Chapter 3

Harmonic Oscillator - Model Systems

3.1 Mass on a spring in a gravitation field

3.1.1 Force Method

The two forces on the mass are due to the spring, \( F_s = -k z \ddot{z} \), and the gravitational field, \( F_g = -m g \ddot{z} \).
As shown, the displacement is negative and therefore the force is in the positive $z$ direction. Evaluating the total force on the mass,

$$F_t = m \frac{\partial^2 z}{\partial t^2} = F_g + F_s = -m g - k z$$  \hspace{1cm} (3.1)

Defining a new variable, $\zeta = z + \frac{m g}{k}$ and noting that $\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 \zeta}{\partial t^2}$ because $m g / k$ is a constant then the equation becomes

$$m \frac{\partial^2 \zeta}{\partial t^2} = -k \zeta$$ \hspace{1cm} (3.2)

From our previous discussions, the general solution is of the form $\zeta = A e^{i \omega t}$ with $\omega^2 = k / m$ and therefore the motion for $z$ is again harmonic but with a displaced zero.

### 3.1.2 Energy Method

The kinetic energy is only due to the motion of the mass whereas the potential energy is a combination of potential energy due to the spring and gravity.

$$KE = \frac{1}{2} m \left( \frac{\partial z}{\partial t} \right)^2$$ \hspace{1cm} (3.3)

$$PE = \frac{1}{2} k z^2 + m g z$$ \hspace{1cm} (3.4)

The equilibrium position of the mass is given by the minimum of the potential energy. Setting
the derivative of the potential energy to zero and solving for $z$

$$\frac{\partial PE}{\partial z} = g m + k z = 0$$  \hspace{1cm} (3.5)

or

$$z_{equilibrium} = -\frac{g m}{k}$$  \hspace{1cm} (3.6)

The effective spring constant is given by the second derivative of the potential energy.

$$k_{effective} = \frac{\partial^2 PE}{\partial z^2} = k$$  \hspace{1cm} (3.7)

Because the effective mass is just $m$, the frequency of oscillation is just $\omega^2 = k / m$.

### 3.2 Pendulum

#### 3.2.1 Force Method

The forces on the mass of a pendulum are the gravitational force pointing in the $-\hat{z}$ direction, $-m g \hat{z}$, and the tension in the pendulum rod that points along the rod towards the pivot point. A component of the gravitational force perpendicular to the rod, $-m g \sin[\theta] \hat{\theta}$, provides a torque, $-m g L \sin[\theta] \hat{\theta}$, that results in a changing angular momentum of the body.
The torque equation then becomes

\[
\frac{dL}{dt} = mL^2 \frac{d^2 \theta}{dt^2} = -mgL\sin[\theta]
\]  

(3.8)

We see then that the resulting equation does not describe a simple harmonic oscillator except in the limit of small angle. From the Taylor expansion of the Sine,

\[
\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + O[\theta]^7
\]  

(3.9)

we see that when \( \frac{\theta^3}{6} \ll \theta \) or \( \theta \ll \text{Sqrt}[6] = 2.45R \), this system will exhibit simple harmonic motion. The frequency of oscillation will be given by

\[
\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{gLm}{mL^2}} = \sqrt{\frac{g}{L}}
\]  

(3.10)

which is independent of the mass of the pendulum and only depends on its length and the acceleration due to gravity.

For larger amplitudes, the period of oscillation is no longer independent of the amplitude and increases. At very large amplitudes, the motion is no longer sinusoidal.
3.2.2 Energy Method

