}}\right)^2 = \frac{7.0227 \times 10^{10} \text{ kg}}{\text{m sec}^2}
\]  
(7.12)

\[
= \frac{7.02 \times 10^{10} \text{ Nt}}{\text{m}^2} = 7.02 \times 10^{10} \text{ Pascals}
\]

Because each end of the rod is free to move (there is no force generated by the air in contact with the rod), both ends have open boundary conditions. The most general standing wave is
\[
y[x, t] = A \sin[kx + \phi] \sin[\omega t]
\]  
(7.13)

where the phase \(\phi\) is used to adjust the boundary conditions. To satisfy the open boundary condition at \(x = 0\), we require that
\[
\left.\left(\frac{\partial y}{\partial x}\right)\right|_{x=0} = \left.\left(A k \cos[kx + \phi] \sin[\omega t]\right)\right|_{x=0} = A k \cos[\phi] \sin[\omega t] = 0
\]  
(7.14)

Taking \(\phi = \pi / 2\), then the standing wave becomes
\[ y(x, t) = A \sin \left( kx + \frac{\pi}{2} \right) \sin(\omega t) = A \cos(kx) \sin(\omega t) \] (7.15)

To satisfy the boundary condition at \( x = L \),

\[ \left( \frac{\partial y}{\partial x} \right)_{x=L} = (-A k \sin(kx) \sin(\omega t))|_{x=L} = -A k \sin(kL) \sin(\omega t) = 0 \] (7.16)

we must choose \( kL = n \pi \). The wavelengths and frequency of the normal modes are therefore

\[ \lambda = \frac{2\pi}{k} = \frac{2L}{n} = \frac{3.048 m}{n} \]
\[ \nu = \frac{c}{\lambda} = \frac{1673.23 n}{1673 n \text{ Hz}} = 1673 n \text{ Hz} \] (7.17)

The frequency of the first few normal modes are

\[ \nu_1 = 1673.22 \text{ Hz} \]
\[ \nu_2 = 3346.44 \text{ Hz} \]
\[ \nu_3 = 5019.66 \text{ Hz} \]
\[ \nu_4 = 6692.88 \text{ Hz} \]
\[ \nu_5 = 8366.1 \text{ Hz} \] (7.18)

### 7.3 Derivation of the Wave Equation for Sound Waves in a Gas

A similar derivation can be done for compressional waves in a gas.

The bulk modulus of a gas is defined as
\[ B = -V \frac{\partial P}{\partial V} \quad \text{(units of pressure)} \quad (7.19) \]

Alternatively, one can describe the behavior in terms of the compressibility. Depending upon the condition imposed as the gas is compress, one can define two compressibilities: the isothermal compressibility if the temperature of the gas is kept constant or the adiabatic compressibility if no heat is exchanged with the environment or that the entropy of the gas is kept constant.

\[ \kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \quad \text{isothermal} \]

\[ \kappa_S = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \quad \text{adiabatic} \quad (7.20) \]

Consider the forces on a small volume of gas induced by the neighboring volumes

<table>
<thead>
<tr>
<th>( x + \Delta x )</th>
<th>( x - \frac{\Delta x}{2} )</th>
<th>( x + \frac{\Delta x}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_0 )</td>
<td>( P_0 )</td>
<td>( P_0 )</td>
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<tr>
<td>( P_1 )</td>
<td>( F_1 )</td>
<td>( F_2 )</td>
</tr>
<tr>
<td>( P(x) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y(x - \Delta x) )</td>
<td>( y\left(x - \frac{\Delta x}{2}\right) )</td>
<td>( y\left(x + \frac{\Delta x}{2}\right) )</td>
</tr>
</tbody>
</table>

The change of the pressure in the sections (1) and (2) is given by the change in the corresponding volumes

\[ \delta P_2 = -B \frac{\Delta V}{V} = -\frac{B}{A} \Delta x \left( y[x + \Delta x] - y\left[x + \frac{\Delta x}{2}\right]\right) \]

\[ = -B \left( \frac{\partial y}{\partial x} \right)_{x + \frac{\Delta x}{2}} \quad (7.21) \]

and

\[ \delta P_1 = -B \frac{\Delta V}{V} = -\frac{B}{A} \Delta x \left( y\left[x - \frac{\Delta x}{2}\right] - y[x - \Delta x]\right) \]

\[ = -B \left( \frac{\partial y}{\partial x} \right)_{x - \frac{\Delta x}{2}} \quad (7.22) \]

