1.1 Wave Equation

Consider the transverse motion of a section of a string

\[ T \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial x^2} \sin \phi - \frac{\partial^2}{\partial x^2} \cos \phi \]

\[ T \frac{\partial^2}{\partial x^2} \cos \phi + \frac{\partial^2}{\partial x^2} \sin \phi \]

Figure 1.1  Forces on a section of a string between \( x \) and \( x + \Delta x \)

The total vertical and horizontal forces on this section of the string are

\[ F_V = T[x + x \Delta] \sin[\phi[x + x \Delta]] - T[x] \sin[\phi[x]] \]
\[ F_H = T[x + x \Delta] \cos[\phi[x + x \Delta]] - T[x] \cos[\phi[x]] \] \hspace{1cm} (1.1)

For transverse modes, there is no horizontal motion. Consequently the force in the horizontal direction must be equal to zero.

\[ F_H = T[x + x \Delta] \cos[\phi[x + x \Delta]] - T[x] \cos[\phi[x]] = 0 \] \hspace{1cm} (1.2)

or

\[ T[x + x \Delta] \cos[\phi[x + x \Delta]] = T[x] \cos[\phi[x]] = T_0 \] \hspace{1cm} (1.3)

Substituting for \( T[x + \Delta x] \) and \( T[x] \) in the expression for the vertical forces

\[ F_V = T_0 \tan[\phi[x + x \Delta]] - T_0 \tan[\phi[x]] \] \hspace{1cm} (1.4)

where the Tangent of the angle is simply the slope of the line.
Substituting

\[ F_V = T_0 \left( \frac{\partial y}{\partial x} \right)_{x+x\Delta} - T_0 \left( \frac{\partial y}{\partial x} \right)_x \]

\[ = T_0 \frac{\partial^2 y}{\partial x^2} \Delta x \quad (1.5) \]

The net force should be equal to the mass of the section of the string, \( \mu \Delta x \), where \( \mu = \frac{M}{L} \) is the mass of the string per unit length times the acceleration.

\[ T_0 \frac{\partial^2 y}{\partial x^2} \Delta x = \frac{\partial^2 y}{\partial t^2} \mu \Delta x \quad (1.6) \]

or

\[ T_0 \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \quad (1.7) \]

Consequently, the transverse motion of the string satisfies the wave equation. Note that the quantity \( T_0 / \mu \) has the dimensions of \( \text{Force (mass/length)} = \text{Energy (mass)} = (\text{velocity})^2 \).

### 1.2 Wave Equation Solutions

To solve the wave equation, try a solution of the form

\[ y[x, t] = f[x - c t] \quad (1.8) \]

where the function \( f[z] \) is any reasonable function of \( z \). Taking the derivatives with respect to \( t \)
\[
\frac{\partial y}{\partial t} = \frac{\partial f[x - ct]}{\partial t} = -c \left( \frac{\partial f[z]}{\partial z} \right)_{z=x-ct} = -c f'[x - ct] \quad (1.9)
\]

where \( f'(x - ct) \) is the functional derivative of \( f \) evaluated at \( x - ct \). The second derivative with respect to \( t \) becomes

\[
\frac{\partial^2 y}{\partial t^2} = (-c)^2 f''[x - ct] = c^2 f''[x - ct] \quad (1.10)
\]

Similarly the derivatives with respect to \( x \) become

\[
\frac{\partial y}{\partial x} = c f'[x - ct]
\]

\[
\frac{\partial^2 y}{\partial x^2} = f''[x - ct] \quad (1.11)
\]

Substituting these expressions into the wave equation

\[
\frac{\partial^2 y}{\partial t^2} = \frac{T_0}{\mu} \frac{\partial^2 y}{\partial x^2} \quad \rightarrow \quad c^2 f''[x - ct] = \frac{T_0}{\mu} f''[x - ct] \quad (1.12)
\]

Therefore any function of the form \( f[x - ct] \) is a solution of the wave equation if

\[
c = \pm \sqrt{\frac{T_0}{\mu}} \quad (1.13)
\]

Similarly \( f[x + ct] \) is also a solution. Moreover because the differential equation is linear, the sum of any two solutions is also solution.

\[
y = Af[x - ct] + Bf[x + ct] \quad (1.14)
\]

How does \( f[x - ct] \) behave a function of time? Consider \( f[z] \) defined as

\[
f(z) = \begin{cases} 
0 & z < 0 \\
0.5 cm (1 - z) & 0 \leq z \leq 1 \\
0 & z > 1 
\end{cases}
\]

At \( t = 0 \), \( y[x, t = 0] = f[x] \)
If $c = 3 \text{ m/s}$, then at $t = 1/3 \text{ sec}$, $y(x, t = 1/3 \text{ s}) = f(x - 1)$, the function looks like

In $1/3 \text{ sec}$, the pattern has moved unchanged a distance of 1 meter. Therefore we identify $c = 3 \text{ m/s}$ as the velocity of propagation. Similarly, we can interpret $f(x + ct)$ as a pattern that propagates to the left with velocity $c$.

