Chapter 10

Interference and Diffraction

10.1 Interference

10.1.1 Two Slit Interference Pattern

We have already seen examples of the interference between two waves. Given two counter propagating waves of the same frequency and wavelength on a string, there exist fixed position that have zero amplitudes for all time.

\[
y[x, t] = A \sin[k x - \omega t] + A \sin[k x + \omega t] \\
= 2A \cos[\omega t] \sin[k x]
\]  

(10.1)

The relative phase of the two waves at the node positions will odd multiples of \( \pi \) (180° out of phase) whereas the positions of the antinodes correspond to relative phases that are multiples of \( 2\pi \) (0° phase shift).

If we now consider two sources of waves embedded in 2 or 3 dimensions, interference can only occur if the two waves are coherent; the relative phase between the waves of the two source must be fixed for all time. An easy way to accomplish this is to have two slits in a screen that is illuminated by a single plane wave source.
Consider two rays that reach a point on a distant screen from the two sources
To determine the relative phase between the two rays that reach the screen at point C, one can use the law of Cosines to determine the distance that each ray has traveled. Noting that \( \cos \alpha = \cos (90 - \vartheta) = \sin \vartheta \) and \( \cos \beta = \cos (90 + \vartheta) = -\sin \vartheta \)

\[
\begin{align*}
    r_1 &= \sqrt{\left( r^2 + \frac{d^2}{4} - 2 \left( \frac{d}{2} \right) r \cos \alpha \right)} = \sqrt{\left( r^2 + \frac{d^2}{4} - d r \sin \vartheta \right)} \\
    r_2 &= \sqrt{\left( r^2 + \frac{d^2}{4} - 2 \left( \frac{d}{2} \right) r \cos \beta \right)} = \sqrt{\left( \frac{d^2}{4} + r^2 + d r \sin \vartheta \right)}
\end{align*}
\]  

(10.2)

For \( D \gg d \), the above expressions can be expanded in a Taylor series in \( d \)

\[
\begin{align*}
    r_1 &= r - \frac{1}{2} \sin \vartheta \, d + \frac{(r - r \sin \vartheta)^2}{8 \, r^2} + O(d^3) \\
    r_2 &= r + \frac{1}{2} \sin \vartheta \, d + \frac{(r - r \sin \vartheta)^2}{8 \, r^2} + O(d^3)
\end{align*}
\]  

(10.3)
To lowest order in $d$, the difference between the paths is given by

$$\Delta r = r_2 - r_1 = d \sin[\theta]$$

(10.4)

If the two waves are in phase at the sources, then each wave will accumulate a phase given by $2\pi$ times the number of wavelengths contained within the corresponding path

$$\Delta \phi_1 = \frac{2\pi r_1}{\lambda}, \quad \Delta \phi_2 = \frac{2\pi r_2}{\lambda}$$

(10.5)

so that the phase difference becomes

$$\Delta \phi = \frac{2\pi (r_2 - r_1)}{\lambda} = \frac{2\pi d \sin[\theta]}{\lambda}$$

(10.6)

If $\Delta \phi = 0, \pm 2\pi, \pm 4\pi \ldots = m(2\pi)$, the two waves will interfere constructively and if $\Delta \phi = \pm\pi, \pm 3\pi \ldots = (2m + 1)\pi$, then the two waves will interfere destructively. Therefore, one obtains

Constructive Interference whenever $d \sin[\theta] = m \lambda$

Destructive Interference whenever $d \sin[\theta] = \frac{1}{2} (1 + 2m) \lambda = \left(\frac{1}{2} + m\right) \lambda$

The intensity at positions with constructive interference should be large (or bright) and at position of destructive interference should be zero (or dark).

To see such effects, it is necessary that $d$ should be comparable to the wavelength. For example

If $\lambda = 5000 \text{\,Å}$ and $d = 1 \mu\text{m}$, then the first non-central peak (or maximum) corresponding to $m = 1$ occurs at

$$\sin[\theta] = \frac{\lambda}{d} = \frac{5000 \text{\,Å}}{1 \mu\text{m}} = \frac{5 \times 10^{-7} \text{\,m}}{1 \times 10^{-6} \text{\,m}} = 2 \quad \text{or} \quad \theta = \frac{\pi}{6} \text{ or } 30^\circ$$

(10.7)

If the screen is at a distance of 1 meter, then the distance between the center of this peak and the central peak is

$$\Delta x = D \tan[\theta] = 1 \text{m} \tan\left[\frac{\pi}{6}\right] = 0.57735 \text{m}$$

(10.8)

If however, $\lambda = 5000 \text{\,Å}$ and $d = 0.5 \text{mm}$, then the first non-central peak (or maximum) corresponding to $m = 1$ occurs at

$$\sin[\theta] = \frac{\lambda}{d} = \frac{5000 \text{\,Å}}{0.5 \text{\,mm}} = \frac{5 \times 10^{-7} \text{\,m}}{5 \times 10^{-4} \text{\,m}} = \frac{1}{1000} \quad \text{or} \quad \theta = 0.001$$

(10.9)

and the distance between the center of this peak and the central peak is
\[ \Delta x = D \tan[\vartheta] = 1 \, m \tan\left[ \frac{1}{1000} \right] = 0.001 \, m \quad (10.10) \]

In this example of a large separation one would not be able to resolve the separate peaks and the result would appear to be a single spot of light.

10.1.2 Fermat’s Principle

One can improve the analysis a bit by introducing a lens between the screen and the sources. Consider a point source of light placed at the focus, \( F_1 \), of a converging lens.

The rays from the point source are refracted by the first lens into parallel rays. The second lens then focuses these rays at its focus, \( F_2 \). Although ray \( A \) travels a larger distance in air than the ray \( B \), ray \( B \) travels larger distance in the two lenses. Because the propagation velocity is slower within the glass of the lenses, it is conceivable that the two rays actually take the same time in going from \( F_1 \) to \( F_2 \). This is a statement of Fermat’s Principle

Light traveling from one point to another will follow a path such that when compared to other paths, the time required to transverse the path will either be a minimum, a maximum or remain the same.

In the above figure, all rays take the same amount of time to transverse their separate paths.