The force on central volume is equal to its mass, \( \rho A \Delta x \), times the acceleration, \( \left( \frac{\partial^2 y}{\partial x^2} \right) \).
\[ A \Delta x \rho \frac{\partial^2 y}{\partial t^2} = A (\delta P_1 - \delta P_2) = A B \left( -\left( \frac{\partial y}{\partial x}\right)_{x-\frac{\Delta x}{2}} + \left( \frac{\partial y}{\partial x}\right)_{x+\frac{\Delta x}{2}} \right) = A B \Delta x \frac{\partial^2 y}{\partial x^2} \]  

(7.23)

or

\[ \rho \frac{\partial^2 y}{\partial t^2} = B \frac{\partial^2 y}{\partial x^2} \]  

(7.24)

Now the propagation velocity of sound is given by

\[ c = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\text{Pressure}}{\text{Mass/Volume}}} \]  

(7.25)

These are longitudinal waves whereas the displacements on the string were perpendicular to the propagation direction.

The change in the pressure at a point is

\[ \delta P = -B \frac{\partial y}{\partial x} \]  

(7.26)

Consider a triangular displacement wave

\[ y(x - c t) \]

\[ \delta P = -B \frac{\partial y}{\partial x} \]

\[ v_t = \frac{5 \text{ cm}}{s} \]

\[ v_t = -\frac{5 \text{ cm}^4}{s^2} \]

Consider a pipe with one closed end. At the closed end \( y[x = 0, t] = 0 \). At the open end,
\[ \frac{\partial y}{\partial x} [x = L, t] = 0. \] The lowest normal mode is

where the black curve represents the displacement, \( y[x, t] \) and the red curve represents the pressure oscillations. At the closed end, there is no displacement but the pressure oscillations are at a maximum. At the open end there are no pressure oscillations (the pressure is kept constant by the atmosphere) but the displacements are at a maximum.

### 7.4 Power and Energy Density

The kinetic energy per unit volume is rather straightforward; the mass per unit volume times the velocity squared.

\[ KE = \frac{1}{2} \rho \left( \frac{\partial y}{\partial t} \right)^2 \]  \hfill (7.27)

The potential energy density can be understood in terms of the work done by the force associated with the change in pressure.

\[ Work = \text{Force} \times \frac{\Delta V}{\text{Area}} = \left( \text{Area} \delta P \right) \left( \Delta x \frac{\partial y}{\partial x} \right) = B \frac{\partial y}{\partial x} \frac{\partial y}{\partial x} \text{Area} \Delta x \]  \hfill (7.28)

so that the work per unit volume becomes

\[ PE = \frac{Work}{Volume} = B \left( \frac{\partial y}{\partial x} \right)^2 \]  \hfill (7.29)

Similarly, the power should be the force times the velocity

\[ Power = \text{Force} \times \frac{\partial y}{\partial t} = -B \frac{\partial y}{\partial x} \text{Area} \frac{\partial y}{\partial t} \]  \hfill (7.30)

and therefore the energy flowing past a surface per unit area per unit time becomes the energy flux

\[ P = -B \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \]  \hfill (7.31)

For a sound wave or pulse of the form \( f[x \mp c t] \) where \( c^2 = B/\rho \), the total energy per unit volume becomes
\[ \mathcal{E} = \frac{1}{2} \rho \left( \frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} B \left( \frac{\partial y}{\partial x} \right)^2 \]
\[ = \frac{1}{2} c^2 \rho f'[x \mp c t]^2 + \frac{1}{2} B f'[x \mp c t]^2 \]
\[ = B f'[x \mp c t]^2 \]

The energy flux becomes
\[ \mathcal{P} = -B \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} = \pm c B f'[x \mp c t]^2 \quad (7.33) \]

The flux of energy is just the energy density times the propagation velocity. This is reasonable. Consider a block of energy moving through a surface.