How does a particular point on the string behave as a function of time?

Plotting the pulse as it moves along the string and the value of the displacement at a fixed position as a function of time
It takes $2/3$ sec for the “front” of the pulse to reach the position $x = 3 \text{ m}$ and takes an additional $1/3$ sec to pass this point. We can distinguish two distinct velocities,

1. Velocity of propagation down the string
2. Transverse velocity of a particular point on the string

In the above example, the velocity of the point on the string at $x = 3 \text{ m}$ for $2/3 \text{s} < t < 1 \text{s}$ is $v_t = \frac{0.5 \text{ cm}}{1/3 \text{ sec}} = 1.5 \text{ cm/sec}$. To calculate this velocity formally, remember that

$$\left. \left( \frac{\partial y}{\partial t} \right) \right|_{x=x_0} = \left. \left( \frac{\partial f[x \pm ct]}{\partial t} \right) \right|_{x=x_0} = \pm c f'[x_0 \pm ct]$$

$$= \pm c \left. \left( \frac{\partial f[x \pm ct]}{\partial x} \right) \right|_{x=x_0} = \pm c \left. \left( \frac{\partial y}{\partial x} \right) \right|_{x=x_0}$$

(1.15)

Plotting the displacement and the velocity as functions of the position along the string for a triangular waveform.
1.3 Power and Energy Density

To derive the conservation of energy for a harmonic oscillator, we multiplied the force equation

\[ m \frac{d^2 x}{dt^2} = m \frac{dv}{dt} = -kx \] (1.16)

becomes

\[ m v \frac{dv}{dt} = -k v x = -k x \frac{dx}{dt} \] (1.17)

by the velocity, to obtain

\[ m v \frac{dv}{dt} = -k v x = -k x \frac{dx}{dt} \] (1.18)

Collecting terms, we obtained that the time rate of change of the kinetic and potential energies is zero

\[ \frac{d}{dt} \left( \frac{m v^2}{2} + \frac{kx^2}{2} \right) = 0 \] (1.19)

One can derive a similar conservation of energy for the string. Multiplying the wave equation by
the transverse velocity

\[
T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial^2 y}{\partial x^2} \right) = \mu \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial^2 y}{\partial t^2} \right)
\] (1.20)

or

\[
T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial^2 y}{\partial x^2} \right) = \frac{\partial}{\partial t} \left( \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 \right)
\] (1.21)

The right side of the equation certainly looks like a kinetic energy. To identify the potential energy on the left side of the equation, we should integrate over a section of the string from \(x_1\) to \(x_2\)

\[
\int_{x_1}^{x_2} T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial^2 y}{\partial x^2} \right) \, dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t} \left( \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 \right) \, dx
\] (1.22)

Integrating the integral on the left by parts

\[
\int_{x_1}^{x_2} T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial^2 y}{\partial x^2} \right) \, dx = \left[ T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial y}{\partial x} \right) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} T_0 \left( \frac{\partial^2 y}{\partial t \partial x} \right) \left( \frac{\partial y}{\partial x} \right) \, dx
\]

\[
= \left[ T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial y}{\partial x} \right) \right]_{x_1}^{x_2} + \frac{\partial}{\partial t} \left( \int_{x_1}^{x_2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 \, dx \right)
\] (1.23)

Bringing the integrals to the same side

\[
\frac{\partial}{\partial t} \left( \int_{x_1}^{x_2} \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 \right) \, dx = \left[ T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial y}{\partial x} \right) \right]_{x_1}^{x_2}
\] (1.24)

The first term of the integral is clearly the total kinetic energy in the string from \(x_1\) and \(x_2\). The second term has the interpretation as a potential energy. As a section of the string moves vertically, it changes its length and therefore the tension does work on the section.
The change in length is given by

\[ \Delta L = -dx + \sqrt{dx^2 + dy^2} = \left( -1 + \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \right) dx \]  

