Problem 1. Name.  
Write down your name. Clearly. In block letters!

Problem 2. Almost van der Waals.  
The energy of the gas depends on temperature and volume according to  
\[ E = f_1(T) + f_2(V), \]
Where \( f_1 \) and \( f_2 \) are some known/given functions.

a. (11pt.) What can you say about temperature dependence of \( \alpha/\beta_T \)?

b. (11pt.) If \( \alpha/\beta_T \) is given how the temperature of this gas will change if the gas undergoes the expansion by \( dV \) into vacuum?

c. (11pt.) If \( \alpha/\beta_T \) is given how the temperature of this gas will change if the gas undergoes the expansion by \( dV \) by the adiabatic process?

Consider a cyclic engine operating with one mole of an ideal monoatomic gas in the cycle which consists of three legs \( a \to b \to c \to a \). \( T_a \) is the temperature of the gas in point \( a \).

\( a \to b \) is increase of temperature from \( T_a \) to \( 4T_a \) in such a process that \( P = \lambda V \).

\( b \to c \) is compression at constant pressure back to \( V_a \).

\( c \to a \) is cooling at constant volume back to \( T_a \).

All processes are reversible.

a. (5pt.) What is the net entropy change of the gas in one cycle?

b. (5pt.) What is the net change of the energy of the gas in one cycle?

c. (6pt.) What is the change of entropy of the gas during each leg?

d. (6pt.) What work is done by the gas during each leg?

e. (6pt.) What is the total heat that the gas obtained/gave up during the full cycle?

f. (5pt.) What is the total work done by the gas during the full cycle?
Consider $N$ identical and independent particles of spin $1/2$ in a magnetic field so that the energy of each particle is either $+\epsilon$ or $-\epsilon$ depending upon the orientation of the spin: down or up respectively. The probability of the energy $-\epsilon$ is $p$ and of $+\epsilon$ is $q = 1 - p$.

a. (11pt.) Find the probability that $N_1$ spins are up and $N_2$ are down, with $N_1 + N_2 = N$.

b. (11pt.) From this probability find the average or expected value of the total energy in terms of $p$, $\epsilon$, and $N$.

c. (11pt.) By considering that $N_1$ and $N_2$ are continuous rather than discrete variables, find the most likely (i.e., maximal probability) for the values $N_1$ and $N_2$ using the approximation that $\log(x!) \approx x \log(x) - x$. 

2 – Exam 1
EXAM 2. Monday, April 8, 2013

Problem 1. Name. 1pt
Write down your name. Clearly. In block letters!

Problem 2. Qualifying Exam. 1987 Fall. Graduate level. 33pt
1. 10pt Consider a particle which can take one of only two energy levels \( E = 0 \) or \( E = \epsilon \). Find and sketch as a function of \( T \) the average energy \( \bar{E}(N,T) \) and the heat capacity \( C_V(N,T) \) for a system of \( N \) particle. The particles are distinguishable.

2. 23pt Suppose that the value of the excited state energy \( \epsilon \) varies for different particles in a system of \( N \) particles. Futhermore, assume that the number of particles with a particular value of the excited state energy between \( \epsilon \) and \( \epsilon + d\epsilon \) is \( n(\epsilon)d\epsilon \) and suppose that \( n(\epsilon) \) is a constant for all energies between 0 and \( \epsilon_0 \), and that there is no states with \( \epsilon > \epsilon_0 \), i.e.

\[
n(\epsilon) = \begin{cases} 
  n_0, & \text{for } 0 < \epsilon \leq \epsilon_0 \\
  0, & \text{for } \epsilon > \epsilon_0 
\end{cases}
\]

and \( N = \int_0^{\infty} n(\epsilon)d\epsilon = n_0\epsilon_0 \). Find the temperature dependence of the specific heat of these \( N \) particles at low temperature, \( T \ll \epsilon_0/k_B \).

Problem 3. Qualifying Exam. 1992 Fall. Graduate level. 33pt
In experiments on the absorption spectrum of gases at finite temperature, atoms are always moving either towards or away from the light source with distribution of velocities \( v_x \). As a consequences the frequency of the photon seen by an electron in a Bohr atom is Doppler shifted to a value \( \nu_D \) according to the classical formula

\[
\nu_D = \nu_0(1 + v_x/c).
\]

1. 10pt Write down the normalized Maxwell distribution for \( v_x \).

2. 10pt Determine the distribution function \( g(\nu_D) \) for the fraction of gas atoms that will absorb light at frequency \( \nu_D \).

3. 10pt Determine the fractional linewidth

\[
\left( \frac{\Delta \nu}{\nu_0} \right)_{\text{rms}} \equiv \sqrt{\left< \frac{(\nu_D - \nu_0)^2}{\nu_0^2} \right>}. \]

4. 3pt Estimate the fractional linewidth for a gas of hydrogen atoms \( (M \approx 938 MeV/c^2, \ 1 eV/k_B \approx 11600 K) \) at room temperature.