The amount of energy that moves through the surface in time \( \Delta t \) is \((c \Delta t A) (\text{Energy/Volume})\) and so the amount of energy that passes through the surface per unit time per unit area is

\[ \text{Energy Flux} = \frac{\Delta V}{A \Delta t} \frac{\text{Energy}}{\text{Volume}} = A c \Delta t \frac{\text{Energy}}{\Delta t \text{Volume}} = c \mathcal{E} \quad (7.34) \]
7.5 Hot Chocolate

Remembering that the definition of the bulk modulus is

\[ B = -V \left( \frac{\partial P}{\partial V} \right) \]  

(7.35)

the change in volume for a change in pressure is

\[ \Delta V = -\frac{1}{B} V \Delta P \]  

(7.36)

For a small element of the hot chocolate, the total change in the volume due to a change in the pressure has contributions from the liquid and from the air bubbles in the liquid.

\[ \Delta V = \Delta V_{\text{air}} + \Delta V_{\text{liq}} = - \left( \frac{1}{B_{\text{liq}}} V_{\text{liq}} + \frac{1}{B_{\text{air}}} V_{\text{air}} \right) \Delta P = - \left( \frac{V_{\text{air}}}{B_{\text{air}}} + \frac{V_{\text{liq}}}{B_{\text{liq}}} \right) V \Delta P \]  

(7.37)

Therefore the effective bulk modulus of the system is

\[ \frac{1}{B_{\text{eff}}} = \frac{1}{B_{\text{air}}} V_{\text{air}} + \frac{1}{B_{\text{liq}}} V_{\text{liq}} \]  

(7.38)

The compressibilities of the water and air are very different.

\[ B_{\text{air}} = 10^5 \text{ Pascals} \]
\[ B_{\text{water}} = 2 \times 10^9 \text{ Pascals} \]  

(7.39)

Assuming that only 0.1% of the volume is air bubbles, then \( V \approx V_{\text{liq}} \) and so

\[ B_{\text{eff}} = \frac{1}{V \left( \frac{V_{\text{air}}}{B_{\text{air}}} + \frac{V_{\text{liq}}}{B_{\text{liq}}} \right)} \Delta P = \frac{1}{0.001 B_{\text{air}} + 1 B_{\text{liq}}} = 9.52381 \times 10^7 \text{ Pascals} \]  

(7.38)

The effective speed of sound then is

\[ c_{\text{eff}} = \sqrt{\frac{B_{\text{eff}}}{\rho_{\text{water}}}} = \sqrt{\frac{9.52 \times 10^7 \text{ Pascals}}{1000 \text{ kg/m}^3}} = \frac{308.545 \text{ m}}{\text{sec}} \]  

(7.40)

This is compared with the velocity in pure water,

\[ c_{\text{water}} = \sqrt{\frac{B_{\text{water}}}{\rho_{\text{water}}}} = \sqrt{\frac{2 \times 10^9 \text{ Pascals}}{1000 \text{ kg/m}^3}} = \frac{1414.21 \text{ m}}{\text{sec}} \]  

(7.41)

If we choose closed boundary conditions at the surfaces of the cup (\( L = 2.8 \text{ inches} = 7.1 \text{ cm} \)) then the fundamental normal mode (\( \lambda = 2L \)) has frequencies
\[
\nu_{\text{mix}} = \frac{c_{\text{eff}}}{\lambda} = \frac{308.5 \text{ m}}{0.071 \text{ m}} = 4345.07 \text{ Hz}
\]

\[
\nu_{\text{water}} = \frac{c_{\text{water}}}{\lambda} = \frac{1414.21 \text{ m}}{0.071 \text{ m}} = 19918.5 \text{ Hz}
\]

which is a factor of 4.58415!

7.6 Plane Waves and Spherical Waves

7.6.1 Plane Waves

The pressure in a gas can be written as

\[
P(x, y, z, t) = \delta P \sin(kx - \omega t) + P_0 \quad (7.43)
\]

This wave moves in the x-direction and has surfaces of constant pressure that are planes perpendicular to the x-axis.
A pressure wave traveling in an arbitrary direction is given by

\[ P(x, y, z, t) = \delta P_0 \sin[k(x + y + z) - \omega t] + P_0 \]

where the vector \( \mathbf{k} \) points in the propagation direction.