(1.25)

The work done is therefore

\[ \text{Work} = \text{Distance Force} = \Delta L \cdot T[x] = \frac{T_0 \Delta L}{\cos[\phi(x)]} \]

\[ = \Delta x T_0 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \left( -1 + \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \right) \]

(1.26)

Expanding for small amplitudes and therefore small slopes, \( \left( \frac{dy}{dx} \right)^2 \ll 1 \)

\[ \text{Work} = \frac{1}{2} \Delta x T_0 \left( \frac{dy}{dx} \right)^2 + O \left( \frac{dy}{dx} \right)^4 \]

(1.27)

Therefore to lowest order the amount of potential energy stored in the stretched string due to the work done on that section is \( \frac{1}{2} T_0 \left( \frac{dy}{dx} \right)^2 \Delta x \). Integrating \( \frac{1}{2} T_0 \left( \frac{dy}{dx} \right)^2 \) from \( x_1 \) to \( x_2 \) gives the total potential energy in that section of the string. The left hand side of Eq. 6.23 is the total energy contained within the section of string from \( x_1 \) to \( x_2 \). We see that the total energy of the section is not constant but can increase or decrease depending on the value of \( T_0 \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial y}{\partial x} \right) \) evaluated at...
the end points of the section. Consider a pulse traveling down the string.

To understand this term, consider the forces at the end of a section of the string.

The vertical force acting on the right hand side of the center section is

$$F_V = T[x] Sin[\phi[x]] = T_0 Tan[\phi[x]] = T_0 \frac{\partial y}{\partial x}$$  (1.28)
The force on the next section of the string must be equal and opposite

\[ F_+ = -T_0 \frac{\partial y}{\partial x} \]  

(1.29)

The rate at which work is done by the center section on the next section is therefore

\[ Power = \text{Force} \times \text{Velocity} = F_+ \times \frac{\partial y}{\partial t} = -T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \]  

(1.30)

Comparing this with our previous expression, we see that

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 \right] dx \bigg|_{x_1}^{x_2} = - \left( T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right)_{x_1} - \left( T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right)_{x_2} \]  

(1.31)

The energy of the section will increase if the power at \( x_1 \) is positive (the previous section is doing work on this section) and the energy of the section will decrease if the power at \( x_2 \) is positive (this section is doing positive work on the next section).

For a general waveform \( f[x \pm ct] \), the kinetic energy density becomes

\[ \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 = \frac{1}{2} \mu \left( \frac{\partial f[x \pm ct]}{\partial t} \right)^2 = \frac{1}{2} c^2 \mu f'[x \pm ct]^2 = \frac{1}{2} T_0 f'[x \pm ct]^2 \]  

(1.32)

where we have used the fact that \( c^2 = T_0 / \mu \). The potential energy density becomes

\[ \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 = \frac{1}{2} T_0 \left( \frac{\partial f[x \pm ct]}{\partial x} \right)^2 = \frac{1}{2} T_0 f'[x \pm ct]^2 \]  

(1.33)

The total energy density becomes

\[ E = \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 = T_0 f'[x \pm ct]^2 \]  

(1.34)

Similarly the power at a given point is

\[ P = -T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} = -T_0 \frac{\partial f[x \pm ct]}{\partial t} \frac{\partial f[x \pm ct]}{\partial x} = \mp c T_0 f'[x \pm ct]^2 \]  

(1.35)

This expression is very general and will be seen in different wave phenomena. If one has a pulse moving down the string with velocity \( c \) then the energy associated with the pulse will also move down the string with velocity \( c \).
\[ y(x, t) = f'(x \mp c t) \]

\[ \frac{\partial y}{\partial x} = f'(x \mp c t) \]

\[ \frac{\partial y}{\partial t} = \mp c f'(x \mp c t) \]

\[ \mathcal{E} = \left( \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 \right) = T_0 f'(x \mp c t)^2 \]

\[ \mathcal{P} = -T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} = \pm c T_0 f'(x \pm c t)^2 \]

Using a smooth function
1.4 Sinusoidal Wave Solutions

One particular solution of the wave equation has the form

\[ y(x, t) = A \sin[\beta(x - c t)] \]  \hspace{1cm} (1.36)

