1. Determine the energy in eV (electron volts) of 
   (a) a γ-ray photon with wavelength \( \lambda = 2 \times 10^{-15} \text{ m} \) 
   (b) an x-ray photon with wavelength \( \lambda = 2 \text{ Å} \) 
   (c) a visible photon with wavelength \( \lambda = 3500 \text{ Å} \). What color is this photon?

2. Using the uncertainty principle \( \Delta x \Delta p_x \geq \hbar / 2 \), determine the uncertainty of the momentum and the average kinetic energy \( \langle p_x^2 \rangle = \left( \frac{p_x^2 + p_y^2 + p_z^2}{2m} \right) \) of the following particles. Express the kinetic energy in eV’s. Hint: By definition \( \Delta p_x = \sqrt{\langle (p_x - \langle p_x \rangle)^2 \rangle} = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} \) and if a particle is confined to a sphere \( \Delta x \sim \Delta y \sim \Delta z \sim 2r \) and \( \langle p_x \rangle = 0 \), \( \langle p_y \rangle = 0 \), \( \langle p_z \rangle = 0 \). 
   (a) an electron at rest confined to a sphere of radius of the first Bohr orbit, \( a_0 = 0.5 \times 10^{-10} \text{ m} \) 
   (b) a proton at rest confined to a sphere of radius of the first Bohr orbit, \( a_0 = 0.5 \times 10^{-10} \text{ m} \) 
   (c) a proton at rest confined to a sphere of radius 2 Fermi \( (2 \times 10^{-15} \text{ m}) \)

3. Consider a dial indicator whose needle is free to rotate so that if you give the needle a flick it is equally likely to come to rest at any angle between 0 and \( \pi \). 
   (a) What is the probability density, \( \rho(\theta) \)? Graph \( \rho(\theta) \) as a function of \( \theta \) from 0 to \( \pi \). [Note: \( \rho(\theta) \, d\theta \) is the probability that the needle will come to rest between \( \theta \) and \( \theta + d\theta \)] 
   (b) Compute \( \langle \theta \rangle \), \( \langle \theta^2 \rangle \) and \( \sigma = \sqrt{\langle (\theta - \langle \theta \rangle)^2 \rangle} \) 
   (c) Compute \( \langle \sin(\theta) \rangle \) and \( \langle \cos^2(\theta) \rangle \)

4. Using the same device as in Prob. 3, consider the x-coordinate of the needle point i.e. the projection of the needle on the horizontal axis. 
   (a) What is the probability density, \( \rho(x) \). Graph \( \rho(x) \) as a function of \( x \) from \( -2r \) to \( 2r \) where \( r \) is the length of the needle. [Note: \( \rho(x) \, dx \) is the probability that the projection lies between \( x \) and \( x + dx \). Hint: From Prob. 3 you know the probability that \( \theta \) falls within a certain interval. What interval \( dx \) corresponds to the interval of \( d\theta \).] 
   (b) Compute \( \langle x \rangle \), \( \langle x^2 \rangle \) and \( \sigma = \sqrt{\langle (x - \langle x \rangle)^2 \rangle} \)

5. A particle is defined in space by \( \psi(x, t) = \frac{A}{1 + x^2} e^{-i(k_0 x - \omega t)} \) where \( k_0 \) and \( \omega \) are constants. 
   (a) Find the normalization constant, \( A \), up to an undetermined phase angle. 
   (b) What is the probability of finding the particle in the interval \( x \) and \( x + dx \) at time \( t \)? Does this value depend on the value of \( t \)? 
   (c) Sketch the probability density as a function of \( x \). 
   (d) Determine the uncertainty in position i.e. \( \sqrt{\langle (x - \langle x \rangle)^2 \rangle} \)

6. Using the wavefunction defined in Prob. 5 
   (a) Determine the k-space wavefunction \( \phi(k) \) and verify that it is normalized. To evaluate the integral, you may need to look it up, do a contour integral or use Mathematica. 
   (b) Determine the average value of \( k \) using the momentum space wavefunction \( \phi(k) \). 
   (c) Determine the uncertainty in momentum i.e. \( \sqrt{\langle (k - \langle k \rangle)^2 \rangle} \). 
   (d) Does this wavefunction satisfies the uncertainty relationship \( \Delta x \Delta p_x > \hbar / 2 \)

7. Show by explicit integration that the following representation for the delta function does in fact agree with the definition of the delta function \( \delta(x - x_0) = \frac{1}{\sqrt{\pi}} \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} e^{(x-x_0)^2/\epsilon} \). Hint: Multiply the delta function by a general
function and expand the function in a Taylor series and integrate term by term. Then take the limit in the delta function definition.

8. By using the operational definition of the delta function \( \int_{-\infty}^{\infty} f[x] \delta[x] \, dx = f(0) \) where \( f[x] \) is a “well-behaved” function, verify the following operational statements:

\[
\begin{align*}
\delta[-x] &= \delta[x] \\
\delta[a \, x] &= \frac{1}{|a|} \delta[x] \quad \text{where } a \neq 0 \\
\delta'[-x] &= -\delta'[x] \\
f[x] \, \delta'[x] &= -f'[x] \, \delta[x] \quad \text{where } \delta'[x] = \frac{d \delta[x]}{dx} \\
\delta[f[x]] &= \frac{\delta[x - x_0]}{[\partial f[x] / \partial x]_{x_0}} \quad \text{where the only zero of } f[x] \text{ is at } x_0 \quad \text{i.e. } f[x_0] = 0
\end{align*}
\]

Hint: Consider the Taylor expansion of \( f \)
Photon Energies, Wavelengths and Color

Determine the energy in eV (electron volts) of

(a) a γ-ray photon with wavelength $\lambda = 2 \times 10^{-15} \text{ m}$.
(b) an x-ray photon with wavelength $\lambda = 2. \text{ Å}$.
(c) a visible photon with wavelength $\lambda = 3500 \text{ Å}$. What is the color of this photon.