7.6.2 Spherical Waves

If sound (or any 3D wave) is emitted from a point source isotropically, then one might expect the resulting waves to have the form

\[ P(x, y, z, t) = \delta P \sin[k \cdot \mathbf{r} - \omega t] + P_0 \]  

(7.45)
where \( r = \sqrt{x^2 + y^2 + z^2} \).

Now the surfaces of constant pressure are spheres. But in order to conserve energy the amplitude of the wave must decrease with distance. Consider the flux radiating outwards. For any sphere of radius \( r \), the total flux passing through the sphere must be a constant. Otherwise, the amount of energy that enters and leaves any spherical shell would not be equal. Energy would then either be appearing or disappearing and energy would not be conserved.

In order to conserve energy, the flux must decrease as \( 1/r^2 \). Because the flux depends on the product of \( \frac{\partial y}{\partial t} \) and \( \frac{\partial y}{\partial x} \), we should chose the displacement or the pressure to decrease as \( 1/r \). The pressure can be written as

\[
P[x, y, z, t] = \frac{\delta P \sin(kr - \omega t)}{r} + P_0
\]

(7.46)

corresponding to a displacement of

\[
y[x, y, z, t] = \frac{A \cos(kr - \omega t)}{r}
\]

(7.47)

To see that this is a solution of the wave equation, consider any function of \( kr - \omega t \) of the form

\[
f(kr - \omega t) = f\left(k \sqrt{x^2 + y^2 + z^2} - \omega t\right)
\]

\[
= \frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2 + z^2}}.
\]
\[ c^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) y[x, y, z, t] = c^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{f[k x^2 + y^2 + z^2 - \omega t]}{\sqrt{x^2 + y^2 + z^2}} \] (7.48)

but

\[ \frac{\partial^2}{\partial x^2} \frac{f[k x^2 + y^2 + z^2 - \omega t]}{\sqrt{x^2 + y^2 + z^2}} = \frac{\partial}{\partial x} \left( -\frac{x f[k x^2 + y^2 + z^2 - \omega t]}{(x^2 + y^2 + z^2)^{3/2}} + \frac{k x f[k x^2 + y^2 + z^2 - \omega t]}{(x^2 + y^2 + z^2)^{3/2}} \right) \] (7.49)

\[ = \frac{2 x^2}{(x^2 + y^2 + z^2)^{3/2}} - \frac{y^2}{(x^2 + y^2 + z^2)^{3/2}} - \frac{z^2}{(x^2 + y^2 + z^2)^{3/2}} \frac{f[k x^2 + y^2 + z^2 - \omega t]}{\sqrt{x^2 + y^2 + z^2}} + \frac{2 k x^2}{(x^2 + y^2 + z^2)^2} + \frac{k y^2}{(x^2 + y^2 + z^2)^2} + \frac{k z^2}{(x^2 + y^2 + z^2)^2} \] (7.50)

\[ f''[k x^2 + y^2 + z^2 - \omega t] \]

and so

\[ c^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{f[k x^2 + y^2 + z^2 - \omega t]}{\sqrt{x^2 + y^2 + z^2}} = \] (7.51)

\[ \left( c^2 k f''[k x^2 + y^2 + z^2 - \omega t] \right) / \left( \sqrt{x^2 + y^2 + z^2} \right) \]

and

\[ \frac{\partial^2}{\partial t^2} \frac{f[k x^2 + y^2 + z^2 - \omega t]}{\sqrt{x^2 + y^2 + z^2}} = \left( \omega^2 f''[k x^2 + y^2 + z^2 - \omega t] \right) / \left( \sqrt{x^2 + y^2 + z^2} \right) \] (7.52)

Therefore \( \frac{f(k r - \omega t)}{r} \) is a solution of the wave equation if \( c k = \omega \). One can make a simplification if one expresses the second derivatives in spherical coordinates.
\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2} \frac{\partial (r^2 \frac{\partial}{\partial r})}{\partial r} + \frac{1}{r^2 \sin[\theta]} \frac{\partial (\sin[\theta] \frac{\partial}{\partial \theta})}{\partial \theta} + \frac{1}{r^2 \sin[\theta]} \frac{\partial^2}{\partial \phi^2}
\]