The potential energy of the pendulum relative to its equilibrium position is given by

\[ U(\theta) = mgh = mgL(1 - \cos(\theta)) \]
The equilibrium position is given by

$$\frac{\partial U[\theta]}{\partial \theta} = g L m \sin[\theta] = 0$$

or

$$z_{equilibrium} = 0$$

For small oscillations about the equilibrium position, expand $U[\theta]$ about the equilibrium position, $\theta = 0$,

$$U[\theta] = \frac{1}{2} g L m \theta^2 + O[\theta]^4$$

The kinetic energy is $\frac{1}{2} m v^2$ but $v = L \theta$ so that

$$KE = \frac{1}{2} m L^2 \left( \frac{d \theta}{dt} \right)^2$$

or equivalently $KE = \frac{1}{2} I \frac{d^2 \theta}{dt^2}$ where $I$ is the moment of inertia. The total energy becomes

$$E_{total} = \frac{1}{2} L^2 m \left( \frac{d \theta}{dt} \right)^2 + \frac{1}{2} g L m \theta^2$$

Comparing this to the general form of the energy of a harmonic oscillator,
\[ E = \frac{1}{2} m_{\text{eff}} \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} k_{\text{eff}} x^2, \] we see that the oscillation frequency becomes

\[
\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{g L m}{m L^2}} = \sqrt{\frac{g}{L}}
\]

(3.16)

### 3.3 3.3 Capacitor and Inductor Circuit

For a parallel circuit of an inductor and capacitor, the energy oscillate between the magnetic field of the inductor and the electric field of the capacitor.

#### 3.3.1 Energy Method

The total energy is given by the energy of the capacitor, \( \frac{1}{2} C V^2 \), and the energy of the inductor, \( \frac{1}{2} L I^2 \) or noting that the voltage of the capacitor is \( V = \frac{Q}{C} \) where \( I = \frac{dQ}{dt} \)

\[
E_{\text{total}} = \frac{1}{2} L \left( \frac{dQ}{dt} \right)^2 + \frac{1}{2} \frac{1}{C} Q^2
\]

(3.17)

Comparing this to the general form of the energy of a harmonic oscillator, \( E_{\text{total}} = \frac{1}{2} m_{\text{eff}} \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} k_{\text{eff}} x^2 \), we see that the oscillation frequency becomes

\[
\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{1}{L C}}
\]

(3.18)

#### 3.3.2 Circuit Equation
One could also have solved this problem using the circuit equations. The voltage of the capacitor is equal to that of the inductor, \( V_C = V_L \) or \( \frac{Q}{C} = -L \frac{dI}{dt} \). Differentiating once we have that

\[
-L \frac{d^2 I}{dt^2} = \frac{1}{C} \frac{dQ}{dt} = \frac{1}{C} I \tag{3.19}
\]

or

\[
\frac{d^2 I}{dt^2} = -\frac{1}{LC} I \tag{3.20}
\]

### 3.4 Gas Piston

The ideal gas law states that \( PV = N k_B T \). For an isothermal process, the product of the pressure and volume will be a constant, i.e. \( PV = P_0 V_0 \).

The forces on the piston will be the downwards gravitational force and the upwards force due to the pressure

\[
F = PA - mg
\]

In general the pressure will depend upon the volume of the gas or the length of the chamber, \( P = \frac{P_0 V_0}{V} = \frac{P_0 V_0}{AL} \). The force equation then becomes

\[
F = PA - mg = \frac{P_0 V_0}{L} - mg \tag{3.22}
\]

In equilibrium the force on the piston will be zero and so the equilibrium length of the chamber
will be $L_0 = \frac{P_0 V_0}{m g}$. For deviations about this equilibrium length, $L = L_0 + y$, the force equation becomes

$$F = \frac{P_0 V_0}{y + L_0} - m g$$

(3.23)

If $y$ is small, we can expand in a Taylor’s series. Note that the 0th order terms cancel. The first order terms gives up a force proportional to the displacement.

$$F = \left(-g m + \frac{P_0 V_0}{L_0} \right) - \frac{P_0 V_0 y}{L_0^2} + \frac{P_0 V_0 y^2}{L_0^3} + O[y]^3$$

$$= -\frac{y P_0 V_0}{L_0^2}$$

(3.24)

Equating this force to the acceleration of the piston, we see that we will give harmonic motion with a frequency of $\omega = \sqrt{\frac{P_0 V_0}{L_0^2 m}}$. But $\frac{P_0 V_0}{L_0} = m g$ and so the frequency can also be written as $\omega = \sqrt{\frac{g}{L_0}}$ which resembles the frequency of a pendulum.