(a) a γ-ray photon with wavelength $\lambda = 2 \times 10^{-15} \text{ m}$.

The energy of a photon is given by

$$\varepsilon = \omega = c k = \frac{2\pi}{\lambda} = \frac{\hbar}{c}$$

where a very useful number to remember is $\hbar c = 197.327 \text{ eV nm}$.

Substituting

$$\varepsilon = \frac{2\pi}{\lambda} \hbar c = \frac{2\pi}{\lambda} \frac{197.327 \text{ eV nm}}{2 \times 10^{-15} \text{ m}} = 6.19921 \times 10^8 \text{ eV}$$

(b) an x-ray photon with wavelength $\lambda = 2. \text{ Å}$.

Substituting

$$\varepsilon = \frac{2\pi}{\lambda} \hbar c = \frac{2\pi}{\lambda} \frac{197.327 \text{ eV nm}}{2 \times 10^0 \text{ Å}} = 6199.21 \text{ eV}$$

(c) a visible photon with wavelength $\lambda = 3500 \text{ Å}$. What is the color of this photon.

Substituting

$$\varepsilon = \frac{2\pi}{\lambda} \hbar c = \frac{2\pi}{\lambda} \frac{197.327 \text{ eV nm}}{3.5 \times 10^3 \text{ Å}} = 3.54241 \text{ eV}$$

and the color is
3500 Å
Estimating Kinetic Energy of a Particle Confined to a Sphere

Using the uncertainty principle $\Delta x \Delta p_x > \frac{\hbar}{2}$ determine the uncertainty of the momentum and the average kinetic energy $\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{p_x^2 + p_y^2 + p_z^2}{2m} \right\rangle$ of the following particles. Express the kinetic energy in eV's. Hint: By definition $\Delta p_x = \sqrt{\left(\left\langle (p_x) + p_x \right\rangle \right)^2} = \sqrt{-\left(\left\langle p_x \right\rangle \right)^2 + \left\langle p_x^2 \right\rangle}$ and if a particle is confined to a sphere of radius $2r$ then $\Delta x \approx \Delta y \approx \Delta z \approx 2r$ and $\left\langle p_x \right\rangle = 0 \left\langle p_y \right\rangle = 0 \left\langle p_z \right\rangle = 0$. (a) an electron at rest confined to a sphere of radius, $a_0 = 5. \times 10^{-11}$ m. (b) a proton at rest confined to a sphere of radius, $a_0 = 5. \times 10^{-11}$ m. (c) a proton at rest confined to a sphere of radius $r_N = 2$ Fermi $= 2. \times 10^{-15}$ m.

The average kinetic energy of the particle is just $\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{p_x^2 + p_y^2 + p_z^2}{2m} \right\rangle$. From the definition of the momentum uncertainty

$$\Delta p_x = \sqrt{\left(\left\langle (p_x) + p_x \right\rangle \right)^2} = \sqrt{\left\langle p_x^2 \right\rangle - \left\langle p_x \right\rangle^2}$$

we see that $\left\langle p_x^2 \right\rangle = \left\langle p_x^2 \right\rangle + \left\langle p_x \right\rangle^2$. Moreover, if the particle is confined to a sphere, its average momentum must be zero, $\left\langle p_x \right\rangle = 0$ so that $\left\langle p_x^2 \right\rangle = \left\langle \Delta p_x \right\rangle^2$. Therefore we can estimate the average kinetic energy from the uncertainty in the momenta.

$$\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{p_x^2}{2m} \right\rangle + \left\langle \frac{p_y^2}{2m} \right\rangle + \left\langle \frac{p_z^2}{2m} \right\rangle = \frac{(\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2}{2m}$$

If a particle is confined to a sphere of radius, $r$, then the uncertainty in the $x$ direction is roughly $\Delta x \approx 2r$. From the uncertainty relationship

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

we have that the uncertainty in the $x$ momentum is $\Delta p_x \geq \frac{\hbar}{4r}$. Similarly for the uncertainty in the $y$ and $z$ directions, $\Delta p_y \geq \frac{\hbar}{4r}$ and $\Delta p_z \geq \frac{\hbar}{4r}$. Therefore

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{(\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2}{2m} = \frac{1}{2m}((\hbar / 4r)^2 + (\hbar / 4r)^2 + (\hbar / 4r)^2) = \frac{3\hbar^2}{32mr^2}$$

One can rationalize the expression by multiplying both the top and bottom with $c^2$ so that the average kinetic energy is

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{3(\hbar c)^2}{32mc^2r^2}$$

where $\hbar c = 197.327$ eV nm.

