RELATIVISTIC MOMENTUM AND ENERGY

We have derived the addition of velocity equation for motion parallel to the motion of the moving frame

\[ u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \]

Now we need the equation for motion perpendicular to the direction of motion of the moving frame.

From the Lorentz-Einstein Equation we have

\[ y' = y \]

\[ t' = \gamma \left( t - \frac{vx}{c^2} \right) \]
Differentiate to get

\[ dy' = dy \]

\[ dt' = \gamma \left( dt - \frac{vdx}{c^2} \right) \]

Divide to get

\[ \frac{dy'}{dt'} = \frac{dy}{\gamma \left( dt - \frac{vdx}{c^2} \right)} \]

Or

\[ u'_y = \frac{u_y}{\gamma \left( 1 - \frac{vux}{c^2} \right)} \]
And for $z$ the same

$$u_z' = \frac{u_z}{\gamma \left(1 - \frac{vu_x}{c^2}\right)}$$

We will use these equations to see how momentum is different from what we have used classically.
Consider two observers each with a ball. The balls are identical. The two observers are moving with respect to each other. For example:
Or using the railroad car example.

Looking down on the railroad tracks:

(a) As seen by O
There are two identical balls. One is with the observer on the railroad car (moving frame) and one is with the observer on the ground.

The balls are thrown perpendicular to the tracks in a way such that they will collide as shown.

From the view of the observer on the ground the equation for conservation of momentum is

$$2(mu_y) = 2(mu_y')$$
Momentum is not conserved if \( u_y \neq u_y' \)

But \( u_y' \) is

\[
u'_y = \frac{u_y}{\gamma \left(1 - \frac{nu_x}{c^2}\right)}\]

\( u_x = 0 \)

Thus

\[
u'_y = \frac{u_y}{\gamma}\]

Not

\[
u_y = u_y'\]

Momentum is not conserved.
To conserve momentum we must define momentum in a different way.

Let momentum be

\[ p = \gamma mu \]

Where

\[ \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \]

Keep in mind that this is different from the quantity we have been using for \( \gamma \) where we used \( v \) the velocity of the moving frame but it is convenient to do so.

Proving this definition of momentum solves the problem of conservation of momentum but requires considerable work.
We will show it from a different point of view and do what some other books do.

Consider the equation

\[ p = \gamma mu \]

We can interpret this as the mass increasing with velocity.

\[ p = (\text{relativistic mass})u = (\gamma m)u \]

Then
For the observer in rest frame

Change in momentum for ball in rest frame

\[ 2(m_{\text{rest}}u_y) \]

Change in momentum for ball in moving frame
\[2(m_{\text{moving}}u_y') = 2 \left[ m_{\text{moving}} \frac{u_y}{\gamma \left(1 - \frac{vu_x}{c^2}\right)}\right]\]

\[u_x = 0\]

\[2(m_{\text{moving}}u_y') = 2 \left[ m_{\text{moving}} \frac{u_y}{\gamma (1 - 0)}\right]\]

\[2(m_{\text{moving}}u_y') = 2m_{\text{moving}} \frac{u_y}{\gamma}\]

Conservation of momentum the two changes must be equal
\[ 2(m_{\text{rest}} u_y) = 2m_{\text{moving}} \frac{u_y}{\gamma} \]

\[ m_{\text{moving}} = \gamma m_{\text{rest}} \]

This states that a moving mass has a larger mass than the same object when moving.

The conservation of momentum holds if we have a different definition of momentum

\[ p = \gamma m u \]

Or

Have a different definition for mass

\[ m_{\text{moving}} = \gamma m_{\text{rest}} \]
Either interpretation is proved with the experiment of accelerating electrons and measuring the momentum.
KINETIC ENERGY

WHAT WE DID IN BASIC MECHANICS

\[ KE = K = \text{WORK TO BRING OBJECT FROM REST TO STATE OF MOTION WITH VELOCITY} \]

CALCULATE THIS WORK AND IT IS THE KINETIC ENERGY

\[ \text{WORK} = \text{FORCE} \times \text{DISTANCE} \]

\[ K = \int F \, ds \]

AND
\[ F = m \frac{dv}{dt} \]

SO

\[ K = m \frac{dv}{dt} ds = \int m \frac{ds}{dt} dv \]

OR

\[ K = \int mv dv \]

If the object goes from 0 to some speed \( v \)

\[ K = \int_0^v mv dv = m \int_0^v vdvdv = m \frac{v^2}{2} \]
Or

\[ K = \frac{1}{2}mv^2 \]

BUT IF WE USE THE RELATIVISTIC MOMENTUM

\[ p = \gamma mv \]

WE GET SOMETHING DIFFERENT
START WITH NEWTON’S SECOND LAW

\[ F = \frac{d}{dt} p \]

Relativistic momentum

\[ p = \gamma m u \]

\[ F = \frac{d}{dt} \gamma m u \]

Kinetic Energy, \( K \), then is force times distance

\[ K = W = \int F \, ds = \frac{d}{dt} (\gamma m u) \, ds \]

\[ s = ut \]
\[ ds = u \, dt \]

So

\[ K = \int \frac{d}{dt} (\gamma m u) \, u \, dt = m \int u \frac{d}{dt} (\gamma u) \, dt \]

\[ K = m \int u \, d(\gamma u) \]

\[ d(\gamma u) = u \, dy + \gamma \, du \]

\[ dy = d \left( 1 - \frac{u^2}{c^2} \right)^{\frac{1}{2}} \]

\[ dy = -\frac{1}{2} \left( 1 - \frac{u^2}{c^2} \right)^{-\frac{3}{2}} \left( -\frac{2u}{c^2} \right) \, du \]
\[ d\gamma = \frac{u}{c^2} \left( 1 - \frac{u^2}{c^2} \right)^{-3/2} \, du \]

\[ d(\gamma u) = u \left[ \frac{u}{c^2} \left( 1 - \frac{u^2}{c^2} \right)^{-3/2} \, du \right] + \gamma du \]

\[ d(\gamma u) = \frac{u^2}{c^2} \left( 1 - \frac{u^2}{c^2} \right)^{-3/2} \, du + \left( 1 - \frac{u^2}{c^2} \right)^{-1/2} \]

Find common denominator
\[ d(\gamma u) = \frac{du}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}} \]

Put in KE Equation

\[ K = m \int u \frac{du}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}} \]

\[ K = mc^3 \int \frac{udu}{(c^2 - u^2)^{3/2}} \]

\[ K = mc^3 \left[ \frac{1}{\sqrt{c^2 - u^2}} \right]_0^u \]
Put in the limits

\[ K = mc^3 \left[ \frac{1}{\sqrt{c^2 - u^2}} - \frac{1}{c} \right] \]

\[ K = mc^2 \left[ \frac{c}{\sqrt{c^2 - u^2}} - \frac{c}{c} \right] \]

\[ K = mc^2 \left[ \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \right] \]

\[ K = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 \]

\[ K = \gamma mc^2 - mc^2 \]

Or

\[ K = (\gamma - 1)mc^2 \]
\[ K = (\gamma - 1)mc^2 \]

\[ K = \gamma mc^2 - mc^2 \]

\[ \gamma mc^2 = K + mc^2 \]

**ALL ENERGY UNITS**

**TOTAL ENERGY = KE + MASS ENERGY**

**TOTAL ENERGY = \( \gamma mc^2 \)**

\[ E = \gamma mc^2 \]