(7.53)

Because \( \frac{f(kr - \omega t)}{r} \) only depends on \( r \) and not \( \theta \) and \( \phi \), we only need to evaluate the derivatives with respect to \( r \)

\[
\frac{1}{r^2} \frac{\partial \left( r^2 \frac{\partial \left( \frac{f(kr-t\omega)}{r} \right)}{\partial r} \right)}{\partial r} = \frac{1}{r^2} \frac{\partial \left( r^2 \left( -\frac{f(kr-t\omega)}{r^2} + \frac{rf'(kr-t\omega)}{r} \right) \right)}{\partial r}
\]

\[
= \frac{1}{r^2} \frac{\partial}{\partial r} \left( -\frac{f(kr-t\omega)}{r} + rf'(kr-t\omega) \right)
\]

\[
= \frac{k^2 f''(kr-t\omega)}{r}
\]

\[
= \frac{\partial^2 \left( \frac{f(kr-t\omega)}{r} \right)}{\partial t^2} = \frac{\omega^2 f''(kr-t\omega)}{r}
\]

(7.54)

### 7.7 Doppler Shift

Given a stationary source and observer, the propagation of sound between them can be indicated by plotting the lines or surfaces of constant phase at some instant in times. In the following one can treat each phase line as indicating the crest of the wave.
If the source is moving towards the observer, then the wave crests moving towards the observer will be separated by a distance smaller than $\lambda$. 
The distance between the crest in the direction of the observer becomes
The shorter wavelength becomes

$$\lambda' = \lambda - \tau v_s = \tau (c - v_s) = \frac{c - v_s}{v_s}$$  \hspace{1cm} (7.55)\]

Solving for $v_{\text{obs}} = c / \lambda'$

$$v_{\text{obs}} = \frac{v_s}{1 - \frac{v_s}{c}}$$  \hspace{1cm} (7.56)\]

For the source approaching the observer, $v_s > 0$ and $v_{\text{obs}} > v_s$

For the source receding from the observer, $v_s < 0$ and $v_{\text{obs}} < v_s$

For the observer moving towards the source, the observer will see the crests at shorter intervals. If $t = 0$, the observer is at a crest then he will see the next crest at
Therefore

\[ \frac{c}{v_s} = \lambda = c \tau' + v_{\text{obs}} \tau' = \frac{c + v_{\text{obs}}}{v_{\text{obs}}} \]  \hspace{1cm} (7.57)

and therefore

\[ v_{\text{obs}} = \left(1 + \frac{v_{\text{obs}}}{c}\right)v_s \] \hspace{1cm} (7.58)

For the observer approaching the source, \( v_{\text{obs}} > 0 \) and \( v_{\text{obs}} > v_s \)

For the observer moving away from the source, \( v_{\text{obs}} < 0 \) and \( v_{\text{obs}} < v_s \)

If both the source and observer are moving, the observed frequency is a combination of both effects

\[ v_{\text{obs}} = \frac{(1 + \frac{v_{\text{obs}}}{c})v_s}{1 - \frac{v_s}{c}} \] \hspace{1cm} (7.59)

If both are moving in the same direction (for example if the source is approaching the observer and the observer is moving away from the source) then

\[ v_s = \text{Abs}[v_s] \hspace{0.5cm} \text{and} \hspace{0.5cm} v_{\text{obs}} = -\text{Abs}[v_{\text{obs}}] = -\text{Abs}[v_s] \] \hspace{1cm} (7.60)
If the source and observer were standing on a moving flatcar moving at a velocity $v$, there would be no frequency shift. The two people would not expect the pitch of their voices to shift due to the common motion. This situation is also equivalent by a Galilean transformation to a stationary source and observer with a wind with velocity $-v$.

To treat a general problem with a wind present, one can transform to a frame in which there is no wind and calculate the frequency shift.

Assume that the source is stationary and that the observer is moving towards the source in the presence of a wind with a velocity pointing towards the observer.

Going to the frame in which the wind is stationary, both the source and observer are moving.