(a) an electron at rest confined to a sphere of radius, $a_0 = 5. \times 10^{-11}$ m
The uncertainty in the momentum, $\Delta p_x \geq \hbar / 4r$ can be written as $c \Delta p_x \geq \hbar c / 4r$ where $\hbar c = 197.327 \text{ eV nm}$. For an electron localized to the first Bohr orbit, $r = 0.5 \AA = 0.05 \text{ nm}$.

For an electron localized to a sphere of radius, $a_0 = 5 \times 10^{-11} \text{ m}$

$$c \Delta p_x > \frac{\hbar c}{4r} = \frac{197.327 \text{ eV nm}}{4(5 \times 10^{-11} \text{ m})} = 986.635 \text{ eV}$$

or

$$\Delta p_x > \frac{\hbar}{4r} = \frac{3.29106 \times 10^{-6} \text{ eV s}}{m} = \frac{5.27286 \times 10^{-25} \text{ kg m}}{s}$$

The average kinetic energy is then

$$\langle \frac{p^2}{2m} \rangle = \frac{3(hc)^2}{32 m c^2 r^2}$$

where for an electron $m c^2 = 0.511 \text{ MeV}$. Therefore

$$\langle \frac{p^2}{2m} \rangle = \frac{3(hc)^2}{32 m c^2 r^2} = \frac{(3(197.327 \text{ eV nm})^2)}{(32(0.511 \text{ MeV})(5 \times 10^{-11} \text{ m})^2)} = 2.85748 \text{ eV}$$

As we shall see, this is comparable to the kinetic energy determine by solving Schrodinger equation for the hydrogen atom.

(b) an proton at rest confined to a sphere of radius, $a_0 = 5 \times 10^{-11} \text{ m}$

The uncertainty in the momentum, $\Delta p_x \geq \hbar / 4r$ for an proton localized to a sphere of radius, $r_n = 2 \times 10^{-15} \text{ m}$ has the same value as before

$$c \Delta p_x > \frac{\hbar c}{4r} = \frac{197.327 \text{ eV nm}}{4(5 \times 10^{-11} \text{ m})} = 986.635 \text{ eV}$$

or

$$\Delta p_x > \frac{\hbar}{4r} = \frac{3.29106 \times 10^{-6} \text{ eV s}}{m} = \frac{5.27286 \times 10^{-25} \text{ kg m}}{s}$$

However the mass of the proton is given by $mc^2 = 938 \text{ MeV}$ and so

$$\langle \frac{p^2}{2m} \rangle = \frac{3(hc)^2}{32 m c^2 r^2} = \frac{(3(197.327 \text{ eV nm})^2)}{(32(938 \text{ MeV})(5 \times 10^{-11} \text{ m})^2)} = 0.00155669 \text{ eV}$$

As expected the larger mass causes the average kinetic energy to be greatly reduced. Given such a
large uncertainty in position compared to the size of a proton \( r_p \sim 1 \text{ Fermi} = 10^{-15} m \), the particle can be treated classically.

(c) a proton at rest confined to a sphere of radius, \( r_N = 2 \times 10^{-15} m \)

Now the uncertainty in the momentum, \( \Delta p_x \geq \hbar / 4 r \) becomes much larger

\[
c \Delta p_x > \frac{\hbar c}{4 r} = \frac{197.327 \text{ eV nm}}{4 (2 \times 10^{-15} m)} = 2.46659 \times 10^7 \text{ eV}
\]  

or

\[
\Delta p_x > \frac{\hbar}{4 r} = \frac{0.0822765 \text{ eV s}}{m} = \frac{1.31821 \times 10^{-20} \text{ kg m}}{s}
\]  

Now the average kinetic energy is typical of a nuclear system.

\[
\left\langle \frac{p^2}{2 m} \right\rangle = \frac{3 (\hbar c)^2}{32 m c^2 r^2} = \frac{3 (197.327 \text{ eV nm})^2}{32 (938 \text{ MeV}) (2 \times 10^{-15} m)^2} = 0.97293 \text{ MeV}
\]

\[ \text{(15)} \]
Dial Indicator - Probability density $\rho[\theta]$

Consider a dial indicator whose needle is free to rotate so that if you give the needle a flick it is equally likely to come to rest at any angle between 0 and $\pi$.

(a) What is the probability density, $\rho[\theta]$? Graph $\rho[\theta]$ as a function of $\theta$ from 0 to $\pi$. [Note: $\rho[\theta]d\theta$ is the probability that the needle will come to rest between $\theta$ and $\theta + d\theta$]

(b) Compute $\langle \theta \rangle$, $\langle \theta^2 \rangle$ and $\sigma = \sqrt{\langle (\theta - \langle \theta \rangle)^2 \rangle}$

(c) Compute $\langle \sin[\theta] \rangle$ and $\langle \cos^2[\theta] \rangle$

(a) What is the probability density, $\rho[\theta]$? Graph $\rho[\theta]$ as a function of $\theta$ from 0 to $\pi$.