If the waves in each ray leave \( F_1 \) in phase, they will arrive at \( F_2 \) in phase. This enables one to simplify the analysis of the 2-slit interference pattern. By placing a lens in front of the slits with the screen at the focal plane of the lens, one need only consider parallel rays that leave the slits. These rays will all be focused at the same point on the screen. Moreover, from our previous discussion, we see that the two rays will accumulate the same phase from a point perpendicular to the rays.
The two rays accumulate the same phase starting from the points $A$ and $B$ to the focus at $C$. The overall phase difference between two rays is due to the difference in the path length which is just $d \sin \vartheta$.

10.1.3 Intensity of the Double Slit Pattern

To determine the variation of the intensity on the screen as a function of $\vartheta$, the sum of the electric fields from the two rays. For convenience, assume that the electric field points perpendicular to the paper ($z$-axis). The electric fields leave the slits in phase. As they travel to the screen, they accumulate a phase related to the total distance traveled.
The sum of the electric fields is
\[
E_{\text{total}} = E_1 + E_2
\]
\[
= E_0 \cos(kr_1 - \omega t) \hat{z} + E_0 \cos(kr_2 - \omega t) \hat{z}
\]
\[
= 2E_0 \cos\left( \frac{1}{2}(r_1 + r_2) - \omega t \right) \cos\left( \frac{1}{2}k(r_1 - r_2) \right) \hat{z}
\]
(10.11)

Corresponding to this electric field, the magnetic field is
\[
B_{\text{total}} = \frac{2E_0}{c} \cos\left( \frac{1}{2}(r_1 + r_2) - \omega t \right) \cos\left( \frac{1}{2}k(r_1 - r_2) \right) \hat{y}
\]
(10.12)

(Note: because the spacing between the slits is small and the screen is a large distance away, we can assume that both magnetic fields point in the same direction.)

The intensity at the screen is determined by the time average of the Poynting vector
\[
I[\theta] = \overline{\mathbf{S}} = \frac{(E \times B)}{\mu_0} = \frac{4E_0^2}{c\mu_0} \cos\left( \frac{1}{2}k(r_1 + r_2) - \omega t \right)^2 \cos\left( \frac{1}{2}k(r_1 - r_2) \right)^2
\]
(10.13)

but \((r_1 - r_2) = d \sin[\theta]\) and \(k = 2\pi/\lambda\) so that
\[
I[\theta] = \frac{2E_0^2}{c\mu_0} \cos\left( \frac{\pi d \sin[\theta]}{\lambda} \right)^2
\]
(10.14)

At \(\theta = 0\), the intensity is \(I[\theta = 0] = \frac{2E_0^2}{\mu_0 c}\). The general form for the intensity as a function of \(\theta\) is
\[
I[\theta] = I[0] \cos\left( \frac{\pi d \sin[\theta]}{\lambda} \right)^2
\]
(10.15)
Plotting the intensity as a function of $\varphi$ and $\frac{\chi}{D} = \tan(\varphi)$ we obtain

$$\frac{d}{\lambda} = 2.32$$

$$\frac{\lambda}{d} = 0.431034$$

10.2 Interference Due To Differences in Optical Path Length

In some problems, the interfering rays travel through media of different indices of refraction. Consider the two rays shown. Over the physical length $L$, the two rays travel through media of indices $n_1$ and $n_2$. 

Although the two waves enter with the same phase, they accumulate different phases because now within the length $L$, the two waves have different wavelengths. The phase difference is given by

$$\Delta \phi = \frac{2 L \pi}{\lambda_1} - \frac{2 L \pi}{\lambda_2}$$

(10.16)

where $\lambda_1 = \frac{v_1}{c} = \frac{\nu_1}{c} = \frac{\lambda_{\text{vac}}}{n_1}$ and $\lambda_2 = \frac{v_2}{c} = \frac{\nu_2}{c} = \frac{\lambda_{\text{vac}}}{n_2}$, i.e. the wavelengths inside the material is smaller. Using these definitions, the phase difference is

$$\Delta \phi = \frac{2 L \pi n_1}{\lambda_{\text{vac}}} - \frac{2 L \pi n_2}{\lambda_{\text{vac}}} = \frac{2 \pi (L n_1 - L n_2)}{\lambda_{\text{vac}}}$$

(10.17)

The product of the physical length, $L$, and the appropriate index of refraction is called the optical path length. As before

If $\Delta \phi = 2 m \pi \ (m = 0 \pm 1 \pm 2...)$ or $(L n_1 - L n_2) = m \lambda_{\text{vac}}$ then the two waves will interfere constructively.
If $\Delta \phi = (1 + 2m)\pi \ (m = 0 \pm 1 \pm 2 \ldots)$ or $(L n_1 - L n_2) = (1 + 2m)\lambda_{vac}$ then the two waves will interfere destructively.

10.2.1 Soap Bubbles

Consider the interference between two rays reflecting from the front and back interfaces of a thin film.

The rays are drawn at an angle (away from normal incidence) for clarity. At the first surface, the incident ray is both reflected and refracted. The transmitted ray is also partially reflected at the second surface. The ray reflected from the second surface will be refracted through the first surface where it combines with the original reflected ray, $A$. (In principle, there will be additional reflections as each ray encounters an interface.)

Ray $B$ will no longer be in phase with ray $A$ because it has traveled a different distance. The phase difference should be

$$\Delta \phi = \frac{2\pi}{\lambda_{vac}} \left( 2d n_{vac} - 0 \right)$$

because ray $B$ travels a distance $2d$ in a medium with an index of refraction, $n$, whereas ray $A$ does not.

Based on this calculation, one would expect that as $d \to 0$, $\Delta \phi \to 0$ and there would be no phase shift between the two rays. Consequently, the two rays interfere constructively and one would expect to always see a strong reflection from very thin films such as found in a soap bubble. Experimentally, soap bubbles become transparent as the soap film becomes thinner.

To resolve this problem, one must take into account the discrete phase shift associated with a reflection. From the analysis of a pulse on a string, we found that the phase shift upon reflection depended on the relative velocity of propagation.
If a wave goes from a medium with a high propagation speed to a medium with a low propagation speed, there will be a $\pi$ phase shift. If the wave goes from a medium with a low propagation speed to one with a high propagation speed then the reflected waves have no phase shift.