Plotting this solution as a function of \( x \) at a fixed time \( t = 0 \), \( y(x, t = 0) = A \sin[\beta x] \)
we see that the motion is periodic as a function of $x$. Solving for this periodic length

$$y[-ct + x_2] = y[-ct + x_1] \rightarrow A \sin[\beta(-ct + x_2)] = A \sin[\beta(-ct + x_1)]$$

or because $\sin[a] - \sin[b] = 2 \sin \left[ \frac{a}{2} - \frac{b}{2} \right] \cos \left[ \frac{a}{2} + \frac{b}{2} \right]$

$$A \left( \sin[\beta(-ct + x_2)] - \sin[\beta(-ct + x_1)] \right) =$$

$$2A \cos \left[ \frac{1}{2} \beta(-2ct + x_1 + x_2) \right] \sin \left[ \frac{1}{2} \beta(-x_1 + x_2) \right] = 0$$

$$\frac{1}{2} \beta(-x_1 + x_2) = n\pi \rightarrow -x_1 + x_2 = \frac{2n\pi}{\beta}$$

The smallest periodic length defines the wavelength, $\lambda = \frac{2\pi}{\beta}$. Substituting back into our equation then

$$y[x, t] = A \sin \left[ \frac{2\pi}{\lambda} (-ct + x) \right]$$

If we examine the displacement at a fixed position $x = 0$, the displacement is also periodic in time. Plotting $y(x = 0, t) = A \sin \left[ \frac{2\pi}{\lambda} (-c t) \right] = -A \sin \left[ \frac{2\pi}{\lambda} ct \right]$
we see that the motion is periodic as a function of $t$. Solving for this periodic time interval

$$y(x - ct_2) = y(x - ct_1) \rightarrow \sin\left(\frac{2\pi(x - ct_2)}{\lambda}\right) = \sin\left(\frac{2\pi(x - ct_1)}{\lambda}\right)$$

or

$$A\left(-\sin\left(\frac{2\pi(x - ct_1)}{\lambda}\right) + \sin\left(\frac{2\pi(x - ct_2)}{\lambda}\right)\right) =$$

$$-2A\cos\left(\frac{\pi(2x - ct_1 - ct_2)}{\lambda}\right)\sin\left(\frac{\pi(-t_1 + t_2)}{\lambda}\right) = 0$$

$$\frac{c\pi(-t_1 + t_2)}{\lambda} = n\pi \rightarrow -t_1 + t_2 = \frac{n\lambda}{c}$$

The smallest periodic time interval defines the period, $\tau = \frac{\lambda}{c}$ or the natural frequency $\nu = \frac{1}{\tau} = \frac{c}{\lambda}$
or the radial frequency $\omega = \frac{2\pi}{\tau} = \frac{2\pi c}{\lambda} = c\ k$ where $k$ is the wavenumber or wave vector. Substituting back into our equation, the general form of the displacement becomes
\[ y[x, t] = A \sin\left[\frac{2\pi x}{\lambda} - 2\pi \nu t\right] = A \sin[k x - \omega t] \tag{1.44} \]

or more generally

\[ y[x, t] = A \sin[k x \mp \omega t + \phi] \tag{1.45} \]

### 1.5 Energy Density and Power

The general form of the energy density is

\[ E = \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 \tag{1.46} \]

However for traveling waves of the form \( f[x \mp ct] \) we have that

\[ \frac{\partial y}{\partial t} = \mp c \frac{\partial y}{\partial x} \tag{1.47} \]

and the energy density reduces to

\[ E = \frac{T_0}{2} + \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 \frac{1}{c^2} \left( \frac{\partial y}{\partial t} \right)^2 = \mu \left( \frac{\partial y}{\partial t} \right)^2 \tag{1.48} \]

where we have used the fact that \( c^2 = T_0 / \mu \). Similarly the power can be expressed as

\[ P = -T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} = \pm T_0 \left( \frac{\partial y}{\partial t} \right)^2 = \pm c \mu \left( \frac{\partial y}{\partial t} \right)^2 \tag{1.49} \]

For waves of the form \( y(x, t) = A \sin(k x \mp \omega t + \phi) \), the energy density and power become

\[ E = \mu \left( \frac{\partial y}{\partial t} \right)^2 = A^2 \mu \omega^2 \cos[k x \mp \omega t + \phi]^2 \]

\[ P = \pm c \mu \left( \frac{\partial y}{\partial t} \right)^2 = \pm c A^2 \mu \omega^2 \cos[k x \mp \omega t + \phi]^2 \tag{1.50} \]