If the needle is equally likely to come to rest at any angle between 0 and $\pi$, the probability density must be a constant. To determine the constant we use the fact that the probability density is normalize, i.e. the probability of finding it between 0 and $\pi$ is unity.

$$\int_{0}^{\pi} \rho[\theta] d\theta = \int_{0}^{\pi} c d\theta = c \pi = 1$$

(1)

and therefore $c = \frac{1}{\pi}$

and the probability that the needle comes to rest in the interval between $\theta$ and $d\theta$ is given by

$$\rho(\theta) d\theta = \begin{cases} 
0 & \theta < 0 \text{ and } \theta > \pi \\
\frac{1}{\pi} d\theta & 0 \leq \theta \leq \pi 
\end{cases}$$

(2)

Plotting the probability density
(b) Compute $\langle \vartheta \rangle$, $\langle \vartheta^2 \rangle$ and $\sigma = \sqrt{\langle (\vartheta - \langle \vartheta \rangle)^2 \rangle}$

The average value of any function of $\vartheta$ is determined by integrating that function times the probability $\rho(x) \, dx$ over all possible values of $\vartheta$ ($0 \leq \vartheta \leq \pi$).

$$\langle f \rangle = \int_0^\pi f(\vartheta) \rho(\vartheta) \, d\vartheta$$

(3)

Therefore the average of $\vartheta$ becomes

$$\langle \vartheta \rangle = \int_0^\pi \vartheta \rho(\vartheta) \, d\vartheta = \int_0^\pi \frac{1}{\pi} \, d\vartheta = \left[ \frac{\vartheta^2}{2\pi} \right]_0^\pi = \frac{\pi}{2}$$

(4)

The average of $\vartheta^2$ becomes

$$\langle \vartheta^2 \rangle = \int_0^\pi \vartheta^2 \rho(\vartheta) \, d\vartheta = \int_0^\pi \frac{1}{\pi} \, d\vartheta = \left[ \frac{\vartheta^3}{3\pi} \right]_0^\pi = \frac{\pi^2}{3}$$

(5)

Using the relationship that $\sqrt{\langle (\vartheta - \langle \vartheta \rangle)^2 \rangle} = \sqrt{\langle \vartheta^2 \rangle - \langle \vartheta \rangle^2}$ we have that

$$\sqrt{\langle (\vartheta - \langle \vartheta \rangle)^2 \rangle} = \sqrt{\langle \vartheta^2 \rangle - \langle \vartheta \rangle^2} = \sqrt{\frac{\pi^2}{3} - \left( \frac{\pi}{2} \right)^2} = \frac{\pi}{2\sqrt{3}}$$

(6)

(c) Compute $\langle \sin[\vartheta] \rangle$ and $\langle \cos^2[\vartheta] \rangle$

Similarly, the average of $\sin[\vartheta]$ becomes

$$\langle \sin[\vartheta] \rangle = \int_0^\pi \sin[\vartheta] \rho(\vartheta) \, d\vartheta = \int_0^\pi \frac{1}{\pi} \, d\vartheta = \left[ \frac{\cos[\vartheta]}{\pi} \right]_0^\pi = \frac{2}{\pi}$$

(7)

and the average of $\cos^2[\vartheta]$ is

$$\langle \cos^2[\vartheta] \rangle = \int_0^\pi \cos^2[\vartheta] \rho(\vartheta) \, d\vartheta = \int_0^\pi \frac{1}{\pi} \, d\vartheta = \left[ \frac{(\vartheta + \cos[\vartheta] \sin[\vartheta])}{2\pi} \right]_0^\pi = \frac{1}{2}$$

(8)
Dial Indicator - Probability density $\rho[x]$

Using the same device as in the previous problem, consider the x-coordinate of the needle point i.e. the projection of the needle on the horizontal axis.

(a) What is the probability density $\rho[x]$. Graph $\rho[x]$ as a function of $x$ from $-2r$ to $2r$ where $r$ is the length of the needle. [Note: $\rho(x) \, dx$ is the probability that the projection lies between $x$ and $x + dx$. Hint: From the previous you know the probability that $\theta$ falls within a certain interval. What interval $dx$ corresponds to the interval of $d\theta$.]

(b) Compute $\langle x \rangle$, $\langle x^2 \rangle$ and $\sigma = \sqrt{(\langle x - \langle x \rangle \rangle^2)}$

The probability that the needle comes to rest in the interval between $\theta$ and $d\theta$ is given by

$$
\rho[\theta] \, d\theta = \begin{cases} 
0 & \theta < 0 \text{ and } \theta > \pi \\
\frac{1}{2\pi} & 0 \leq \theta \leq \pi 
\end{cases}
$$

To find the probability that the projection of the needle falls between $x$ and $x + dx$, we need to consider the relationship between $x$ and $\theta$ i.e. $x = r \cos[\theta]$. Because the probability that the needle comes to rest within the $d\theta$ shown is $\frac{1}{\pi} \, d\theta$, this must also be the probability that the needle falls within the corresponding $dx$.

Therefore

$$
\rho[x] \, dx = \frac{1}{\pi} \, d\theta \text{ where } dx = r \sin[\theta] \, d\theta
$$

and

$$
\rho[x] \, dx = \frac{1}{\pi} \frac{dx}{r \sin[\theta]}
$$
Expressing Sin[θ] in terms of x, Sin[θ] = \((1 - \cos(x)^2)^{1/2}\) = \((1 - (x/r)^2)^{1/2}\) we obtain

\[
\rho[x] dx = \frac{1}{\pi} \frac{dx}{r(1 - (x/r)^2)^{1/2}}
\]

Checking the normalization \(-r \leq x \leq r\)

\[
\int_{-r}^{r} \frac{1}{\pi} \frac{1}{r \sqrt{1 - \frac{x^2}{r^2}}} dx = \left. \left( \frac{\text{ArcSin}[\frac{x}{r}]}{\pi} \right) \right|_{-r}^{r} = 1
\]

Plotting the function we note that the probability diverges at \(x = \pm r\).