The frequency heard by the observer is therefore

$$v_{\text{obs}} = \left(1 - \frac{\frac{v_{\text{obs}}}{c} + \frac{v_{\text{wind}}}{c}}{1 - \frac{v_{\text{wind}}}{c}}\right) \frac{v_s}{c} = \frac{\frac{v_s}{c}}{c + v_{\text{wind}}} = \left(1 - \frac{\frac{v_{\text{obs}}}{c} + \frac{v_{\text{wind}}}{c}}{1 - \frac{v_{\text{wind}}}{c}}\right) \frac{v_s}{c}$$

We see that in the original frame in which only the observer is moving, their frequency shift can be calculated by noting that they effective speed of sound is actually $c + v_{\text{wind}}$. The speed of sound $c$ is the propagation velocity relative to the medium in which the sound is traveling. If the air is moving then its velocity is added to that of the sound. In this example, if the wind were blowing in the opposite direction and has a velocity equal to the speed of sound, the observer would eventually hear an infinite frequency because all the crests of the emitted sound waves pile up in one location.

If the source is moving and the observer is fixed then the resulting frequency would be
\[ v_{\text{obs}} = \frac{1 + \frac{v_{\text{obs}}}{c}}{1 - \frac{v_s}{c}} \quad v_s = \frac{1 + \frac{v_{\text{wind}}}{c}}{1 + \frac{v_s + v_{\text{wind}}}{c}} \quad v_s = \frac{c + v_{\text{wind}}}{1 + \frac{v_{\text{wind}}}{c + v_s}} \quad (7.63) \]

Again if the wind were blowing in the opposite direction and has a velocity equal to the speed of sound, the observer would hear zero frequency because all the crests of the emitted sound waves again pile up in one location and never reach the observer.

Returning to the situation in which the source is moving. The expression for the observed frequency has a singularity when the source is moving at the speed of sound.

If the source is moving at the speed of sound all the wavefronts overlap. In this region, there will be overlapping crests and valleys resulting in large pressure fluctuations. Therefore if a plane were to go faster than the speed of sound, it would have to pass through this region of pressure fluctuations and would experience large stresses on its structure. This explains why some planes were torn apart when they tried to go past the “sound barrier”.

If the source is moving faster than the speed of sound, then the waves reinforce each other along a cone following the source (red lines). These lines can be seen behind boats when they move faster than the speed of the water waves. If the source is a plane then as the edge of the cone passes an observer on the ground, he will hear the sonic boom, the superposition of all the sound. The sonic boom does not happen only once but can be heard by observers on the ground as the cone edge passes them.

This phenomenon also occurs when charged particles move through a medium at a speed faster than the speed of light in that medium (Cherenkov radiation). This type of radiation can been
seen in the water surrounding a nuclear reactor. The astronauts occasionally see flashes of light when cosmic rays traverse the vitrious humor in their eyeballs. This is also how the neutrinos from supernovae were detected. The detectors consisted of large volumes of water surrounded by sensitive photodetectors. If a neutrino collides with a nucleus, it can eject a charged particle that produces Cherenkov radiation.

### 7.8 Dispersion and Group Velocity

The most general form for a standing wave consists of two counter propagating waves

\[
\sin(x - t) + \sin(1.15x - 1.15t) = 2 \cos\left(\frac{1}{2}(0.15x - 0.15t)\right) \sin\left(\frac{1}{2}(2.15x - 2.15t)\right)
\]

\[
c_{envelope} = 1 \quad c_{phase} = 1
\]

If however, the velocity of the waves depends on the frequency or wavelength then the short wavelength oscillations will travel at a different velocity than the envelope. If the velocity
decreases with increasing frequency, then the envelope will travel at a slower velocity.

If however, the velocity increases with increasing frequency, then the envelope will travel at a faster velocity.
To see this consider the superposition of two waves with different velocities

\[ A \sin((x - t \cdot c_1) k_1) + A \sin((x - t \cdot c_2) k_2) = 2A \cos \left[ \frac{1}{2} \left( x (k_1 - k_2) - t (\omega_1 - \omega_2) \right) \right] \sin \left[ \frac{1}{2} \left( x (k_1 + k_2) - t (\omega_1 + \omega_2) \right) \right] \] (7.64)

The envelope travels at a velocity \( c_{envelope} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \) whereas the internal oscillations is given by \( c_{phase} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \).