Chapter 4

Damped Harmonic Oscillator

4.1 Friction

In the absence of any form of friction, the system will continue to oscillate with no decrease in amplitude. However, if there is some form of friction, then the amplitude will decrease as a function of time.

If the damping is sliding friction, \( F_{sf} = \text{constant} \), then the work done by the frictional is equal to the difference in the potential energy at the turning points. At the turning points, the velocity is zero and therefore the energy is given by the potential energy.

\[
(A_0 + A_1) F_{sf} = \frac{1}{2} k (A_0^2 - A_1^2) \tag{4.1}
\]

or

\[
F_{sf} = \frac{1}{2} k (A_0 - A_1) \tag{4.2}
\]

Solving for \( A_1 \)
4.2 Velocity Dependent Friction

Another common form of friction is proportional to the velocity of the object. This includes, air drag (at low velocities), viscous drag and magnetic drag.

Formally, the frictional force points in the opposite direction to the velocity

\[ F_f = -b v \]  \hspace{1cm} (4.4)
The total force on the mass is then \( F = -kx - bv \). Therefore the equation of motion becomes

\[
F = m \frac{d^2 x}{dt^2} = -kx - \frac{dx}{dt}
\]  

(4.5)

or

\[
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0
\]  

(4.6)

Let's try our previous trick to find the energy. Multiply the equation by \( v \) and rewrite

\[
\frac{d^2 x}{dt^2} = \frac{dv}{dt}
\]

\[
m v \frac{dv}{dt} + b v^2 + k v x = 0
\]  

(4.7)

or

\[
m v \frac{dv}{dt} + k x \frac{dx}{dt} = -b v^2
\]  

(4.8)

The quantity on the left hand side is just the time rate of change of the energy

\[
\frac{d}{dt} \left( \frac{m v^2}{2} + \frac{k x^2}{2} \right) = \frac{dE}{dt} = -b v^2
\]  

(4.9)

Previously we had \( b = 0 \) and consequently the energy was a constant in time. Now we see that
\[
\frac{dE}{dt} < 0 \text{ i.e. the energy decreases with time. To estimate the rate of the decrease of } E, \text{ we can rewrite the above equation as}
\]
\[
\frac{dE}{dt} = -\frac{2b}{m} \left( \frac{1}{2} mv^2 \right)
\]

(4.10)

and replace both sides of the equation by the average value of the quantities. This is not precisely correct as we are neglecting some terms.

\[
\frac{d\bar{E}}{dt} = -\frac{2b}{m} \left( \frac{m v^2}{2} \right) = -\frac{2b}{m} \left( \frac{\bar{E}}{2} \right) = -\frac{b}{m} \bar{E}
\]

(4.11)

where we have used the fact that for the undamped harmonic oscillator \( \frac{1}{2} m v^2 = E \). Now we can solve the equation

\[
\bar{E} = E_0 e^{-\frac{bt}{m}} = E_0 e^{-t/\gamma} = E_0 e^{ \frac{t}{\tau} }
\]

(4.12)

The average energy decreases exponentially with a characteristic time \( \tau = 1/\gamma \) where \( \gamma = b/m \). From this equation, we see that the energy will fall by \( 1/e \) of its initial value in time \( \tau \).

For an undamped harmonic oscillator, \( \frac{1}{2} kx^2 = E \). Therefore one would expect that
\[ x \propto \sqrt{E} \propto e^{-\frac{1}{2} \gamma t} \]  

\[ (4.13) \]

### 4.3 Formal Solution

Dividing Equation 4.6 by \( m \), we have that

\[
\frac{d^2 x}{dt^2} + \left( \frac{b}{m} \right) \frac{dx}{dt} + \left( \frac{k}{m} \right) x = \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0
\]

\[ (4.14) \]

Substitute a solution of the form \( A e^{i \omega t} \)

\[
-A \omega^2 e^{i \omega t} + i A \omega e^{i \omega t} \gamma + \omega_0^2 A e^{i \omega t} = 0
\]

\[ A e^{i \omega t} (i \gamma \omega - \omega^2 + \omega_0^2) = 0 \]

\[ (4.15) \]

Because \( e^{i \omega t} \neq 0 \) or \( A \neq 0 \), \( A e^{i \omega t} \) will be a solution if

\[
-i \gamma \omega + \omega^2 - \omega_0^2 = 0
\]

\[ (4.16) \]

or

\[
\omega_1 = \frac{1}{2} \left( i \gamma + \sqrt{-\gamma^2 + 4 \omega_0^2} \right)
\]

\[
\omega_2 = \frac{1}{2} \left( i \gamma - \sqrt{-\gamma^2 + 4 \omega_0^2} \right)
\]

\[ (4.17) \]

The complete solution then is

\[
x(t) = A_1 e^{i \omega_1 t} + A_2 e^{i \omega_2 t}
\]

\[
= A_1 e^{-\gamma t \frac{\gamma + 1}{2} t} \sqrt{-\gamma^2 + 4 \omega_0^2} + A_2 e^{-\gamma t \frac{\gamma - 1}{2} t} \sqrt{-\gamma^2 + 4 \omega_0^2}
\]

\[ (4.18) \]

### 4.3.1 No Damping \( \gamma = 0 \)

For \( b = 0 \), we obtain our previous solution

\[
x(t) = A_1 e^{i \omega_0 t} + A_2 e^{-i \omega_0 t}
\]

\[ (4.19) \]

For initial condition at \( t = 0 \), \( x(0) = x_0 \) and \( v(0) = v_0 \), we have that
\[ x[0] = A_1 + A_2 = x_0 \]
\[ v[0] = i A_1 \omega_0 - i A_2 \omega_0 = v_0 \]  \hspace{1cm} (4.20)

Solving for \( A_1 \) and \( A_2 \), we obtain
\[
A_1 = \frac{x_0}{2} - \frac{i v_0}{2 \omega_0} \\
A_2 = \frac{x_0}{2} + \frac{i v_0}{2 \omega_0} \]  \hspace{1cm} (4.21)

Clearly \( A_1 \) is the complex conjugate of \( A_2 \). Rewriting them in exponential form
\[
A_1 = \frac{\sqrt{v_0^2 + x_0 \omega_0^2}}{2 \omega_0} e^{-i \phi} = \frac{1}{2} \sqrt{\frac{v_0^2}{\omega_0^2} + \frac{x_0^2}{\omega_0^2}} e^{-i \phi} \\
A_2 = \frac{\sqrt{v_0^2 + x_0 \omega_0^2}}{2 \omega_0} e^{i \phi} = \frac{1}{2} \sqrt{\frac{v_0^2}{\omega_0^2} + \frac{x_0^2}{\omega_0^2}} e^{i \phi} \]  \hspace{1cm} (4.22)

where \( \tan[\phi] = \frac{v_0}{x_0 \omega_0} \)

The displacement then becomes
\[
x[t] = A_1 e^{i t \omega_0} + A_2 e^{-i t \omega_0} = \sqrt{\frac{v_0^2}{\omega_0^2} + \frac{x_0^2}{\omega_0^2}} \left( e^{i \phi - i t \omega_0} + e^{-i \phi + i t \omega_0} \right) \\
= \frac{\cos[\phi - t \omega_0]}{\omega_0} \sqrt{\frac{v_0^2}{\omega_0^2} + \frac{x_0^2}{\omega_0^2}} \]  \hspace{1cm} (4.23)

4.3.2 Small Damping \( \frac{\gamma}{2} < \omega_0 \)

For \( \frac{\gamma}{2} < \omega_0 \), the term in the exponent, \( \sqrt{4 \omega_0^2 - \gamma^2} \) will be real. Defining a new frequency,
\[ \omega_D = \frac{1}{2} \sqrt{4 \omega_0^2 - \gamma^2} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \], we obtain
We can now identify $\omega_D$ as the frequency of oscillations of the damped harmonic oscillator. As before we can rewrite the exponentials in terms of Cosine function with an arbitrary phase.

$$x(t) = A e^{-\frac{t\gamma}{2}} \cos[\omega_D t - \phi]$$

For initial condition at $t = 0$, $x[0] = x_0$ and $v[0] = v_0$, we have that

$$x[0] = A \cos[\phi] = x_0$$
$$v[0] = -\frac{1}{2} A \gamma \cos[\phi] + A \sin[\phi] \omega_D = v_0$$

Substituting, for $\cos[\phi]$ from the first equation into the second, we have that

$$-\frac{\gamma x_0}{2} + A \sin[\phi] \omega_D = v_0$$

Therefore

$$A \cos[\phi] = x_0$$
$$A \sin[\phi] = \frac{2 v_0 + \gamma x_0}{2 \omega_D}$$

Solving for $A$ and $\phi$

$$A = \sqrt{x_0^2 + \frac{(2 v_0 + \gamma x_0)^2}{4 \omega_D^2}}$$
$$\tan[\phi] = \frac{2 v_0 + \gamma x_0}{2 x_0 \omega_D}$$

One can check this answer by taking the limit as $\gamma \to 0$ (no damping), $\omega_D \to \omega_0$ and the expressions for $A$ and $\phi$ should reduce to our previous result.
4.3.3 Critical Damping $\frac{\gamma}{2} = \omega_0$

At $\frac{\gamma}{2} = \omega_0$, the frequency $\omega_D$ vanishes and the expression in the exponent reduces to

$$-\frac{\gamma}{2} \pm \frac{1}{2} i \sqrt{-\gamma^2 + 4 \omega_0^2} \rightarrow -\frac{\gamma}{2}$$

(4.31)

The solution no longer has an oscillatory part. In addition one no longer has two solutions that can be used to fit arbitrary initial conditions.

The general solution as a function of time becomes

$$x[t] = e^{-\frac{t\gamma}{2}} (A + B t)$$

(4.32)

The second term is necessary to satisfy all possible initial conditions. Differentiating

$$v[t] = -\frac{1}{2} e^{-\frac{t\gamma}{2}} (A \gamma + B (-2 + t \gamma))$$

(4.33)

$$a[t] = \frac{1}{4} e^{-\frac{t\gamma}{2}} \gamma (A \gamma + B (-4 + t \gamma))$$
Substituting in the differential equation

\[
\frac{d^2 x}{dt^2} + \frac{d x}{dt} + \omega_0^2 x = 0
\]

\[
1 - \frac{t \gamma}{4} \gamma (A \gamma + B (-4 + t \gamma)) = \frac{1}{2} e^{-\frac{t \gamma}{2} (A \gamma + B (-2 + t \gamma))} + \omega_0^2 e^{-\frac{t \gamma}{2} (A + B)} \tag{4.34}
\]

\[
\frac{1}{4} e^{-\frac{t \gamma}{2} (A + B t) (\gamma^2 - 4 \omega_0^2)} = 0
\]

which by the definition, \(\gamma^2 = 4 \omega_0^2\), is identically equal to zero.

To determine \(A\) and \(B\) from the initial conditions, equate

\[
x[0] = \left. \left( e^{-\frac{t \gamma}{2} (A + B t)} \right) \right|_{t=0} = A
\]

\[
v[t] = \left. \left( -\frac{1}{2} e^{-\frac{t \gamma}{2} (A \gamma + B (-2 + t \gamma))} \right) \right|_{t=0} = \frac{1}{2} (2 B - A \gamma)
\]

Solving for \(A\) and \(B\)

\[
A = x_0
\]

\[
B = \frac{1}{2} (2 v_0 + \gamma x_0) \tag{4.36}
\]

and therefore the general solution becomes

\[
x[t] = e^{-\frac{t \gamma}{2}} \left( x_0 + \frac{1}{2} t (2 v_0 + \gamma x_0) \right) \tag{4.37}
\]

### 4.3.4 Overdamping \(\frac{\gamma}{2} > \omega_0\)

For \(\frac{\gamma}{2} > \omega_0\), the argument of the square root becomes negative. Taking the \(-1\) out of the squareroot so that its argument is now positive, the frequency, \(\omega\), becomes completely imaginary

\[
\omega_1 = \frac{i \gamma}{2} + \sqrt{-\frac{\gamma^2}{4} + \omega_0^2} \quad \rightarrow \quad \frac{i \gamma}{2} + i \sqrt{\frac{\gamma^2}{4} - \omega_0^2}
\]

\[
\omega_1 = \frac{i \gamma}{2} - \sqrt{-\frac{\gamma^2}{4} + \omega_0^2} \quad \rightarrow \quad \frac{i \gamma}{2} - i \sqrt{\frac{\gamma^2}{4} - \omega_0^2}
\]

\[
\omega_1 = \frac{i \gamma}{2} - \sqrt{-\frac{\gamma^2}{4} + \omega_0^2} \quad \rightarrow \quad \frac{i \gamma}{2} + i \sqrt{\frac{\gamma^2}{4} - \omega_0^2}
\]
Again, there are no oscillatory terms. Both solutions are dying exponentials

\[ x(t) = A_1 e^{-\frac{\gamma}{2}t \left( -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \right)} + A_2 e^{-\frac{\gamma}{2}t \left( -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \right)} \]  

(4.39)

Although the second term contains a growing exponential, \( \frac{\gamma D}{2} = \sqrt{\left( \frac{\gamma}{2} \right)^2 - \omega_0^2} < \frac{\gamma}{2} \) and so the overall function is still a dying exponential.

As before, to determine \( A_1 \) and \( A_2 \) from the initial conditions, equate

\[ x[0] = \left( e^{-\frac{\gamma D}{2}t} A_1 + e^{-\frac{\gamma D}{2}t} A_2 \right) \bigg|_{t=0} = A_1 + A_2 \]  

(4.40)

and

\[ u[0] = \left( \frac{1}{2} e^{-\frac{\gamma D}{2}t} \left( e^{\frac{\gamma D}{2}t} A_2 (-\gamma + \gamma D) - A_1 (\gamma + \gamma D) \right) \right) \bigg|_{t=0} \]

\[ = \frac{1}{2} (A_2 (-\gamma + \gamma D) - A_1 (\gamma + \gamma D)) \]

(4.41)

Solving for \( A_1 \) and \( A_2 \)

\[ A_1 = \frac{-2 v_0 + x_0 (-\gamma + \gamma D)}{2 \gamma D} \]

(4.42)

\[ A_2 = \frac{2 v_0 + x_0 (\gamma + \gamma D)}{2 \gamma D} \]

and therefore the general solution becomes

\[ x[t] = \frac{1}{2} e^{-\frac{\gamma D}{2}t} \left( 2 (1 + e^{\gamma D}) v_0 + x_0 (-\gamma + \gamma D + e^{\gamma D} (\gamma + \gamma D)) \right) \]

(4.43)

Note that this solution looks very much like our original solution for the underdamping case. In particular if we let \( \sqrt{\gamma^2 - 4 \omega_0^2} = i \sqrt{4 \omega_0^2 - \gamma^2} = 2 i \omega_D \), then the above solution becomes

\[ x[t] = \frac{1}{2 i \omega_D} e^{-\frac{1}{2}i (\gamma D)} \left( \text{Cosh} \left[ \frac{2}{2} t i \omega_D \right] 2 i \omega_D x_0 + \text{Sinh} \left[ \frac{1}{2} t 2 i \omega_D \right] (2 v_0 + \gamma x_0) \right) \]

(4.44)

Now, \( \text{Cosh}[i y] = \frac{e^{i y} + e^{-i y}}{2} = \cos[y] \) and \( \text{Sinh}[i y] = \frac{e^{i y} - e^{-i y}}{2} = i \sin[y] \) so that the above expression reduces to
\[ x[t] = -\frac{i e^{-\frac{t \gamma}{2}} (i \sin[t \omega_D] (2v_0 + \gamma x_0) + 2i \cos[t \omega_D] x_0 \omega_D)}{2 \omega_D} \]
\[ = e^{-\frac{t \gamma}{2}} \left( \cos[t \omega_D] x_0 + \frac{\sin[t \omega_D] (v_0 + \frac{\gamma x_0}{2})}{\omega_D} \right) \]

Equating \( x_0 = \cos[\phi] \) and \( \frac{v_0 + \frac{\gamma x_0}{2}}{\omega_D} = \sin[\phi] \), then the above solution becomes the solution for an underdamped harmonic oscillator

\[ x[t] = \cos[\phi + t \omega_D] \sqrt{x_0^2 + \frac{(v_0 + \frac{\gamma x_0}{2})^2}{\omega_D^2}} \] (4.46)

### 4.4 Quality Factor or Q

A dimensionless quantity used to measure the amount of damping is given by for a harmonic oscillator is defined as

\[ Q = \frac{2 \pi (\text{average energy stored})}{(\text{energy lost per cycle})} \] (4.47)

As defined the \( Q \) will be large if the damping is small. We know that the average energy is given approximately by \( \bar{E}[t] \approx E_0 e^{-\gamma t/2} \). The change is the energy in one period \( \tau \) is therefore

\[ \Delta E = \bar{E}[t_0] - \bar{E}[\tau + t_0] = e^{-\gamma \tau} (1 - e^{-\gamma \tau}) E_0 \] (4.48)

For small damping \( \gamma \tau \ll 1 \), one can expand the exponential, \( e^{-\gamma \tau} \approx 1 - \gamma \tau + \ldots \) so that

\[ \Delta E = e^{-\gamma t_0} \gamma \tau E_0 \] (4.49)

Substituting this into the expression for the \( Q \)

\[ Q = \frac{2 e^{-\gamma t_0} \frac{\pi}{\gamma} E_0}{e^{-\gamma t_0} \gamma \tau E_0} = \frac{2 \pi}{\gamma \tau} = \frac{\omega}{\gamma} \] (4.50)

where we have used the fact that \( \omega = \frac{2 \pi}{\tau} \). The following illustrates the behavior for several values of the \( Q \).
4.5 Energy of Damped Harmonic Oscillator

Using the expression for an underdamped harmonic oscillator, \( x(t) = A \cos(\omega_D t + \phi) e^{-\gamma t/2} \), the potential energy becomes

\[
PE = \frac{1}{2} k x(t)^2 = \frac{1}{2} m \omega^2 x(t)^2 = \frac{1}{2} A^2 e^{-\gamma t} m \omega^2 \cos(\phi + t \omega_D)^2
\]  

To calculate the kinetic energy, first calculate the velocity
\[ v[t] = -\frac{1}{2} A e^{-\frac{t \gamma}{2}} \gamma \cos(\phi + t \omega_D) - A e^{-\frac{t \gamma}{2}} \sin(\phi + t \omega_D) \omega_D \]  

(4.52)

and the kinetic energy becomes

\[
PE = \frac{1}{2} m v[t]^2 = \frac{1}{2} m \left(-\frac{1}{2} A e^{-\frac{t \gamma}{2}} \gamma \cos(\phi + t \omega_D) - A e^{-\frac{t \gamma}{2}} \sin(\phi + t \omega_D) \omega_D\right)^2
\]

\[
= \frac{1}{8} A^2 e^{-t \gamma} k (\gamma \cos(\phi + t \omega_D) + 2 \sin(\phi + t \omega_D) \omega_D)^2
\]

(4.53)

The total energy is then

\[
E[t] = \frac{1}{8} A^2 e^{-t \gamma} m
\]

\[
(\gamma^2 + 4 \omega^2) \cos(\phi + t \omega_D)^2 + 2 \gamma \sin(2(\phi + t \omega_D) \omega_D + 4 \sin(\phi + t \omega_D)^2 \omega_D^2)
\]

(4.54)

Substituting \( \omega^2 = \frac{\gamma^2}{4} + \omega_D^2 \)

\[
E[t] = \frac{1}{4} A^2 e^{-t \gamma} m \left( \gamma^2 \cos(\phi + t \omega_D)^2 + \gamma \sin(2(\phi + t \omega_D) \omega_D + 2 \omega_D^2\right)
\]

\[
E[t] = \frac{1}{2} A^2 e^{-t \gamma} m \omega_D^2 \left(1 + \frac{\gamma^2 \cos(\phi + t \omega_D)^2}{2 \omega_D^2} + \frac{\gamma \sin(2(\phi + t \omega_D) \omega_D}{2 \omega_D} \right)
\]

(4.55)

Plotting the total energies assuming \( \phi = 0 \).