(b) Compute \(\langle x \rangle, \langle x^2 \rangle\) and \(\sigma = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}\)

The average value of x is determined by integrating x times the probability \(\rho(x) dx\) and integrating over all values of x \((-r \leq x \leq r)\).

By symmetry this average should be zero.

\[
\langle x \rangle = \int_{-r}^{r} \frac{1}{\pi} \frac{x dx}{r(1 - (x/r)^2)^{1/2}} = 0
\]

The average value of \(x^2\) is determined by integrating \(x^2\) times the probability \(\rho(x) dx\) and integrating over all values of x \((-r \leq x \leq r)\).

For this case the average should not be zero.
\[ \langle x^2 \rangle = \frac{1}{\pi} \int_{-r}^{r} \frac{x^2 \, dx}{r \left(1 - \frac{x^2}{r^2}\right)^{1/2}} = \frac{r^2}{2} \]  

(7)

and finally using the relationship that \( \sigma = \sqrt{\langle x - \langle x \rangle \rangle^2} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \) we have that

\[ \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{r^2}{2} - 0} = \frac{r}{\sqrt{2}} \]  

(8)
Particle Wavefunction in Coordinate Space

A particle is defined in space by \( \psi(x, t) = \frac{A}{(1 + (x/a)^2)} e^{-i(k_0 x - \omega t)} \) where \( k_0 \) and \( \omega \) are constants

(a) Find the normalization constant, \( A \), up to an undetermined phase angle.
(b) What is the probability of finding the particle in the interval \( x \) and \( x + dx \) at time \( t \)? Does this value depend on the value of \( t \)?
(c) Sketch the probability density as a function of \( x \).
(d) Determine the uncertainty in position i.e. \( \sqrt{\langle (x - \langle x \rangle)^2 \rangle} \)

(a) Find the normalization constant, \( A \), up to an undetermined phase angle.

To determine the normalization constant we require that

\[
\int_{-\infty}^{\infty} |\psi(x, t)|^2 \, dx = 1
\]

Using the above form for \( \psi(x, t) \) we obtain

\[
|\psi(x, t)|^2 = \frac{A^2}{1 + \frac{x^2}{a^2}} \cdot \frac{e^{i(-t \omega + x k_0)}}{1 + \frac{x^2}{a^2}} = A^2 \frac{1}{1 + \left(\frac{x^2}{a^2}\right)^2}
\]

Integrating

\[
\int_{-\infty}^{\infty} |\psi(x, t)|^2 \, dx = \int_{-\infty}^{\infty} A^2 \frac{1}{1 + \left(\frac{x^2}{a^2}\right)^2} \, dx
\]

This integral can be done by trig substitution. Let \( x = \alpha \tan(\phi) \), \( dx = \alpha \sec^2(\phi) \, d\phi \) with \( -\pi/2 \leq \phi \leq \pi/2 \)

\[
\int_{-\infty}^{\infty} |\psi(x, t)|^2 \, dx = |A|^2 \int_{-\pi/2}^{\pi/2} \frac{\alpha \sec^2(\phi)}{(1 + \tan^2(\phi))^2} \, d\phi = |A|^2 \int_{-\pi/2}^{\pi/2} \alpha \cos^2(\phi) \, d\phi
\]

\[
= |A|^2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} (\alpha + \alpha \cos(2\phi)) \, d\phi
\]

\[
= |A|^2 \left( \frac{\pi}{2} \left( \frac{\phi}{2} + \frac{1}{4} \sin(2\phi) \right) \right)_{-\pi/2}^{\pi/2} = |A|^2 \frac{a\pi}{2}
\]

Setting this equal to 1, we can define \( A \) up to an undetermined phase angle \( \varphi \)
A = \frac{2}{\pi} \frac{\pi}{a} e^{i\varphi} \tag{5}

For simplicity, we shall assume that A is real.

(b) What is the probability of finding the particle in the interval x and x + dx at time t? Does this value depend on the value of t?

The probability of finding the particle in the interval x and x + dx at time t is given by

\[ \{|\psi(x, t)|^2\} \, dx = |A|^2 \frac{1}{\left(1 + \frac{x^2}{a^2}\right)^2} \, dx = \frac{2}{a \pi} \frac{1}{\left(1 + \frac{x^2}{a^2}\right)^2} \, dx \tag{6} \]

which is clearly time independent. As you can see even though the wavefunction is time dependent, the probability density can be time independent. Note also that the probability is dimensionless as it should be.

(c) Sketch the probability density as a function of x.