- fast $\rightarrow$ slow  $\pi$ phase shift upon reflection
- slow $\rightarrow$ fast  0 phase shift upon reflection

In either case, the transmitted wave has no phase shift at the interface.

Returning to the previous diagram, if the index of refraction changes from $n = 1$ to $n > 1$, ray A picks up a phase shift of $\pi$ upon reflecting from the first interface.
Ray $B$ does not pick up a phase shift upon reflection because now the velocity of light goes from slow to fast at the second interface. The overall phase shift between the two rays is then

$$\Delta \phi = \frac{2 \pi}{\lambda_{\text{vac}}} \left( 2 d n_{\text{vac}} - 0 \right) - \pi = \frac{4 d n \pi}{\lambda_{\text{vac}}} - \pi$$

(10.19)

If $\frac{4 d n \pi}{\lambda_{\text{vac}}} - \pi = 2 \pi m \ (m = 0 \pm 1 \pm 2 \ldots)$ or $d = \left( \frac{1 + 2 m}{4 n} \right) \lambda_{\text{vac}}$ then the two waves will interfere constructively.

If $\frac{4 d n \pi}{\lambda_{\text{vac}}} - \pi = (1 + 2 m) \pi \ (m = 0 \pm 1 \pm 2 \ldots)$ or $d = \left( \frac{1 + m}{2 n} \right) \lambda_{\text{vac}}$ then the two waves will interfere destructively.

Therefore as $d \to 0$, the relative phase shift approaches $-\pi$ and the two rays interfere destructively and not reflection is possible. This explains the “black” specks that appear on a soap bubble just before it bursts.

Consider $\lambda = 5320$ Å and $n = 1.33$ (water).

Constructive interference or bright bands will occur at

$$d = \frac{1 + 2 m}{4 n} \lambda_{\text{vac}}$$

(10.20)

$$m = 0 \quad d = \frac{1}{4 n} \lambda_{\text{vac}} = 1001.88 \text{ Å}$$

(10.21)

$$m = 1 \quad d = \frac{3}{4 n} \lambda_{\text{vac}} = 3005.64 \text{ Å}$$

$$m = 2 \quad d = \frac{5}{4 n} \lambda_{\text{vac}} = 5009.4 \text{ Å}$$
Destructive interference or dark bands will occur at

\[ d = \frac{1 + m}{2n} \lambda_{\text{vac}} \]  \hspace{1cm} (10.22)

\[ m = 0 \quad d = \frac{1}{2n} \lambda_{\text{vac}} = 2003.76 \text{ Å} \]

\[ m = 1 \quad d = \frac{1}{n} \lambda_{\text{vac}} = 4007.52 \text{ Å} \]  \hspace{1cm} (10.23)

\[ m = 2 \quad d = \frac{3}{2n} \lambda_{\text{vac}} = 6011.28 \text{ Å} \]

As the film thickens from the \( d = 0 \) band, the first color should correspond to the smallest visible wavelength, i.e. blue. As the film thickens, blue will have a dark band whereas red will have a bright band. When observed with white light, each wavelength will have its own bright and dark bands at different thicknesses. Consequently, the film will appear to have numerous bands of color.

(*

The 1931 CIE colour matching functions and standard illuminant data in the format: \{wavelength(nm),x-bar,y-bar,z-bar, D65 standard illuminant\}*)

ciedata = {{380, 0.00137, 0.00004, 0.00645, 49.97550},
{385, 0.00224, 0.00006, 0.01055, 52.31180},
{390, 0.00424, 0.00012, 0.02005, 54.64820},
{395, 0.00765, 0.00022, 0.03621, 68.70150},
{400, 0.01431, 0.00040, 0.06785, 82.75490},
{405, 0.02319, 0.00064, 0.11020, 87.12040},
{410, 0.04351, 0.00121, 0.20740, 91.48600},
{415, 0.07763, 0.00218, 0.37130, 92.45890},
{420, 0.13438, 0.00400, 0.64560, 93.43180},
{425, 0.21477, 0.00730, 1.03905, 90.05700},
{430, 0.28390, 0.01160, 1.38560, 86.68230},
{435, 0.32850, 0.01684, 1.62296, 95.77360},
{440, 0.34828, 0.02300, 1.74706, 104.86500},
{445, 0.34806, 0.02980, 1.78260, 110.93600},
{450, 0.33620, 0.03800, 1.77211, 117.00800},
{455, 0.31870, 0.04800, 1.74410, 117.41000},
{460, 0.29080, 0.06000, 1.66920, 117.81200},
{465, 0.25110, 0.07390, 1.52810, 116.33600},
{470, 0.19536, 0.09098, 1.28764, 114.86100},
{475, 0.14210, 0.11260, 1.04190, 115.39200},}
{480, 0.09564, 0.13902, 0.81295, 115.92300},
{485, 0.05795, 0.16930, 0.61620, 112.36700},
{490, 0.03201, 0.20802, 0.46518, 108.81100},
{495, 0.01470, 0.25860, 0.35330, 109.08200},
{500, 0.00490, 0.32300, 0.27200, 109.35400},
{505, 0.00240, 0.40730, 0.21230, 108.57800},
{510, 0.00930, 0.50300, 0.15820, 107.80200},
{515, 0.02910, 0.60820, 0.11170, 106.29600},
{520, 0.06327, 0.71000, 0.07825, 104.79000},
{525, 0.10960, 0.79320, 0.05725, 106.23900},
{530, 0.16550, 0.86200, 0.04216, 107.68900},
{535, 0.22575, 0.91485, 0.02984, 107.68900},
{540, 0.29040, 0.95400, 0.02030, 104.40500},
{545, 0.35970, 0.99495, 0.01340, 104.22500},
{550, 0.43345, 0.99495, 0.00875, 104.04600},
{555, 0.51205, 1.00000, 0.00575, 102.02300},
{560, 0.59450, 0.99500, 0.00390, 100.00000},
{565, 0.67840, 0.97860, 0.00275, 98.16710},
{570, 0.76210, 0.95200, 0.00210, 96.33420},
{575, 0.84250, 0.91540, 0.00180, 96.06110},
{580, 0.91630, 0.87000, 0.00165, 95.78800},
{585, 0.97860, 0.81630, 0.00140, 92.23680},
{590, 1.02630, 0.75700, 0.00110, 88.68560},
{595, 1.05670, 0.69490, 0.00100, 89.34590},
{600, 1.06220, 0.63100, 0.00080, 90.00620},
{605, 1.04560, 0.56680, 0.00060, 89.80260},
{610, 1.00260, 0.50300, 0.00034, 89.59910},
{615, 0.93840, 0.44120, 0.00024, 88.64890},
{620, 0.85445, 0.38100, 0.00019, 87.69870},
{625, 0.75140, 0.32100, 0.00010, 85.49360},
{630, 0.64240, 0.26500, 0.00005, 83.28860},
{635, 0.54190, 0.21700, 0.00003, 83.49390},
{640, 0.44790, 0.17500, 0.00002, 83.69920},
{645, 0.36080, 0.13820, 0.00001, 81.86300},
{650, 0.28350, 0.10700, 0.00000, 80.02680},
{655, 0.21870, 0.08160, 0.00000, 80.12070},
{660, 0.16490, 0.06100, 0.00000, 80.21460},
{665, 0.12120, 0.04458, 0.00000, 81.24620},
{670, 0.08740, 0.03200, 0.00000, 82.77800},
{675, 0.06360, 0.02320, 0.00000, 80.28100},
{680, 0.04677, 0.01700, 0.00000, 78.28420},
\{685, 0.03290, 0.01192, 0.00000, 74.00270\},
\{690, 0.02270, 0.00821, 0.00000, 69.72130\},
\{695, 0.01584, 0.00572, 0.00000, 70.66520\},
\{700, 0.01136, 0.00410, 0.00000, 71.60910\},
\{705, 0.00811, 0.00293, 0.00000, 72.97900\},
\{710, 0.00579, 0.00209, 0.00000, 74.34900\},
\{715, 0.00411, 0.00148, 0.00000, 67.97650\},
\{720, 0.00290, 0.00105, 0.00000, 61.60400\},
\{725, 0.00205, 0.00074, 0.00000, 65.74480\},
\{730, 0.00144, 0.00052, 0.00000, 69.88560\},
\{735, 0.00100, 0.00036, 0.00000, 72.48630\},
\{740, 0.00069, 0.00025, 0.00000, 75.08700\},
\{745, 0.00048, 0.00017, 0.00000, 69.33980\},
\{750, 0.00033, 0.00012, 0.00000, 63.59270\},
\{755, 0.00023, 0.00008, 0.00000, 55.00540\},
\{760, 0.00017, 0.00006, 0.00000, 46.41820\},
\{765, 0.00012, 0.00004, 0.00000, 56.6180\},
\{770, 0.00008, 0.00003, 0.00000, 66.80540\},
\{775, 0.00006, 0.00002, 0.00000, 65.09410\},
\{780, 0.00004, 0.00001, 0.00000, 63.38280\};

(*Transmission/reflection function describing interference phenomena*)
Transmission[retardation_, _\_\_] := (1 - Cos[2 \* Pi \* retardation / \_\_\_]) / 2;
(*nonlinear rotation matrix to convert CIE XYZ coords to RGB(linear)*)
M = {{3.2406, -1.5372, -0.4986}, {-0.9689, 1.8758, .0415},
    {0.557, -.204, 1.057}};
cal = {Total[ciedata[[All, 2]] \* ciedata[[All, 5]]],
        Total[ciedata[[All, 3]] \* ciedata[[All, 5]]],
        Total[ciedata[[All, 4]] \* ciedata[[All, 5]]]};
(*Gamma correction set at \_\_\_ = 2.2 (standard computer monitor)*)
gc[x_] := Clip[If[x \leq 0.0031308, 12.92 \* x, -.055 + 1.055 \*(x^{0.4})],
                {0, 1}];
(*definition for the value of a colour channel = \int I(\_\_)\_\_channel_bar(\_\_)\_\_\_Illuminant(\_\_) d\_\_\_ *)
chan[n_, ret_] :=
    Total[Transmission[ret, ciedata[[All, 1]]] \* ciedata[[All, n]] \* ciedata[[All, 5]] / cal[[n - 1]]];
XYZ[ret_] := {chan[2, ret], chan[3, ret], chan[4, ret]}
Fprint[{Rotate[Plot[\{x / 50, -x / 50\}, \{x, 0, 2000\}, Frame \rightarrow True,
RotateLabel \rightarrow True, \text{PlotRange} \rightarrow \{\{-50, 2000\}, \{-40, 150\}\}, \\
FrameTicks \rightarrow \text{None}, \\
Axes \rightarrow \{\text{False, False}\}, \\
AspectRatio \rightarrow .30, \text{Background} \rightarrow \text{Black}, \\
\text{ColorFunction} \rightarrow \\
\quad \text{Function}[[x, y], \text{RGBColor}[\text{Map}[gc, M.XYZ[2000 x]]]], \\
\text{Filling} \rightarrow \text{Axis}, \text{ImageSize} \rightarrow \{400, 120\}, \\
\text{Epilog} \rightarrow \{\text{White, Arrowheads}[0.02], \text{Arrow}[[\{0, 150\}, \{0, 0\}]], \\
\quad \text{Style}[\text{Text}[d = 0, \{20, 30\}, \{-1, 1\}, \{0, 1\}], 12], \\
\quad \text{Arrow}[[\{500, 150\}, \{500, 10\}]], \\
\quad \text{Style}[\text{Text}[d = 1000 \text{ Å}, \{520, 40\}, \{-1, 1\}, \{0, 1\}], 12], \\
\quad \text{Arrow}[[\{1000, 150\}, \{1000, 20\}]], \\
\quad \text{Style}[\text{Text}[d = 2000 \text{ Å}, \{1020, 40\}, \{-1, 1\}, \{0, 1\}], 12], \\
\quad \text{Arrow}[[\{1500, 150\}, \{1500, 30\}]], \\
\quad \text{Style}[\text{Text}[d = 3000 \text{ Å}, \{1520, 40\}, \{-1, 1\}, \{0, 1\}], 12] \\
\}], -\text{Pi}/2])}
10.2.2 Non-Reflecting Coatings

On some optical elements, a coating is added to suppress reflections. Consider a coating of MgF$_2$ on glass. As before we can look at the interference between the reflections from the first and second surfaces.

Upon reflection ray $A$ has a $\pi$ phase shift because the velocity of light in air is faster than that in the MgF$_2$ layer. Similarly ray $B$ will also obtain a $\pi$ phase shift because the velocity of light is faster in the MgF$_2$ layer than in the glass. The overall phase shift is then

$$\Delta \phi = \Delta \phi_B - \Delta \phi_A = \frac{2\pi}{\lambda_{\text{vac}}} \left( 2 \frac{d}{\lambda_{\text{vac}}} + \pi \right) - \pi$$

$$= \frac{4 d n \pi}{\lambda_{\text{vac}}}$$  \hspace{1cm} (10.24)

In order for there to be no reflections, rays $A$ and $B$ must interfere destructively. The phase difference is therefore

$$\Delta \phi = (1 + 2m) \pi$$  \hspace{1cm} (10.25)

or

$$\frac{4 d n \pi}{\lambda_{\text{vac}}} = (1 + 2m) \pi$$  \hspace{1cm} (10.26)

or

$$d = \frac{(1 + 2m) \lambda_{\text{vac}}}{4 n}$$  \hspace{1cm} (10.27)

Therefore the thinnest coating will correspond to $m = 0$ or $d_{\text{min}} = \frac{\lambda_{\text{vac}}}{4 n}$.
10.3 Multiple Slit Interference Patterns

10.3.1 Intensity of Multiple Slit Pattern

Consider the interference for multiple slits. As before we introduce a lens between the slits and the screen to simplify the analysis.

Assuming that each wave starts at the same phase, the phase of each wave at the point \( C \) is determined by the total length that it propagates from the slits to the screen. Noting that each successive wave travels an additional distance given by \( \Delta L = d \sin[\theta] \), the electric field from each slit can be expressed as

\[
\begin{align*}
E_1 &= E_0 \cos[kr - \omega t] = \text{Re} \left[ E_0 e^{i(kr - \omega t)} \right] \\
E_2 &= \text{Re} \left[ E_0 e^{i(k(r+\Delta L) - \omega t)} \right] \\
E_3 &= \text{Re} \left[ E_0 e^{i(k(r+2\Delta L) - \omega t)} \right] \\
E_4 &= \text{Re} \left[ E_0 e^{i(k(r+3\Delta L) - \omega t)} \right] \\
&\vdots \\
E_N &= \text{Re} \left[ E_0 e^{i(k(N-1)\Delta L) - \omega t)} \right]
\end{align*}
\]  

(10.28)

The total electric field at the point \( C \) is then

\[
\begin{align*}
E_{\text{total}} &= \text{Re} \left[ E_0 e^{i(kr - \omega t)} + E_0 e^{i(k(r+\Delta L) - \omega t)} + \ldots + E_0 e^{i(k(N-1)\Delta L) - \omega t)} \right] \\
&= \text{Re} \left[ E_0 e^{i(kr - \omega t)} \left( 1 + e^{i \Delta L} \ldots + e^{i(N-1) \Delta L} \right) \right]
\end{align*}
\]  

(10.29)
This finite series can easily be summed. Taking a finite geometric series

\[ S = 1 + x + x^2 + x^3 + \ldots x^{1+N} \]

\[ S x = x + x^2 + x^3 + \ldots x^N \]

Subtracting the second series from the first

\[ S (1 - x) = 1 - x^N \]

or

\[ S = \frac{1 - x^N}{1 - x} \]

Applying this result to our series

\[ E_{\text{total}} = \Re \left( E_0 e^{i(kr - \omega t)} \left( 1 - e^{i k N \Delta L} \right) \right) \]

\[ = \Re \left[ E_0 e^{i[kr - \omega t + \frac{1}{2} k \Delta L (N-1)]} \left( \frac{e^{-\frac{1}{2} i k N \Delta L} - e^{\frac{1}{2} i k N \Delta L}}{e^{-\frac{1}{2} i k \Delta L} - e^{\frac{1}{2} i k \Delta L}} \right) \right] \]

\[ = E_0 \cos \left( k r - \omega t + \frac{1}{2} k \Delta L (N-1) \right) \left( \frac{\sin \left( \frac{N k \Delta L}{2} \right)}{\sin \left( \frac{k \Delta L}{2} \right)} \right) \]

The corresponding magnetic field is

\[ B_{\text{total}} = \frac{E_0}{c} \cos \left( k r - \omega t + \frac{1}{2} (N-1) k \Delta L \right) \left( \frac{\sin \left( \frac{N k \Delta L}{2} \right)}{\sin \left( \frac{k \Delta L}{2} \right)} \right) \]

The intensity is therefore

\[ I[\delta] = \mathcal{S} = \frac{(E \times B)}{\mu_0} = \frac{E_0^2}{c \mu_0} \cos \left( k r - \omega t + \frac{1}{2} (N-1) k \Delta L \right)^2 \left( \frac{\sin \left( \frac{N k \Delta L}{2} \right)}{\sin \left( \frac{k \Delta L}{2} \right)} \right)^2 \]

\[ = \frac{E_0^2}{2 c \mu_0} \left( \frac{\sin \left( \frac{N k \Delta L}{2} \right)}{\sin \left( \frac{k \Delta L}{2} \right)} \right)^2 \]
\[
\frac{E_0^2}{2 \, c \, \mu_0} \left( \frac{\sin \left( \frac{N \pi d \sin[\theta]}{\lambda} \right)}{\sin \left( \frac{\pi d \sin[\theta]}{\lambda} \right)} \right)^2
\]

where \( k = 2 \pi / \lambda \) has been substituted. This expression can be simplified by looking at the \( \theta = 0 \) limit.

In the limit as \( \theta \to 0 \), both the numerator and denominator vanish. To evaluate the expression, one can use L’Hospital’s rule or the Taylor expansion. For small \( \theta \), \( \frac{\pi d \sin[\theta]}{\lambda} \) will be small and

\[
\sin \left( \frac{d N \pi \sin[\theta]}{\lambda} \right) \to \frac{d N \pi \sin[\theta]}{\lambda} \quad \text{and} \quad \sin \left( \frac{d \pi \sin[\theta]}{\lambda} \right) \to \frac{d \pi \sin[\theta]}{\lambda}
\]

Substituting

\[
I[\theta = 0] \to \frac{E_0^2}{2 \, c \, \mu_0} \, N^2
\]

and

\[
I[\theta] = \frac{I[\theta]}{N^2} \left( \frac{\sin \left( \frac{N \pi d \sin[\theta]}{\lambda} \right)}{\sin \left( \frac{\pi d \sin[\theta]}{\lambda} \right)} \right)^2 = \frac{I[\theta]}{N^2} \left( \frac{\sin[\alpha]}{\sin[\alpha]} \right)^2
\]

where \( \alpha = \pi d \sin[\theta] / \lambda \)

### 10.3.2 Special Case, \( N = 2 \)

Substituting \( N = 2 \)

\[
I[\theta]_{N=2} = \frac{I[\theta]}{4} \left( \frac{\sin[2 \, \alpha]}{\sin[\alpha]} \right)^2 = \frac{I[\theta]}{4} \left( \frac{2 \sin[\alpha] \cos[\alpha]}{\sin[\alpha]} \right)^2 = I[\theta] \, \cos[\alpha]^2
\]

which is just our previous result for the double slit.

### 10.3.3 Exploring \( N = 4 \)

To understand the dependence of the intensity as a function of \( \theta \) consider the case, \( N = 4 \).

\[
I[\theta]_{N=4} = \frac{I[\theta]}{16} \left( \frac{\sin[4 \, \alpha]}{\sin[\alpha]} \right)^2
\]

At \( \theta = 0 \) (\( \alpha = 0 \)), the intensity is a maximum with \( I[\theta = 0] = I[0] \). As \( \theta \) increases so does \( \alpha \), but the argument of the Sine function in the numerator increases more rapidly. Consequently the numer-
ator will vanish at $\alpha = \pi / 4$ whereas the denominator will have a non-zero value. The numerator will also vanish at $\pi / 2$ and $3\pi / 4$. When $\alpha = \pi$, both numerator and denominator will vanish. To reduce the expression, use L'Hospital's rule

$$\lim_{\alpha \to 4} \left( \frac{\sin[4\alpha]}{\sin[\alpha]} \right)^2 = \lim_{\alpha \to 4} \left( \frac{\partial \sin[4\alpha]}{\partial \alpha} \right)^2 = \lim_{\alpha \to 4} \left( \frac{4 \cos[4\alpha]}{\cos[\alpha]} \right)^2 = 4$$

(10.41)

Therefore at $\alpha = \pi$, $I[\varnothing = \pi] = I[0]$ i.e. another maximum. As a function of $\alpha$, one has

zeros at $\alpha = \pm \pi / 4, \pm \pi / 2, \pm 3\pi / 4, \pm 5\pi / 4 \ldots$

and

maxima at $\alpha = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \pm 4\pi \ldots$

Given that there are 3 minima between $\alpha = 0$ and $\pi$, there should be 2 additional maxima in between the minima as well. However, they will not be as large as the principle maxima because although $\sin[4\alpha] = 1$ and $\sin[\alpha] < 1$, there is the overall factor of 1/16. For example at $\alpha = 3\pi / 8$

$$I\left[\frac{\pi}{4}\right]_{N=4} = \frac{I[0]}{16} \left( \frac{\sin[\frac{4(3\pi)}{8}]}{\sin[\frac{3\pi}{8}]} \right)^2 = 0.0732233 \ I[0.]$$

(10.42)

Plotting the intensity as a function of $\varnothing$, $\sin[\varnothing]$ and $\alpha = \pi d \sin[\varnothing] / \lambda$
10.3.4 Resolution

Note that the condition for a principal maximum is the same as that for a double slit interference pattern, \( d \sin[\vartheta] = m \lambda \). In general all N-slit interference patterns have their principle maxima in the same position. As N increase the principle maxima become sharper as more secondary maxima are introduced.
The widths of the principle peaks is related to the distance between the maximum position and the position of the subsequent zero. The location of the \( m \)th order peaks is \( \alpha = m \pi \) or \( N \alpha = N m \pi \). The next minimum occurs when \( N \alpha \) increments by \( \pi \), i.e. \( \alpha = N M \pi + \pi \). Expressing \( \alpha = \pi d \sin[\theta] / \lambda \) and solving for \( \sin[\theta] \)

\[
N \alpha_{\text{peak}} = \frac{d N \pi \sin[\theta_{\text{peak}}]}{\lambda} = m N \pi \quad \text{or} \quad \sin[\theta_{\text{peak}}] = \frac{m \lambda}{d}
\]  

(10.43)

Similarly

\[
N \alpha_{\text{min}} = \frac{d N \pi \sin[\theta_{\text{min}}]}{\lambda} = \pi + m N \pi \quad \text{or} \quad \sin[\theta_{\text{min}}] = \frac{m \lambda}{d} + \frac{\lambda}{d N}
\]  

(10.44)

For large \( N \), one would expect that \( \theta_{\text{min}} = \theta_{\text{peak}} + \Delta \theta \) where \( \Delta \theta \) is small. Substituting \( \theta_{\text{min}} = \theta_{\text{peak}} + \Delta \theta \) into the above expression and expanding

\[
\sin[\Delta \theta + \theta_{\text{peak}}] = \sin[\theta_{\text{peak}}] + \cos[\theta_{\text{peak}}] \Delta \theta + O[\Delta \theta]^2 = \frac{m \lambda}{d} + \frac{\lambda}{d N}
\]  

(10.45)

Substituting \( \sin[\theta_{\text{peak}}] = m \lambda / d \), and solving for \( \Delta \theta \)
\[ \Delta \theta_{\text{max-min}} = \frac{\lambda}{N \cos[\theta_{\text{peak}}]} \]  

(10.46)

Note: the true half width of the principle maxima is determined by solving for \( \Delta \theta \) at which

\[ I[\Delta \theta + \theta] = \frac{I[0]}{N^2} \left( \frac{\sin[N \alpha]}{\sin[\alpha]} \right)^2 = \frac{I[0]}{2} \]  

(10.47)

Solving for \( \Delta \theta \)

\[ \Delta \theta_{\text{FWHM}} = \frac{2 \sqrt{\frac{3}{2}} \lambda}{d \sqrt{-1 + N^2 \pi \cos[\theta]}} \to \frac{\sqrt{6} \lambda}{d N \pi \cos[\theta]} \]  

(10.48)

in the limit of large \( N \). The essential dependence is the same. The width of the peaks decreases as \( 1/N \). For increasing \( N \), the principal maxima become more well defined. Such large \( N \) multiple slits are call gratings and are typically used to disperse light instead of prisms. To see how a grating can separate different colors, one can determine the angular separation of two principal maxima generated by two wavelengths. The angular positions of the maxima are determined from \( d \sin[\theta] = m \lambda \). Noting that \( \Delta \theta = \left( \frac{\partial \theta}{\partial \lambda} \right) \Delta \lambda \), we have that

\[ \frac{\partial (d \sin[\theta])}{\partial \lambda} = d \cos[\theta] \frac{\partial \theta}{\partial \lambda} = m \]  

(10.49)

and so

\[ \Delta \theta_{\text{max-max}} = \Delta \lambda \frac{\partial \theta}{\partial \lambda} = \frac{m}{d \cos[\theta_{\text{peak}}]} \Delta \lambda \]  

(10.50)

The dispersion of the grating increases with the order of the peak, \( m \). Of course as we shall see, the intensity of the higher order maxima will also be smaller.

One criterion for resolving two neighboring peaks associated with two different wavelengths, is to require that the separation between the peaks be larger than the separation between a peak and the first minimum.
\[ \Delta \vartheta_{\text{max-max}} > \Delta \vartheta_{\text{max-min}} \]
\[ \frac{m \Delta \lambda}{d \cos[\vartheta_{\text{peak}}]} > \frac{\lambda}{N d \cos[\vartheta_{\text{peak}}]} \]  

or

\[ \frac{\Delta \lambda}{\lambda} > \frac{1}{m N} \]

This factor of \( Nm \) is often referred to as the resolving power of the grating. To see this in practice, one can plot the sum of the diffraction pattern for two different wavelengths. Only if the peaks from the two wavelengths are separated at a distance greater than the distance from a principle peak and the first minimum, can one distinguish that there are in fact two peaks.

10.4 Single Slit

To determine the intensity from a single slit, one can artificially subdivide the slit into \( N \) slits and sum the electric field from the individual slits.
To determine the intensity from a single slit, one can artificially subdivide the slit into $N$ slits and add the electric fields from the individual slits.

\[
\text{slits} = \text{Table}[\text{Line}[\{0, -2 + n \times 0.4\}, \{0, -2 + n \times 0.4 + 0.1\}]], \{n, 1, 10\};
\]

\[
\text{rays} = \text{Flatten}[\text{Table}[\text{Line}[\{0, -2 + n \times 0.4 + 0.25\}, \{3, -0.5 + n \times 0.4 + 0.25\}]], \text{Line}[\{3.0, -0.5 + n \times 0.4 + 0.25\}, \{10, 5\}]], \{n, 0, 9\}];
\]

\[
\text{Fprint}[[\text{Graphics}[\{\text{Thick, Line}[\{10, -4\}, \{10, 5\}\}], \text{Line}[\{0, -4\}, \{0, -2\}\}], \text{slits}, \text{Line}[\{0, 2 + 0.02\}, \{0, 5\}\}], \text{Arrowheads}[\{-0.02, 0.02\}], \text{Arrow}[\{-1.0, -2\}, \{-1.0, 2\}\}], \text{Circle}[\{-3, 0\}, 7, \{-\pi/6, \pi/6\}], \text{Circle}[\{9.11, 0\}, 7, \{\pi - \pi/6, \pi + \pi/6\}], \{\text{Blue, Style[Text[TDF[\{a\}], \{-1.6, 0\}], 14]}, \\text{Dashed}, \text{Line}[\{0, -2\}, \{10, -2\}\}], \text{Style[Text[TDF[\{0\}], \{1.978, -1.517\}], 14}\}]], \{\text{Thin, Red, rays}, \text{Style[Text[TDF[\{C\}], \{10.5, 5\}], 14]\}}]
\]
From the analysis of the $N$ slit interference pattern, the intensity was found to be

$$I[\phi] = \frac{E_0^2}{c^2 \mu_0} \frac{\sin \left( \frac{\pi N \Delta L}{\lambda} \right)^2}{\sin \left( \frac{\pi \Delta L}{\lambda} \right)^2}$$  \hspace{1cm} (10.53)

For the single slit, we have that $\Delta L = a \frac{\sin(\phi)}{N}$. The intensity from each section of the single slit should also be reduced from $E_0$ to $E_0 / N$ to insure that the total light from the single slit is fixed.

$$I[\phi] = \frac{E_0^2}{c^2 N^2 \mu_0} \frac{\sin \left( \frac{\pi a \sin(\phi)}{N \lambda} \right)^2}{\sin \left( \frac{\pi a \sin(\phi)}{N \lambda} \right)^2}$$  \hspace{1cm} (10.54)

Taking the limit as $N -> \infty$, the Sine function in the numerator can be approximated by its argument

$$I[\phi] = \frac{E_0^2}{c^2 N^2 \mu_0} \frac{\sin \left( \frac{\pi a \sin(\phi)}{N \lambda} \right)^2}{(\frac{\pi a \sin(\phi)}{N \lambda})^2} = \frac{E_0^2}{c^2 \mu_0} \frac{\sin \left( \frac{\pi a \sin(\phi)}{\lambda} \right)^2}{\frac{\pi a \sin(\phi)}{\lambda}} = \frac{E_0^2}{c^2 \mu_0} \frac{\sin(\beta)^2}{\beta^2}$$  \hspace{1cm} (10.55)

where

$$\beta = \frac{a \pi \sin(\phi)}{\lambda}$$  \hspace{1cm} (10.56)

The intensity at $\phi = 0$ or $\beta \rightarrow 0$ is just
\[ I[0] = \frac{E_0^2}{c^2 \mu_0} \]  

(10.57)

and so

\[ I[\delta] = I[0] \frac{\sin[\beta]^2}{\beta^2} \]  

(10.58)

Plotting the intensity as a function of \( \delta \) and \( \sin[\delta] \) we obtain

The minima on either side occurs at \( \beta = \pm m\pi \) or

\[ \frac{a \pi \sin[\delta]}{\lambda} = \pm \pi \quad \text{or} \quad \sin[\delta_{\text{min}}] = \pm m \frac{\lambda}{a} \]  

(10.59)

Note the similarity of this formula with the formula for the maxima of a multislit, only now the formula predicts the position of minima!

As the slit size approaches \( \lambda \), the first minimum approaches \( \delta = \pi / 2 \) and the central maximum becomes broader and broader. Therefore for \( a << \lambda \), the central maximum illuminates the entire screen with a uniform intensity. This was implicitly assumed when we calculated the intensity of a multislit. The light from each slit could illuminate the screen at all possible angles.
If however, \( a > \lambda \), then the single slit pattern will modulate the \( N \) slit pattern. Moreover, some of the principle maxima of the \( N \) slit pattern may be missing because they occur at angles at which the single slit pattern has a minimum.

Consider 4 slits with a spacing \( d = 4 \lambda \) in which the width of the slits is \( a = 2 \lambda \).

In the above example the principle maxima of the multislit pattern occur at

\[
\sin[\vartheta_{\text{max}}] = \pm m \frac{\lambda}{d} = 0, \pm \frac{1}{4}, \pm \frac{1}{2}, \pm \frac{3}{4}, \pm 1
\]  

(10.60)

There are 9 maxima (one could argue that there are only 7 maxima because \( \sin[\vartheta_{\text{max}}] = 1 \) the maxima are not perfectly developed. The minima of the single slit occur at

\[
\sin[\vartheta_{\text{min}}] = \pm m \frac{\lambda}{a} = \pm \frac{1}{2}, \pm 1
\]  

(10.61)

Consequently, the principle maxima at \( \sin[\vartheta_{\text{min}}] = \pm \frac{1}{2} \) and \( \pm 1 \) are eliminated because no light
from the slits are directed in these directions.

For arbitrary values of $N$, $a$ and $d$, the overall pattern will be more complex.

Taking the width of the central maximum of the single slit pattern to be the width at half maximum, one needs to find the value of $\beta$ such that

$$\left( \frac{\text{Sin}[\beta]}{\beta} \right)^2 = \frac{1}{2} \quad \text{(10.62)}$$
The value of $\beta$ must be solved numerically. To get an estimate, the above equation can be reduced to $\sin(\beta) = \frac{\beta}{\sqrt{2}}$. Clearly one solution of this equation is when $\beta$ is small. Taking the Taylor expansion,

$$\sin(\beta) \approx \beta - \frac{\beta^3}{6} + O(\beta)^4 \approx \frac{\beta}{\sqrt{2}}$$

or

$$\beta - \frac{\beta}{\sqrt{2}} - \frac{\beta^3}{6} = 0$$

(10.63)

Solving for $\beta$

$$-\frac{1}{6} \beta \left(-6 + 3\sqrt{2} + \beta^2\right) = 0$$

(10.64)

or

$$\beta \to 0 = 0.$$

$$\beta \to -\sqrt{3(2 - \sqrt{2})} = -1.32565$$

(10.65)

$$\beta \to \sqrt{3(2 - \sqrt{2})} = 1.32565$$

The exact numerical result is $\beta = \frac{a \pi \sin(\vartheta)}{\lambda} = 1.39156$ or $\sin(\vartheta) = 0.442946 \frac{\lambda}{a}$. The width at half maximum is therefore $\Delta \vartheta_{\text{width}} = 2 \arcsin \left( \frac{0.442946 \lambda}{a} \right)$. If the slit size is large compared to $\lambda$, then $\frac{0.885893 \lambda}{a}$.

10.5 Circular Hole

For a circular hole, the first minimum next to the central maximum is given by

$$\sin(\vartheta_{\text{min}}) = \frac{1.22 \lambda}{D}$$

(10.66)

where $D$ is the diameter of the aperture. If one uses the same criterion as before then the image of two point sources can be resolved if their images have an angular separation that is greater than $1.22 \lambda / D$. 

For distance objects the angular separation of the images is equal to the angular separation of the two objects

\[ \Delta \theta_{sep} = \frac{\Delta x}{D_0} = \Delta \theta_{image} = \frac{\Delta y}{D_i} \]  \hspace{1cm} (10.67)

Therefore to resolve the two images as separate images, \( \frac{\Delta x}{D_0} > 1.22 \lambda / D \). Note the dependence on the wavelength. In order to improve the resolution of a microscope, in addition to making the lenses larger, one could also use shorter wavelengths.

For telescopes, one can not change the wavelengths of the incoming light so one must make the lenses or mirrors larger. The same criterion applies to radio telescopes. In New Mexico, there is the VLA (Very Large Array) and now it is possible to link telescopes across the world to form a baseline (or aperture) of 5000 km.

One can estimate, the resolving power of a human eye. The pupil is about \( D \approx 0.5 \text{ cm} \). Because the diffraction is occurring inside the eye, one needs to include the index of refraction of the vitreous humor (assume the index is that of water, \( n \approx 1.33 \)). Choosing \( \lambda = 5000 \text{ Å} \)

\[ \frac{\Delta x}{D_0} > \frac{1.22 \lambda}{D n} = \frac{1.22 (5000 \text{ Å})}{1.33 \times 0.5 \text{ cm}} = 9.172 \times 10^{-5} \]  \hspace{1cm} (10.68)

The headlights on a car are about 1.4 meters apart and therefore the distance at which one could still resolve the two headlights becomes

\[ D_0 < \frac{\Delta x}{9.172 \times 10^{-5}} = 15262.3 \text{ m} = 50073.1 \text{ ft} \]  \hspace{1cm} (10.69)

So in principle, one should be able to resolve the two headlights at the normal cruising altitude of an airplane (~36000 ft). Unfortunately, the size of a single rod is only about 1 \( \mu \text{m} \) and has an angular size of \( \Delta \theta = \frac{1 \mu \text{m}}{2 \text{ cm}} = 5 \times 10^{-5} \). Consequently, the eye may still be unable to resolve the two headlights because there are not enough cells to image the separate signals.
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FrontEndTokenExecute[nb, "SelectionCloseAllGroups"]

SelectionMove[nb, Next, Cell]

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