Chapter 2

Harmonic Oscillator

1.1 Solving the Differential Equation

The importance of a quadratic potential or equivalently a linear force is that such systems can be analytically solved to obtain the evolution of the system as an explicit function of time.

Assuming a quadratic potential, \( U[x] = \frac{1}{2} k x^2 \), the force equation becomes

\[
F = m \ddot{x} = m \frac{\partial^2 x}{\partial t^2} = m \dot{x}
\]

\[
= - \frac{\partial U}{\partial x} = -k x
\]

(1.1)

Denoting the velocity to be \( \frac{dx}{dt} = v \), then the acceleration can be rewritten as \( \frac{d^2 x}{dt^2} = \frac{dv}{dt} \) and therefore

\[
m \frac{\partial^2 x}{\partial t^2} = m \frac{\partial v}{\partial t} = -k x
\]

(1.2)

but by the chain rule

\[
m \frac{\partial v}{\partial t} = m \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} = m v \frac{\partial v}{\partial x} = -k x
\]

(1.3)

or

\[
m \frac{dv}{dt} = -k x \frac{dx}{dt}
\]

(1.4)

Integrating both sides from an initial velocity, \( v_0 \) and an initial position, \( x_0 \)

\[
\int_{v_0}^{v} m v' \; dv' = - \int_{x_0}^{x} k x' \; dx'
\]

(1.5)

\[
\frac{1}{2} m (v^2 - v_0^2) = - \frac{1}{2} k (x^2 - x_0^2)
\]

or

\[
E_{\text{Total}} = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2
\]

(1.6)

That is the total energy, kinetic plus potential energy, is always a constant! The above equation defines a closed or periodic orbit (an ellipse) in the \( x - v \) plane. Note also that one needs two
pieces of information (in this case the initial position and velocity) to specify the relationship between velocity and position.

To obtain the position as a function of time, $x[t]$, we can substitute in for the definition of the velocity in the above equation. Setting $E_0$ to be the energy of the system we have that

$$\frac{1}{2} m v^2 = E_0 - \frac{1}{2} k x^2$$

or

$$v = \frac{dx}{dt} = \sqrt{-\frac{k x^2}{m} + \frac{2 E_0}{m}} = \sqrt{\frac{2 E_0}{m}} \sqrt{1 - \frac{k x^2}{2 E_0}}$$

(1.8)

Now at the maximum extent of the motion (turning points), all of the energy is potential energy so that $x_{\text{max}} = \sqrt{\frac{2 E_0}{k}}$. Similarly the maximum velocity is $v_{\text{max}} = \sqrt{\frac{2 E_0}{m}}$. Substituting these definitions into the equation and collecting all the $x$ terms on the left and the $t$ terms on the right, we can then integrate both sides.

$$\int_{x_0}^{x[t]} \frac{1}{\sqrt{1 - \frac{(x')^2}{x_{\text{max}}^2}}} dx' = \int_0^t v_{\text{max}} \, dt'$$

(1.9)

where the limits are chosen to be the initial time ($t = 0$) and position, $x_0$. To integrate substitute $x = x_{\text{max}} \cos[z]$, $z = \text{ArcCos}\left[\frac{x}{x_{\text{max}}}\right]$ and $dx = -x_{\text{max}} \sin[z] \, dz$
\[
\int_{x_0}^{x_f} \frac{1}{\sqrt{1 - \frac{\{x(t)\}^2}{x_{\text{max}}}^2}} \, dx' = \int_{ArcCos\left[\frac{x_0}{x_{\text{max}}}\right]}^{ArcCos\left[\frac{x_f}{x_{\text{max}}}\right]} \frac{\sin(z) x_{\text{max}}}{\sqrt{1 - \cos(z)^2}} \, dz
\]

or

\[
x[t] = x_{\text{max}} \cos\left[t \frac{v_{\text{max}}}{x_{\text{max}}} - \text{ArcCos}\left[\frac{x_0}{x_{\text{max}}}\right]\right]
\]

(1.11)

From the previous definitions of \(x_{\text{max}}\) and \(v_{\text{max}}\), the particular ratio, \(\frac{v_{\text{max}}}{x_{\text{max}}} = \sqrt{\frac{2 E_0}{m}} = \sqrt{\frac{k}{m}}\)

depends only on the spring constant and the mass. Denoting \(\omega = \sqrt{\frac{k}{m}}\), the general solution has the form

\[
x[t] = x_{\text{max}} \cos(\omega t - \phi)
\]

(1.12)

where we have defined the phase as \(\phi = \text{ArcCos}\left[\frac{x_0}{x_{\text{max}}}\right]\). As one can see, the most general solution is a periodic function consisting of a Cosine function with amplitude \(x_{\text{max}}\) and a phase that depends on the initial displacement at \(t = 0\), \(\phi = \text{ArcCos}\left[\frac{x_0}{x_{\text{max}}}\right]\). As before, the complete solution depends upon two constants. Both sets of constants are equally valid and can be expressed in terms of each other. By differentiating \(x[t]\), one obtains \(v[t]\)

\[
v[t] = \frac{\partial x[t]}{\partial t} = -\omega x_{\text{max}} \sin(\omega t - \phi)
\]

(1.13)

Therefore the initial position and velocity at \(t = 0\) is

\[
x_0 = x[0] = \cos(\phi) x_{\text{max}}
\]

(1.14)

\[
v_0 = v[0] = \omega \sin(\phi) x_{\text{max}}
\]

(1.15)

and therefore
\[ x_{\text{max}} \rightarrow \frac{\sqrt{v_0^2 + \omega^2 x_0^2}}{\omega} \quad \text{and} \quad \phi = \text{ArcTan}\left[ \frac{v_0}{\omega x_0} \right] \quad (1.16) \]

To obtain the same solution, using exponentials, we can try a solution of the form, \( x(t) = A e^{i \omega t} \) where \( A \) can be complex. Substituting this into the force equation

\[ m \frac{\partial^2 x}{\partial t^2} = -k x \]

\[ m \frac{\partial^2 (A e^{i \omega t})}{\partial t^2} = -k (A e^{i \omega t}) \quad (1.17) \]

\[ -A e^{i \omega t} m \omega^2 = -k (A e^{i \omega t}) \]

Collecting terms

\[ A e^{i \omega t} (k - m \omega^2) = 0 \quad (1.18) \]

\( e^{i \omega t} \) cannot be zero. \( A \) can be zero but that would give a trivial solution, \( x(t) = 0 \) for all time. A non-trivial solution can be obtained by choosing \( \omega^2 = \frac{k}{m} \). But what determines the value of \( A \)? If \( A \) is complex then it can always be expressed in terms of an exponential.

\[ A = C e^{i \phi} = C \cos[\phi] + i C \sin[\phi] \quad (1.19) \]

where

\[ C = |A| = \sqrt{\text{Im}[A]^2 + \text{Re}[A]^2} \]

\[ \tan[\phi] = \frac{\text{Im}[A]}{\text{Re}[A]} \quad (1.20) \]

so that

\[ x(t) = A e^{i \omega t} = |A| e^{i(\omega t - \phi)} \quad (1.21) \]

Of course, the true displacement is a real quantity. We can therefore take the real part of the above expression to give

\[ x(t) = \text{Re}(|A| e^{i(\omega t - \phi)}) = |A| \cos[\omega t - \phi] \quad (1.22) \]

which is equivalent to our previous expression where \( |A| = x_{\text{max}} \) and at \( t = 0 \), the initial displacement becomes \( x_0 = |A| \cos[\phi] \) and the initial velocity is \( v_0 = |A| \sin[\phi] \).

A alternate approach is to note that the requirement for a non-trivial solution, \( \omega^2 = \frac{k}{m} \), has two solutions, \( \omega = \pm \sqrt{\frac{k}{m}} \). Therefore the most general solution can be written as
\[ x(t) = A_+ e^{i(\omega t)} + A_- e^{-i(\omega t)} \]  

(1.23)

To satisfy the initial conditions, \( x(t=0) = x_0 \) and \( v(t=0) = v_0 \) we have that

\[
\begin{align*}
  x(t=0) &= x_0 = (A_- + A_+) \\
  v(t=0) &= v_0 = (-i\omega A_- + i\omega A_+) 
\end{align*}
\]

(1.24)

Solving for \( A_+ \) and \( A_- \)

\[
\begin{align*}
  A_+ &= -\frac{i v_0}{2\omega} + \frac{x_0}{2} \\
  A_- &= \frac{i v_0}{2\omega} + \frac{x_0}{2}
\end{align*}
\]

(1.25)

so that

\[
\begin{align*}
  x(t) &= e^{i\omega t} \left( -\frac{i v_0}{2\omega} + \frac{x_0}{2} \right) + e^{-i\omega t} \left( \frac{i v_0}{2\omega} + \frac{x_0}{2} \right)
\end{align*}
\]

(1.26)

Simplifying

\[
\begin{align*}
  x(t) &= \left( \frac{i e^{-i\omega t} - i e^{i\omega t}}{2\omega} \right) v_0 + \left( \frac{1}{2} e^{-i\omega t} + \frac{1}{2} e^{i\omega t} \right) x_0 = \frac{\sin(\omega t) v_0}{\omega} + \cos(\omega t) x_0
\end{align*}
\]

(1.27)

Writing \( x_0 = A \cos(\phi) \) and \( \frac{v_0}{\omega} = A \sin(\phi) \) then

\[
\begin{align*}
  x(t) &= A \cos(\phi) \cos(\omega t) + A \sin(\phi) \sin(\omega t) = A \cos(\omega t - \phi)
\end{align*}
\]

(1.28)

which is the same equation that was found previously.

### 1.2 General Solution

Plotting the general solution as a function of time
The solution is periodic: \( x(t + \tau) = x(t) \) for any time \( t \).

\[
A \cos(\omega t - \phi) = A \cos(\omega(t + \tau) - \phi) = A \cos((\omega t - \phi) + \omega \tau) = A(\cos(\omega t - \phi) \cos(\omega \tau) - \sin(\omega \tau) \sin(\omega t - \phi))
\]

(1.29)

The right side is equal to the left side when \( \cos(\omega \tau) = 1 \) and \( \sin(\omega \tau) = 0 \) or \( \omega \tau = 2n\pi \). The shortest non-trivial solution is therefore for \( n = 1 \) or \( \tau = 2\pi/\omega = 1/f \) where \( f \) is the natural frequency. Note the dimensions of these quantities:

\[
\begin{align*}
\tau & \quad \text{(Time in seconds)} \\
f & \quad \text{(Frequency or cycles per second)} \\
\omega & \quad \text{(Angular Frequency or Radians per second)}
\end{align*}
\]

1.3 Initial Conditions

To determine the amplitude \(|A|\) and the phase \(\phi\), one can either

1. Specify the position and velocity at some instant in time, \( t_0 \)
2. Specify the position or velocity at two separate times.

We have already used method (1) at \( t = 0 \). Given that

\[
\begin{align*}
x(t) &= A \cos(\omega t + \phi) \\
v(t) &= \frac{\partial x(t)}{\partial t} = -A \omega \sin(\omega t + \phi)
\end{align*}
\]

(1.30)

and generalizing to an arbitrary time, \( t_0 \), the initial position and velocity become
\[ x_0 = x[t_0] = A \cos(\phi + \omega t_0) \]  
\[ v_0 = v[t_0] = -A \omega \sin(\phi + \omega t_0) \]  

and therefore
\[ x_0^2 + \frac{v_0^2}{\omega^2} = A^2 \cos(\phi + \omega t_0)^2 + A^2 \sin(\phi + \omega t_0)^2 = A^2 \]  

and
\[ \frac{v_0}{\omega x_0} = \frac{-A \omega \sin(\phi + \omega t_0)}{\omega (A \cos(\phi + \omega t_0))} = -\tan(\phi + \omega t_0) \]  

or
\[ \phi \rightarrow -\arctan\left(\frac{v_0}{\omega x_0}\right) - \omega t_0 \]  

Using the positions at \( t = t_1 \) and \( t_2 \) in method (2) is a bit more complicated.
\[ x_1 = x[t_1] = A \cos(\phi + \omega t_1) \]  
\[ x_2 = x[t_2] = A \cos(\phi + \omega t_2) \]  

Using trigonometric identities
\[ A \rightarrow \csc(\omega (t_1 - t_2)) \sqrt{x_1^2 - 2 \cos(\omega (t_1 - t_2))x_1 x_2} \]  
\[ \phi \rightarrow \arccos\left[\frac{-\sin(\omega t_2 x_1 + \sin(\omega t_1) x_2)}{\sqrt{x_1^2 - 2 \cos(\omega (t_1 - t_2))x_1 x_2}}\right] \]  

1.4 Energy of a Harmonic Oscillator

The kinetic and potential energies of the harmonic oscillator can be written as
\[ KE[t] = \frac{1}{2} m v[t]^2 = \frac{1}{2} A^2 m \omega^2 \sin(\omega t + \phi)^2 \]  
\[ PE[t] = \frac{1}{2} k x[t]^2 = \frac{1}{2} A^2 k \cos(\omega t + \phi)^2 \]  

Both are explicit functions of time. However, the total energy becomes
Using the identity, \( \omega^2 = \frac{k}{m} \)

\[
E(t) = \frac{1}{2} A^2 m \omega^2 \cos(\omega t + \phi) + \frac{1}{2} A^2 m \omega^2 \sin(\omega t + \phi)^2
\]

or

\[
E(t) = \frac{A^2 k}{2}
\]

This is exactly what one would expect. The total energy is conserved and therefore should be a constant independent of time. Furthermore, the total energy can be expressed as the potential energy at its maximum extent of travel, \( x_{\text{max}} = |A| \) when the kinetic energy is zero or the kinetic energy at the minimum of the potential (where the potential energy is zero) when the velocity takes on its maximum value, \( v_{\text{max}} = \omega x_{\text{max}} \).

1.5 Time Averages

Although the total energy is time independent, other quantities are time dependent. Therefore it makes sense to define time average quantities. Because the motion is periodic, this averaging can be done over a single period. For any time dependent function, \( F(t) \), the average is therefore defined as

\[
\overline{F} = \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} F(t') dt'
\]

where \( \tau = 2\pi/\omega = 1/f \) is the period and \( t_0 \) is an arbitrary time. For most functions, the average will not dependent on the choice of \( t_0 \).

For example, the average position becomes

\[
\overline{x} = \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} x(t') dt' = \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} A \cos(\phi + \omega t') dt'
\]

\[
= \frac{1}{\tau} \frac{A(-\sin(\phi + \omega t_0) + \sin(\phi + \omega(\tau + t_0)))}{\omega}
\]

But \( \omega \tau = 2\pi \)
\[ \bar{x} = \frac{A(Sin[\phi + \omega t_0 + 2\pi] - Sin[\phi + \omega t_0])}{2\pi} \]
\[ = 0 \quad (1.45) \]

This make sense. The harmonic oscillator spends an equal amount of time to the left and right of zero (see graph).

A more interesting average would be \( x^2 \) because \( x^2 \) is always positive.

\[ \bar{x^2} = \frac{1}{\tau} \int_{t_0}^{\tau + t_0} x'[t']^2 \, dt' = \frac{1}{\tau} \int_{t_0}^{\tau + t_0} A^2 \cos[\phi + \omega t']^2 \, dt' \]
\[ = \frac{1}{\tau} \int_{t_0}^{\tau + t_0} \frac{1}{2} A^2 (1 + \cos[2\phi + 2\omega t']) \, dt' \]
\[ = \frac{1}{\tau} A^2 \left( 2\tau \omega - \sin[2(\phi + \omega t_0)] + \sin[2(\phi + \omega(\tau + t_0))]) \right) \]
\[ = \frac{A^2}{2} \quad (1.46) \]

Again setting \( \omega \tau = 2\pi \)
\[ = \frac{A^2}{2} \quad (1.47) \]

This can be seen graphically.
We can immediately use this result to determine the average potential energy. Because the potential energy can be written as

\[ PE(t) = \frac{kx^2}{2} \]  

we immediately have that

\[ \bar{PE} = \frac{k\bar{x}^2}{2} = \frac{A^2 k}{4} = \frac{E_{Total}}{2} \]  

We can also see that because

\[ E_{Total} = KE + PE = \frac{A^2 k}{2} \]  

we also have that

\[ \bar{KE} = -\bar{PE} + \frac{E_{Total}}{2} = \frac{A^2 k}{4} = \frac{E_{Total}}{2} \]  

and therefore

\[ \bar{KE} = \frac{m\bar{v}^2}{2} = \frac{A^2 k}{4} = \frac{1}{4} \frac{A^2 m \omega^2}{4} \]  

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