Because the function includes an unknown parameter a, one must used scaled axes. Plotting the probability density is

\[ \{|\psi(x, t)|^2\} = \frac{A^2}{\left(1 + \frac{x^2}{a^2}\right)^2} = \frac{2}{a \pi} \frac{1}{\left(1 + \frac{x^2}{a^2}\right)^2} \tag{7} \]

(d) Determine the uncertainty in position i.e. \( \Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle} \)

As was shown in class, the above expression can be reduced to

\[
\frac{1}{2} \pi a |\psi(x, t)|^2
\]

\[
\begin{align*}
\frac{1}{2} \pi a |\psi(x, t)|^2 &= \frac{1}{2} \pi a \frac{2}{a \pi} \frac{1}{\left(1 + \frac{x^2}{a^2}\right)^2} \\
&= \frac{1}{2} \pi a \frac{1}{\left(1 + \frac{x^2}{a^2}\right)^2}
\end{align*}
\]
\[
\sqrt{\langle (x - \langle x \rangle)^2 \rangle} = \sqrt{\langle x^2 - 2x \langle x \rangle + \langle x \rangle^2 \rangle} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}
\] (8)

To calculate \( \langle x \rangle \) we evaluate
\[
\int_{-\infty}^{\infty} x |\psi(x, t)|^2 \, dx = \int_{-\infty}^{\infty} |A|^2 x \left( \frac{1}{\left( 1 + \frac{x^2}{a^2} \right)^2} \right) \, dx = 0
\] (9)
as expected from the symmetry of the probability distribution.

\[
\int_{-\infty}^{\infty} x^2 |\psi(x, t)|^2 \, dx = \int_{-\infty}^{\infty} |A|^2 x^2 \left( \frac{1}{\left( 1 + \frac{x^2}{a^2} \right)^2} \right) \, dx = \frac{2}{a\pi} \int_{-\infty}^{\infty} \frac{x^2}{\left( 1 + \frac{x^2}{a^2} \right)^2} \, dx
\] (10)

This integral can be solved in various ways. One is to make the trig substitution \( x = a \tan(\phi), \, dx = a \sec^2(\phi) \, d\phi \) with \(-\pi / 2 \leq \phi \leq \pi / 2\)

\[
\langle x^2 \rangle = \frac{2}{a\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a \sec^2(\phi) \left( 1 + a \tan^2(\phi) \right) \, d\phi = \frac{2}{a\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a \cos^2(\phi) \, d\phi
\]

\[
= \frac{2}{a\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \left( a + a \cos(2\phi) \right) \, d\phi
\]

\[
= \frac{2}{a\pi} \left( a \left( \frac{\phi}{2} + \frac{1}{4} \sin(2\phi) \right) \right)_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = a^2
\] (11)

and so
\[
\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{a^2 - 0} = a
\] (12)
Particle Wavefunction in k-Space

Using the wavefunction \( \psi(x, t) = A e^{-(x/a)^2} e^{i(k_0 x - \omega t)} \)
(a) Determine the momentum space wavefunction \( \phi(k_x) \) and verify that it is normalized.
(b) Determine the average value of \( k_x \) using the momentum space wavefunction \( \phi(k_x) \).
(c) Determine the uncertainty in momentum i.e. \( \sqrt{\langle (k_x - \langle k_x \rangle)^2 \rangle} \).
(d) Does this wavefunction satisfies the uncertainty relationship \( \Delta x \Delta p_x > h/2 \)

(a) Determine the momentum space wavefunction \( \phi(k_x) \) and verify that it is normalized.

Previously we found that the normalization constant is \( A = \frac{\sqrt{\pi}}{\sqrt{a}} e^{i\varphi} \) where \( \varphi \) can be taken to be zero for convenience in this problem.

The momentum wavefunction \( \phi(k) \) is determined by
\[
\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi[x, t = 0] e^{-i k x} \, dx
\]
where \( k = p_x / h \) (for simplicity, we have dropped the subscript \( x \)).

\[
\phi[k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi[x, 0] e^{-i k x} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} \left(1 + \frac{x^2}{a^2}\right) e^{-i k x} \, dx
\]

This is a difficult integral which can be done using contour integrals or by plugging into Mathematica

Integrating
\[
\phi[k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi[x, 0] e^{-i k x} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} \left(1 + \frac{x^2}{a^2}\right) e^{-i k x} \, dx
\]

\[
= \sqrt{a} \ e^{-a|k+k_0|}
\]

To see whether \( \phi(k_x) \) is normalized, we integrate the probability density, \( |\phi(k_x)|^2 \, dk \), from \(-\infty \) to \( \infty \).

\[
\int_{-\infty}^{\infty} |\phi[k]|^2 \, dk = \int_{-\infty}^{\infty} a e^{-2a|k+k_0|} \, dk
\]
Making the variable change \( z = k + k_0 \) and \( d z = d k \) with \(-\infty \leq z \leq \infty \)

\[
\int |\phi[k]|^2 \, d k = \int a e^{-2a|z|} \, d z = \int a e^{2a z} \, d z + \int a e^{-2a z} \, d z
\]

\[
= \int a e^{-2a z} \, d z = 1
\]

(b) Determine the average value of \( k \) using the momentum space wavefunction \( \phi(k_x) \).

To determine the average value of \( k \), we multiply \( k \) by the probability density, \(|\phi(k_x)|^2 \, d k \), and integrate over all space (\(-\infty \) to \( \infty \)).

\[
\langle k \rangle = \int \int k |\phi[k]|^2 \, d k = \int a e^{-2a|k+k_0|} \, k \, d k
\]

Making the variable change \( z = k + k_0 \) and \( d z = d k \) with \(-\infty \leq z \leq \infty \) we obtain

\[
\langle k \rangle = \int a e^{-2a|z|} (z - k_0) \, d z
\]

The integral over \( z \) vanishes by symmetry because the integrand is odd and we are integrating over a symmetric interval. The constant term is then our original normalization integral

\[
\langle k \rangle = -k_0
\]

This makes sense. The orginal wavefunction is multiplied by \( e^{-i k_0 x} \) which is an eigenstate of momentum with momentum \( k_0 \).