### 3.5 Orbital Mechanics

For a mass, $m$, orbiting a larger mass, $M$, the force of gravity is $F = -\frac{m M G}{r^2} \hat{r}$ which corresponds to a potential energy of the form $V[r] = -\frac{m M G}{r}$. The kinetic energy of the smaller mass is simply $\frac{1}{2} m v^2$. For a central force the angular momentum, $L = m v r$ is conserved and is a constant. For a circular orbit, the kinetic energy can be expressed in terms of this angular momentum.

$$KE = \frac{m v^2}{2} = \frac{1}{2} m \left( \frac{L}{m r} \right)^2 = \frac{L^2}{2 m r^2}$$

(3.25)

The total energy then becomes

$$E = \frac{L^2}{2 m r^2} - \frac{G m M}{r}$$

(3.26)

Graphing the total energy,
At a given angular momentum, $L$, the object will have a specific orbit at $r_0$ given by the minimum of the energy

$$
\frac{dE}{dr} = \frac{d}{dr} \left( \frac{L^2}{2m r^2} - \frac{G m M}{r} \right) = - \frac{L^2}{m r^3} + \frac{G m M}{r^2} = 0
$$

(3.27)

Solving for $r_0$

$$
 r_0 = \frac{L^2}{G m^2 M}
$$

(3.28)

Alternatively Eq. 2.27 is just a statement that the gravitational force is equal to the centripetal force

$$
\frac{G m M}{r_0^2} = \frac{L^2}{m r_0^3} = \frac{m v^2}{r_0}
$$

(3.29)

Now what happens if you nudge the object in the radial direction with an impulse. Because the impulse is in the radial direction, the angular momentum is unchanged. The object will oscillate about the point $r_0$ as it continues in its orbit. The effective spring constant is given by the second derivative of the energy

$$
 k_{eff} = \left( \frac{d^2 E}{dr^2} \right)_{r=r_0} = \left( \frac{3 L^2}{m r^4} - \frac{2 G m M}{r^3} \right)_{r=r_0}
$$

$$
= \frac{G^4 m^7 M^4}{L^6}
$$

(3.30)

or substituting from Eq. 2.29 $G = \frac{L^2}{m^2 M r_0}$
We can reexpress $k_{\text{eff}}$ in terms of the frequency of the orbits. From $L = m r v r = m r_0^2 \omega_0$ we have that

$$k_{\text{eff}} = \frac{G^4 m^7 M^4}{L^6} = m \omega_0^2$$

(3.32)

Therefore the frequency of the radial oscillations becomes

$$\omega_{\text{osc}} = \sqrt{\frac{k_{\text{eff}}}{m}} = \omega_0$$

(3.33)

i.e. the period of the radial oscillations is the same as the circular motion. Plotting the orbit then

The orbit is the form of an ellipse. A complete analysis in mechanics shows that a general orbit is given by an ellipse with eccentricity, $\varepsilon$. 

![Graph showing elliptical orbit](image)
\[ r = \frac{5(1 - \varepsilon^2)}{4(1 + \varepsilon \cos[\theta])} \] 

(3.34)

For small \( \varepsilon \) we have that

\[ r = \frac{5(1 - \varepsilon^2)}{4(1 + \varepsilon \cos[\theta])} \approx \frac{5}{4} - \frac{5}{4} \cos[\theta] \varepsilon + \frac{5}{4} (-1 + \cos[\theta]^2) \varepsilon^2 + O[\varepsilon]^3 \] 

(3.35)

or to lowest order

\[ r = \frac{5}{4} - \frac{5}{4} \varepsilon \cos[\theta] \] 

(3.36)

which is exactly the orbit that we have found.