Problem 4. Qualifying Exam. 1999 Spring. Graduate level. 33pt
A polymer is a molecule composed of a long chain of identical molecular units, called monomers. For simplicity, assume the polymer consists of a long chain of $N$ rod-like monomers, each of length $l$, attached end to end. Assume that the connectors between the monomers are completely flexible so that each monomer can make any angle with respect to its neighbors. One end of the polymer is fixed while the other end is attached to a weight ($mg$) providing a constant force in the negative $\hat{z}$ direction. Except for an arbitrary constant which can be ignored, the potential energy of any configuration of the polymer can be written as,

$$E(\theta_1, \phi_1, \theta_2, \phi_2, \ldots, \theta_N, \phi_N) = \sum_{i=1}^{N} mgl(1 - \cos(\theta_i)),$$

where $\theta_i, \phi_i$ represent angles in the spherical coordinates that the $i^{th}$ monomer makes with the negative $\hat{z}$ direction. Each angle can take all possible values $0 < \theta_i < \pi$ and $0 < \phi_i < 2\pi$.

1. 11pt Determine the classical canonical partition function $Z(T, N)$ for this polymer in contact with a temperature bath at temperature $T$.

2. 11pt Determine the thermodynamic energy $U(T, N)$ and entropy $S(T, N)$ of the system. Find the temperature dependence of $U$ and $S$ for very small and very large temperatures.

3. 11pt If the polymer is isolated from the temperature bath and the mass at the end of the polymer is increased adiabatically from its initial value of $m$ to a final value of $2m$, determine the final temperature of the polymer.
A particular ideal gas is characterized by the following equations of state:

\[ PV = Nk_B T \quad \text{and} \quad U = cNk_B T, \]

where \( c \) is a constant. The gas is used as a medium in a heat engine/refrigerator whose cycle can be represented on a \( P - V \) diagram as shown in the figure.

1. 6pt Determine the change in the internal energy of the gas for one complete cycle (\( A \rightarrow B \rightarrow C \rightarrow D \rightarrow A \)).

2. 6pt Determine the change in the entropy of the gas for one complete cycle (\( A \rightarrow B \rightarrow C \rightarrow D \rightarrow A \)).

3. 6pt Determine the equation of the adiabats.

4. 7pt Determine the thermodynamic efficiency of the above heat engine. Express your answer in terms of \( c \), \( P_1 \), and \( P_2 \).

Problem 2. Qualifying Exam. 1996 Fall. Graduate level. 25pt

Consider a molecular clock that can take on four angular positions \( \theta = n(\pi/2) \), for \( n = 0, 1, 2, 3 \). Its energy is given by

\[ E = -A \cos \theta. \]

It is subject to thermal fluctuations at a temperature \( T \).

1. 12pt Determine the value of \( \langle \cos \theta \rangle \). Give the high-temperature and low-temperature limits. Interpret your results physically.

2. 13pt Determine the value of \( \langle \cos^2 \theta \rangle \). Give the high-temperature and low-temperature limits. Interpret your results physically.
Problem 3. Qualifying Exam. 1990 Fall. Graduate level. 25pt

Let $\epsilon_{\vec{p}} = \rho(|p_x| + |p_y| + |p_z|)$ be the energy-momentum relation of a conducting electrons in a certain (fictitious) metal, where $\rho$ is a constant with the dimensions of velocity, and $|x|$ denotes the absolute value of $x$.

1. 5pt Draw a picture of a typical constant-energy surface $\epsilon_{\vec{p}} = \epsilon$, for the above $\epsilon_{\vec{p}}$.

2. 5pt Express the Fermi energy $\epsilon_F$ as a function of the electron density $n$ for this metal. (Note, that electrons have spin 1/2.)

3. 5pt Calculate the electronic density of states (per unit volume) as a function of $\epsilon$ for this metal. Denote it as $D(E)$.

4. 5pt Calculate the total energy of $N$ electrons at $T = 0$ in this metal. Express $E_0/N$ in terms of $\epsilon_F$.

5. 5pt Calculate the heat capacity per electron of this gas at small temperature. Express result in terms of $T/\epsilon_F$.

Problem 4. Qualifying Exam. 1990 Spring. Graduate level. 25pt

A system of ideal Boltzmann gas containing $N$ monoatomic molecules (of mass $m$ each) is trapped in a three-dimensional harmonic potential well $V(x, y, z) = \frac{1}{2}K(x^2 + y^2 + z^2)$, so it will form a spherical cloud whose room-mean-square radius $\bar{r}_{\text{rms}} \equiv \sqrt{(x^2 + y^2 + z^2)}^{1/2}$ will be the function of temperature. Calculate:

1. 5pt The (quantum) canonical partition function, (which no longer has volume $V$ as an independent variable due to the trapping potential).

2. 5pt The free energy $F(N, T)$;

3. 5pt The total entropy of the cloud $S(N, T)$;

4. 5pt Take the high-temperature approximation of $S(N, T)$ to the leading order in some small parameter, and use it to obtain the specific heat $C(N, T)$ in this limit to the same order. (Note: the specific heat is not of constant volume, nor of constant pressure, but of constant $K$, the force constant, and, of course, constant $N$.) Interpret your result in light of the equipartition theorem.

5. 5pt Given that $\langle n|x^2|n\rangle = (n+1/2)\hbar\omega_0/K$ for a quantum mechanical harmonic oscillator of mass $m$ and force constant $K$, calculate $\bar{r}_{\text{rms}}$, and hence the “mean” volume $\bar{V} \equiv \frac{4\pi}{3}\bar{r}_{\text{rms}}^3$, of the above cloud as a function of $N$ and $T$. What is the smallest $\bar{V}$ at any temperature, and what is the behavior of $\bar{V}$ with respect to $T$ in the high-temperature limit?

THE END!