(c) Determine the uncertainty in momentum i.e. \( \sqrt{\langle (k - \langle k \rangle)^2 \rangle} \).

As before, the above expression can be reduced to

\[
\sqrt{\langle (k - \langle k \rangle)^2 \rangle} = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}
\]

To determine the average value of \( k^2 \), we multiply \( k^2 \) by the probability density, \(|\phi(k_x)|^2 \, d k \), and integrate over all space (\(-\infty \) to \( \infty \)). To simplify the integrals we again substitute \( k \rightarrow z + k_0 \)

\[
\langle k^2 \rangle = \int \int k^2 |\phi[k]|^2 \, d k = \int a e^{-2a|k+k_0|} k^2 \, d k
\]

\[
= \int \sqrt{a} e^{-a|z|} (z - k_0)^2 \, d z = \int \sqrt{a} e^{-a|z|} (z^2 - 2z k_0 + k_0^2) \, d z
\]

Performing the various integrals
\[ \langle k^2 \rangle = \frac{1}{2a^2} + k_0^2 \]  
\hspace{1cm} (11)

and so

\[ \Delta k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2} = \sqrt{\frac{1}{2a^2} + k_0^2 - k_0^2} = \frac{1}{\sqrt{2a}} \]  
\hspace{1cm} (12)

and

\[ \Delta p_x = \Delta k \frac{\hbar}{\sqrt{2a}} \]  
\hspace{1cm} (13)

(d) Does this wavefunction satisfies the uncertainty relationship \( \Delta x \Delta p_x > \hbar / 2 \)?

Previously we found that

\[ \Delta x = a \]  
\hspace{1cm} (14)

and we see then that these wavefunctions do satisfy the uncertainty relationship

\[ \Delta x \Delta p_x = a \frac{\hbar}{a \sqrt{2}} = \frac{\hbar}{\sqrt{2}} > \hbar / 2 \]  
\hspace{1cm} (15)
Gaussian Representation of a Dirac Delta Function

Show by explicit integration that the following representation for the delta function does in fact agree with

the definition of the delta function \( \delta(x - x_0) = \frac{1}{\sqrt{\pi}} \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} e^{-(x-x_0)^2/\epsilon} \)

Hint: Multiply the delta function by a function \( f(x) \) and expand the function in a Taylor series and integrate term by term. Then take the limit in the delta function definition.

The Taylor expansion of \( f[x] \) about the point \( x_0 \) is given by

\[
f[x_0] + f'[x_0] (x - x_0) + \frac{1}{2} f''[x_0] (x - x_0)^2 + O(x - x_0)^3 = \sum_{n=0}^{\infty} \frac{(x - x_0)^n f^n[x_0]}{n!}
\]  

(1)

Multiply by \( \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} e^{-(x-x_0)^2/\epsilon} \) and integrate from \(-\infty\) to \(\infty\)

\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} e^{-(x-x_0)^2/\epsilon} f[x] \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} e^{-(x-x_0)^2/\epsilon} \sum_{n=0}^{\infty} \frac{1}{n!} f^n[x_0] (x - x_0)^n \, dx
\]

\[
= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} \sum_{n=0}^{\infty} \frac{1}{n!} f^n[x_0] \int_{-\infty}^{\infty} (x - x_0)^n e^{-(x-x_0)^2/\epsilon} \, dx
\]

(2)

Change the integration variable to \( \sqrt{\epsilon} \, y = x - x_0 \) and \( \sqrt{\epsilon} \, dy = dx \) and

\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} e^{-(x-x_0)^2/\epsilon} f[x] \, dx = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} \sum_{n=0}^{\infty} \frac{1}{n!} f^n[x_0] \int_{-\infty}^{\infty} \epsilon^{n/2} y^n e^{y^2} \sqrt{\epsilon} \, dy
\]

\[
= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\epsilon^{n/2}}{n!} f^n[x_0] \int_{-\infty}^{\infty} y^n e^{y^2} \, dy
\]

(3)

Now the integral on the right is just a pure number. If \( n \) is odd then the integral is zero by symmetry. If \( n \) is even, then the integral can be related to the Gamma function. For \( n = 2m \) even, the integral is just twice the integral from 0 to \( \infty \). Letting \( z = y^2 \) and \( d\,y = d(z^{1/2}) = (1/2)z^{-1/2} \, dz \) and so

\[
\int_{-\infty}^{\infty} y^{2m} e^{y^2} \, dy = 2 \int_{0}^{\infty} y^{2m} e^{y^2} \, dy = 2 \int_{0}^{\infty} z^m e^{-z} \, (1/2)z^{-1/2} \, dz
\]

\[
= \int_{0}^{\infty} z^{(m-1)/2} e^{-z} \, dz = \Gamma\left[m + \frac{1}{2}\right] = \frac{(2m)\sqrt{\pi}}{m! 2^m}
\]

(4)

Therefore
\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} e^{-(x-x_0)^2/\epsilon} f(x) \, dx = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{\epsilon^m}{2^m m!} f^{(2m)}(x_0) \frac{(2m)! \sqrt{\pi}}{m! 2^m}
\]

\[
= \sum_{m=0}^{\infty} \epsilon^m f^{(2m)}(x_0) \frac{1}{m! 2^m}
\]