Plotting these as a function of \( x \)
The time averages of the energy density and power for these sinusoidal waves are

\[ \overline{E} = \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} \mu \omega^2 A^2 \cos(k x \mp \omega t + \phi)^2 \, dt \]

\[ = \frac{1}{2} \mu \omega^2 A^2 \]  

(1.51)

and

\[ \overline{P} = \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} \pm c \mu \omega^2 A^2 \cos(k x \mp \omega t + \phi)^2 \, dt \]

\[ = \pm \frac{1}{2} c \mu \omega^2 A^2 \]  

(1.52)

Both are independent of \( x \).

1.6 Superposition

Because the wave equation is linear in \( y \), the sum of solutions of the wave equation is also a solution of the wave equation.
If $g(x, t)$ and $f(x, t)$ are solutions of a wave equation then $g(x, t) + f(x, t)$ is also a solution.

Consider the superposition of two sinusoidal waves of equal amplitude traveling in the same direction but with different frequencies and wavelengths.

$$y(x, t) = A \sin(xk_1 - t\omega_1) + A \sin(xk_2 - t\omega_2) = A \sin(-ct + xk_1) + A \sin(-ct + xk_2)$$

(1.53)

If $k_1$ and $k_2$ are very different, then the resulting waveform will appear as a slow modulation of the center line of the rapid oscillations.

More interesting patterns can be obtained by having $k_1 \approx k_2$. Using the addition formula

$$\sin[A] + \sin[B] = 2 \cos\left[\frac{A}{2} - \frac{B}{2}\right] \sin\left[\frac{A}{2} + \frac{B}{2}\right]$$

(1.54)

we have that

$$y(x, t) = A \sin(-ct + x) k_1 + A \sin(-ct + x) k_2 = 2A \cos\left[\frac{1}{2} (k_1 - k_2)(x - ct)\right] \sin\left[\frac{1}{2} (k_1 + k_2)(x - ct)\right]$$

(1.55)

Because $k_1 \approx k_2$ the Cosine factor will vary slowly and will appear as a slow modulation of the amplitude of the faster oscillations.
As a function of time, this pattern will still move down the string with velocity $c$ because $y(x, t)$ is still a function of $x - ct$. The nodes in the envelope of the rapid oscillations are due to the zeros of the Cosine factor.

The distance between the nodes is given by

$$\frac{1}{2} (k_1 - k_2) (-x_1 + x_2) = \pi$$  \hspace{1cm} (1.56)
or

\[ -x_1 + x_2 = \frac{2\pi}{k_1 - k_2} = \frac{2\pi}{\lambda_1 - \lambda_2} = \frac{\lambda_1 \lambda_2}{-\lambda_1 + \lambda_2} \]  \hspace{1cm} (1.57)

What happens at a fixed position? As the wave passes, one observes a slow periodic modulation of the amplitude. Taking \( x = 0 \)

\[ y(x, t) = \left( \frac{A}{2} \sin(k_1(x - c t)) \right) - \left( \frac{A}{2} \sin(k_2(x - c t)) \right) \bigg|_{x=0} 
\]

\[ = -A \cos\left(\frac{1}{2}(k_1 - k_2)ct\right) \sin\left(\frac{1}{2}(k_1 + k_2)ct\right) \]  \hspace{1cm} (1.58)

Writing this expression in terms of the frequency, \( \omega = c k \)

\[ y(x, t) = -2A \cos\left(\frac{1}{2}t(\omega_1 - \omega_2)\right) \sin\left(\frac{1}{2}t(\omega_1 + \omega_2)\right) \]  \hspace{1cm} (1.59)

So at a fixed position, the slow amplitude oscillations are determined by the envelope function \( \cos\left(\frac{1}{2}t(\omega_1 - \omega_2)\right) \). The time between the “nodes” in the amplitude is therefore given by

\[ \frac{1}{2}(-t_1 + t_2)(\omega_1 - \omega_2) = \pi \]  \hspace{1cm} (1.60)

or

\[ -t_1 + t_2 = \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{v_1 - v_2} \]  \hspace{1cm} (1.61)

or

\[ f_{\text{Beats}} = \frac{1}{-t_1 + t_2} = v_1 - v_2 \]  \hspace{1cm} (1.62)

The beat frequency is given by the difference in the two frequencies.

If the two waves have the same frequency and wavelength but propagate in opposite directions, then the resulting wave becomes a standing wave.

\[ y(x, t) = A \sin(kx - t\omega) + A \sin(kx + t\omega) \]

\[ = 2A \cos(t\omega) \sin(kx) \]  \hspace{1cm} (1.63)
The resulting pattern does not move along the string. There are points called nodes at which the string never moves. Midway between the nodes, the string has its largest motion (antinodes).

The kinetic and potential energy densities and the power also do not move along the string but oscillate in place.

\[ y[x, t] = 2A \cos[t \omega] \sin[kx] \]

\[ KE = \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 = 2A^2 \mu \omega^2 \sin[kx]^2 \sin[t \omega]^2 \]
Note that the nodes in the power occur at the nodes and antinodes of the standing wave. The energy therefore can not cross these positions of the string. The energy is always trapped between the nodes and antinodes. As a function of time the energy within each section oscillates between kinetic and potential.

This suggests that if one were to “cut” the string either at the nodes or antinodes, there would be no changes in the oscillations.

For example, if one were to clamp the string at two nodes, the string in between the nodes would not be affected. At these points, the power is equal to zero because \( \frac{\partial y}{\partial t} = 0 \) for all time.

\[
\mathcal{P} = -T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} = 0 \text{ when } \frac{\partial y}{\partial t} = 0
\]

This is called a fixed or closed boundary condition. Similarly, cutting the string at an antinode would also have no effect as long as one can maintain the tension in the string. This can be done
by attaching a massless ring to the end of the string that can ride up and down a vertical post without friction. Because the ring is massless and the ring moves without friction, the vertical force at the end of the string is zero.

If the vertical force is zero, then the slope must also be zero, \( \left( \frac{\partial y}{\partial x} \right) = 0 \). As before, this implies that the power is equal to zero.

\[
\mathcal{P} = -T_0 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} = 0 \quad \text{when} \quad \frac{\partial y}{\partial x} = 0 
\]

(1.66)

1.7 Normal Modes

The most general form for a standing wave consists of two counter propagating waves

\[
y[x, t] = A \sin[kx - t\omega] + A \sin[kx + \phi + t\omega] 
\]

(1.67)

A relative phase is added to one of the waves to shift the position of the nodes or antinodes.

Reducing this expression

\[
y[x, t] = A \sin[kx - \omega t] + A \sin[kx + \omega t + \phi] \\
= 2A \cos\left[\frac{\phi}{2}\right] \sin\left[kx + \frac{\phi}{2}\right] 
\]

(1.68)

1.7.1 Fixed Boundary Conditions

Assuming fixed boundary conditions at \( x = 0 \) and \( x = L \) we require that \( y[x = 0, t] = 0 \) and \( y[x = L, t] = 0 \). In order for these boundary conditions to be satisfied at all times, we require
\[ y[0, t] = 2A \cos\left(\frac{\phi}{2} + t \omega\right) \sin\left(\frac{\phi}{2}\right) = 0 \quad \text{or} \quad \phi = 0 \quad (1.69) \]

Similarly at \( x = L \), we must require

\[ y[L, t] = 2A \cos[t \omega] \sin[kL] = 0 \quad \text{or} \quad k = \frac{n\pi}{L} \quad (1.70) \]

The form of the normal mode is therefore

\[ y[x, t] = 2A \cos[t \omega] \sin\left(\frac{n\pi x}{L}\right) \quad (1.71) \]

where the wavelength of the normal mode, \( \lambda = \frac{2\pi}{k} = \frac{2L}{n} \). Plotting the normal modes

1.7.2 Open Boundary Conditions

Assuming open boundary conditions at \( x = 0 \) and \( x = L \), we require that \( \frac{\partial y}{\partial x}[x = 0, t] = 0 \) and \( \frac{\partial y}{\partial x}[x = L, t] = 0 \). In order for these boundary conditions to be satisfied at all times, we require
\[
\left( \frac{\partial y}{\partial x} \right)_{x=0} = \left( 2Ak \cos\left[ kx + \frac{\phi}{2} \right] \cos\left[ \frac{\phi}{2} + t\omega \right] \right)_{x=0} = 2Ak \cos\left[ \frac{\phi}{2} \right] \cos\left[ \frac{\phi}{2} + t\omega \right] = 0 \quad \text{or} \quad \phi = \pi
\]

Similarly at \( x = L \), we must require
\[
\left( \frac{\partial y}{\partial x} \right)_{x=L} = (2Ak \sin[kx] \sin[t\omega])_{x=L} = 2Ak \sin[kL] \sin[t\omega] = 0 \quad \text{or} \quad kL = n\pi
\]

The form of the normal mode is therefore
\[
y[x, t] = -2A \cos\left[ \frac{n\pi x}{L} \right] \sin[t\omega]
\]

where the wavelength of the normal mode, \( \lambda = \frac{2\pi}{k} = \frac{2L}{n} \). Plotting the normal modes

1.7.3 Mixed Boundary Conditions

Assuming fixed boundary condition at \( x = 0 \) and open boundary condition at \( x = L \), we require that \( x[x = 0, t] = 0 \) and \( \frac{\partial y}{\partial x}[x = L, t] = 0 \). In order for these boundary conditions to be satisfied at
all times, we require
\[ y[x = 0, t] = 2A \cos\left(\frac{\phi}{2} + t \omega\right) \sin\left[k x + \frac{\phi}{2}\right] = 0 \quad \text{or} \quad \phi = 0 \] (1.75)

At \( x = L \), we require
\[ \left. \frac{\partial y}{\partial x} \right|_{x=L} = (2A k \cos[k x] \cos[t \omega]) \big|_{x=L} = \]
\[ 2A k \cos[k L] \cos[t \omega] = 0 \quad \text{or} \quad k L = \frac{1}{2} (-1 + 2n) \pi \] (1.76)

The form of the normal mode is therefore
\[ y[x, t] = 2A \cos[t \omega] \sin\left[\frac{(-1 + 2n) \pi x}{2L}\right] \] (1.77)

where the wavelength of the normal mode, \( \lambda = \frac{2\pi}{k} = \frac{4L}{2n-1} \). Plotting the normal modes

### 1.8 Reflections

The reflection of a pulse from an open or fixed boundary can be understood by imagining that the string continues past the boundary point and asking what pulse must one counter propagate such that the sum of the two pulses will satisfy the boundary condition.
For a fixed boundary, we know that \( y[x = x_0, t] = 0 \). Consequently, the counter propagating pulse must be the negative of the first pulse so that the sum is odd about the boundary point.

For an open boundary condition, \( \left( \frac{\partial y}{\partial x} \right)[x = x_0, t] = 0 \). Consequently the counter propagating pulse must be a positive image of the first pulse so that the sum is even about the boundary point. Because the sum is even, the waveform can not have any odd terms in its Taylor expansion about \( x_0 \) and \( \left( \frac{\partial y}{\partial x} \right)[x = x_0, t] = 0 \).
For a reflection from a fixed boundary condition there is a $\pi$ phase shift of the pulse. For reflection from an open boundary condition, there is no phase shift of the pulse.

For a continuous wave of the form $y_{\text{inc}}[x, t] = A \sin[k x - \omega t]$, there will be a reflected wave of the form $y_{\text{ref}}[x, t] = A \sin[k x + \omega t + \phi]$.

Requiring the combined waves to vanish at a fixed boundary at $x = 0$,

$$y[x = 0, t] = y_{\text{inc}} + y_{\text{ref}} = -A \sin[t \omega] + A \sin[\phi + t \omega]$$

$$= 2 A \cos\left[\frac{\phi}{2} + t \omega\right] \sin\left[\frac{\phi}{2}\right] = 0 \text{ or } \phi = 0 \quad (1.78)$$

The resulting waveform becomes

$$y[x, t] = A \sin[k x - t \omega] + A \sin[k x + t \omega] \quad (1.79)$$

But where is the expected $\pi$ phase shift. It can be seen by plotting the incident and reflected waves separately.
The wave that is reflected has the opposite sign of the original wave or is advance (or delayed by 1/2 of a wavelength) which corresponds to the \( \pi \) phase shift.

Requiring the combined waves to vanish at an open boundary at \( x = 0 \),

\[
\frac{\partial y}{\partial x}[x = 0, t] = \frac{\partial y_{\text{inc}}}{\partial x} + \frac{\partial y_{\text{ref}}}{\partial x} = A k \cos[t \omega] + A k \cos[\phi + t \omega] \\
= 2 A k \cos\left(\frac{\phi}{2}\right) \cos\left[\frac{\phi}{2} + t \omega\right] = 0 \quad \text{or} \quad \phi = \pi
\]

The resulting waveform becomes

\[
y[x, t] = A \sin[k x - t \omega] - A \sin[k x + t \omega]
\]

Plotting the incident and reflected waves separately.
Now the wave that is reflected has the same sign of the original wave which corresponds to zero phase shift.

Now consider the reflection from an interface between two strings.

\[ c_1 = \sqrt{\frac{T_0}{\mu_1}} \quad \quad \quad \quad c_2 = \sqrt{\frac{T_0}{\mu_2}} \]

The horizontal tension \( T_0 \) is the same in both strings. Because \( \mu_2 > \mu_1 \), the speed of propagation in the left string is larger, \( c_1 > c_2 \).

The question is how will a pulse be reflected if it approaches the interface from the left. In your
mind consider the massive string to be so massive that it is hardly moved, i.e. $y[x = 0, t] \approx 0$. Therefore the reflected pulse will be inverted.

Similarly if a pulse approaches from the right then the lighter string will not be able to impose large forces at the point $x = 0$. One could assume that the lighter string is not there. In this case $-T_0 \left( \frac{\partial y}{\partial x} \right) \approx 0$. The boundary appears to have an open boundary condition. Therefore the reflected pulse will have no phase shift.

These extreme limits apply for the intermediate case and helps one to remember when one gets a phase shift. If a wave goes from a medium with a high propagation speed to a medium with a low propagation speed, there will be a $\pi$ phase shift. If the waves goes from a medium with a low propagation speed to one with a high propagation speed then the reflected waves have no phase shift.

$$\text{fast} \rightarrow \text{slow} \quad \pi \text{phase shift upon reflection}$$

$$\text{slow} \rightarrow \text{fast} \quad 0 \text{phase shift upon reflection}$$

The size of the reflected wave will depend upon the particular speeds of propagation in each string. At the junction point the displacement and its derivative must be continuous. If the derivative is not continuous, then there would be a finite force on a differential element of the string which has an infinitesimally small mass resulting in an infinite acceleration at the junction.

Assuming that the wave is incident from the left to the junction point at $x = 0$, the most general wave would be an incident and reflected wave

$$y_{\text{left}} = A_{\text{inc}} e^{-i(-t\omega + xk_1)} + A_{\text{ref}} e^{i(t\omega + xk_1)} \tag{1.82}$$

where the wavevectors are different for the two different strings. However the frequencies are equal. If the first string oscillates at a frequency $\omega$ then the next string must also oscillate at the same frequency. The overall sign was chosen so that both waves would oscillate as $e^{i\omega t}$. On the right side of the junction, there is only a transmitted wave

$$y_{\text{right}} = A_{\text{trans}} e^{-i(-t\omega + xk_2)} \tag{1.83}$$

where the wavevectors are different for the two different strings. However the frequencies are equal. If the first string oscillates at a frequency $\omega$ then the next string must also oscillate at the same frequency.

At $x = 0$, the wave is continuous

$$y_{\text{left}}[x = 0] = e^{i\omega t} A_{\text{inc}} + e^{i\omega t} A_{\text{ref}} = A_{\text{trans}} e^{i\omega t} = y_{\text{right}}[x = 0] \tag{1.84}$$

or

$$A_{\text{inc}} + A_{\text{ref}} = A_{\text{trans}} \tag{1.85}$$

Similarly requiring the derivative to be continuous
\[
\frac{\partial y_{\text{left}}}{\partial x} [x = 0] = -i e^{it\omega} A_{\text{inc}} k_1 + i e^{it\omega} A_{\text{ref}} k_1 = -i e^{it\omega} A_{\text{trans}} k_2 = \frac{\partial y_{\text{right}}}{\partial x} [x = 0]
\]

or

\[
(-A_{\text{inc}} + A_{\text{ref}}) k_1 = -A_{\text{trans}} k_2
\]

Solving for \(A_{\text{ref}}\) and \(A_{\text{trans}}\)

\[
A_{\text{ref}} = \frac{A (k_1 - k_2)}{k_1 + k_2} = \frac{A (-c_1 + c_2)}{c_1 + c_2}
\]

\[
A_{\text{trans}} = \frac{2A k_1}{k_1 + k_2} = \frac{2A c_2}{c_1 + c_2}
\]

As we can see if \(c_2 > c_1\), there is no phase shift but if \(c_2 < c_1\) there will be a \(\pi\) phase shift. Note also that if \(c_2 = c_1\) then there will not be a reflections because there is not change in the properties of the string.