In the limit as \( \epsilon \to 0 \) all terms in the sum vanish except for the terms with \( m = 0 \) which becomes

\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\epsilon}} e^{-(x-x_0)^2/\epsilon} f(x) \, dx = \lim_{\epsilon \to 0} \sum_{m=0}^{\infty} \epsilon^m f^{(2m)}(x_0) \frac{1}{m! 2^m}
\]

\[
= f[x_0]
\]

Operationally, it behaves just like the delta function \( \delta(x - x_0) \)
By using the operational definition of the delta function

$$\int_{-\infty}^{\infty} f(x) \delta(x) \, dx = f(0)$$

where \( f(x) \) is a “well-behaved” function, verify the following operational statements:

- \( \delta(-x) = \delta(x) \)
- \( \delta(ax) = \frac{1}{|a|} \delta(x) \) where \( a \neq 0 \)
- \( x \delta(x) = 0 \)
- \( f(x) \delta'(x) = -f'(x) \delta(x) \) where \( \delta'(x) = \frac{d[\delta(x)]}{dx} \)
- \( \delta'(x) = -\delta'(x) \)
- \( \delta(f(x)) = \frac{\delta(x-x_0)}{|[\partial f(x)/\partial x]|_{x_0}} \) where \( f(x_0) = 0 \) Hint: Consider the Taylor expansion of \( f(x) \)

(a) \( \delta(-x) = \delta(x) \)

\[
\int_{-\infty}^{\infty} f(x) \delta(-x) \, dx \quad \text{let } y = -x \quad \text{Then } -dy = dx
\]

\[
\int_{-\infty}^{\infty} f(-y) \delta(y) (-dy) = \int_{-\infty}^{\infty} f(-y) \delta(y) \, dy = f(0) \quad \text{so}
\]

\( \delta(-x) = \delta(x) \)

(b) \( \delta(ax) = \frac{1}{|a|} \delta(x) \) where \( a \neq 0 \)

\[
\int_{-\infty}^{\infty} f(x) \delta(ax) \, dx \quad \text{let } y = ax \quad dx = \frac{1}{a} \, dy \quad \text{Assume } a > 0
\]

\[
\int_{-\infty}^{\infty} f(y/a) \delta(y) \, \frac{1}{a} \, dy = \frac{1}{a} f(0) \quad \text{but if } a < 0 \quad \text{then } dx = -\frac{dy}{|a|}
\]

\[
\int_{-\infty}^{\infty} -f(y/a) \delta(y) \, \frac{dy}{|a|} = \frac{1}{|a|} \int_{-\infty}^{\infty} f(y/a) \delta(y) \, dy = \frac{1}{|a|} f(0)
\]

\( \delta(ax) = \frac{1}{|a|} \delta(x) \) where \( a \neq 0 \)
Therefore \( \delta(ax) = \frac{1}{|a|} \delta(x) \)

(c) \( x \delta(x) = 0 \)
\[
\int_{-\infty}^{\infty} f(x) x \delta(x) = 0 \quad (\text{unless } f(x) \propto x^n, \ n > 1).
\]

(d) \( f(x) \delta'(x) = -f'(x) \delta(x) \)
\[
\int_{-\infty}^{\infty} f(x) \delta'(x) \, dx \quad \text{integrate by parts}
\]
\[
f(x) \delta(x) \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x) f'(x) \, dx \]

Since \( \delta(x) = 0 \quad \text{everywhere except at } x=0 \)
\[
\int_{-\infty}^{\infty} f(x) \delta'(x) \, dx = -\int_{-\infty}^{\infty} \delta(x) f'(x) \, dx = 50
\]
\[
f(x) \delta'(x) = -f'(x) \delta(x)
\]

(e) \( \delta'(-x) = -\delta'(x) \quad \text{thus is a combination of (b) and (d)} \)
\[
\int_{-\infty}^{\infty} f(x) \delta'(-x) \, dx \quad \text{let } y = -x \quad -\int_{-\infty}^{\infty} f(-y) \delta'(y) \, dy = \int_{-\infty}^{\infty} f(-y) \delta(y) \, dy
\]

\text{Integrate by parts.}
\[
f(-y) \delta(y) \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(-y) \delta(y) \, dy = -f'(0)
\]

Again the first term is zero because \( \delta(y) \) is only non-zero at \( y=0 \).
\( \therefore \)
\[
\delta'(-x) = -\delta'(x)
\]
(f) \( f(f(x)) = \frac{\delta(x-x_0)}{|(\frac{d^2f}{dx^2})|_{x_0}} \)

Assuming \( f(x_0) = 0 \) then it can be expanded in a Taylor's series
\[
f(x) = f(x_0) + (x-x_0)\left(\frac{df}{dx}\right)_{x_0} + \frac{(x-x_0)^2}{2}\left(\frac{d^2f}{dx^2}\right)_{x_0} + \ldots
\]

and so near \( x = x_0 \) \( \delta\left(f(x)\right) = \delta\left(\frac{df}{dx}\right)_{x_0} (x-x_0) \) and so
from (b). \( \delta\left(\frac{d^2f}{dx^2}\right)_{x_0} (x-x_0) = \frac{\delta(x-x_0)}{|(\frac{d^2f}{dx^2})|_{x_0}|} \)