Homework 1. Due Feb.1

Read: **LL** 9, 11-16
Problems to study: **K.1** ex 2
LL 1 means section 1 from Landau and Lifshitz book
K.1 ex 2 means example 2 from section 1 of Kubo's book.

Problem 1. Jacobian

Prove the Jacobian relations used in LL
a.

$$\frac{\partial(u,y)}{\partial(x,y)} = \left(\frac{\partial u}{\partial x}\right)_y$$

b.

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(u,v)}{\partial(y,x)}$$

c.

$$\frac{\partial(g,f)}{\partial(x,y)} = \frac{\partial(g,f)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}$$

Problem 2.

a. Show that

$$\left(\frac{\partial S}{\partial P}\right)_T = -\alpha V,$$

where α is the coefficient of thermal expansion.

b. Show that for an isothermal compression

$$dE = (-T\alpha V + PV\beta_T)dP,$$

where β_T is the isothermal compressibility. To get the full credit for this problem you must start with $dE = TdS - PdV$.

Problem 3. Ideal gas

Using the ideal gas expression for entropy (see **K.**1 ex 2)

$$S(E,V,N) = N\left\{\log\frac{V}{N} + \frac{3}{2}\log\frac{2E}{3N} + \log\frac{(2\pi m)^{3/2}e^{5/2}}{h^3}\right\}$$

a. find $E(S,V,N)$ and derive T, P from it;

b. find the free energy $F(T,V,N) = E - TS$ and derive S, and P from it;

c. What is the equation of state of an ideal gas? What is the condition for an adiabatic process for an ideal gas in terms of variables T and V?

d. find specific heats $C_V(T,V)$ and $C_P(T,P)$ for an ideal gas.

e. Find α, β_T, and β_S for an ideal gas.

Problem 4. C_P and C_V.

Derive the following general relation between specific heats C_P and C_V:

$$C_P - C_V = -T\frac{[(\partial P/\partial T)_V]^2}{(\partial P/\partial V)_T}$$

(see **LL** 16). Check it for the particular case of classical ideal gas.

Problem 5. Joule-Thomson process

Consider a Joule-Thomson process. Let's take the difference in pressure to be small. What is the change of entropy per volume of filtered gas?

Homework 2. Due Feb. 8.

Read: **LL** 19,15,24

Problem 1. A Brick

For a solid body (brick) C_P and C_V are almost the same, as the thermal expansion is small. In a large range of temperatures the heat capacity can be considered to be independent of temperature.

If temperature of a brick with heat capacity C has changed from T_1 to T_2 what is the change of energy ΔE and entropy ΔS?

Problem 2. A Brick and an Iceberg.

The thermodynamic system consists of a hot brick of temperature T_1 and specific heat C and of an iceberg at $T_2 = 0°\text{C}$. What is the maximal work W that can be performed by bringing this system to the state of thermal equilibrium? Consider the effects due to the change of the total volume of the system as negligible.

Problem 3. Two bricks

There are two identical bricks with heat capacitance C each. The temperature of the first brick is T_1, the temperature of the second is T_2. What will be the finite temperature of the bricks

a. if we bring the bricks to a contact and leave them touching for a long time?

b. if we extract as much work as possible from the equilibration process?

c. In which case the final temperature is smaller?

d. What is the maximal amount of work we were able to extract in the part **b**?

Problem 4. Three bricks

There are three identical bricks with temperatures T_1, T_2, and T_3. You are allowed to use any engines, but you cannot use external work or heat. What is the larges temperature you can give to one of the bricks?

Homework 3. Due date Feb. 17.

Read: **LL** 19,15,24

Problem 1. Magnetization

In the presence of magnetic field H one defines magnetization of the body as $M = -(\partial E/\partial H)_{S,V}$. Then one thinks about the energy as of function of three independent thermodynamic variables $E = E(S, V, H)$ with $dE = TdS - PdV - MdH$.

Assume that we know the complete equation of state $P(T, V, H)$, the temperature dependence of (constant volume, constant magnetic field) specific heat $C_V(T, V_0, H_0)$ at some given V_0 and H_0 and temperature and magnetic field dependence of magnetization $M(T, V_0, H)$ at the same given V_0 .

Find the full dependence of $C_V(T, V, H)$ and $M(T, V, H)$ from this data.

Problem 2. Heat

Show that for any infinitesimal slow process the amount of heat transferred is given by

$$đQ = \frac{C_P}{C_P - C_V} PdV + \frac{C_V}{C_P - C_V} VdP$$

Problem 3. String

A string of length L has a temperature dependent tension $F(T)$. How does the heat capacity of the string depend on the displacement $x \ll L$ of the center of the string perpendicular to the string?

Problem 4. Equilibrium

Let's consider a macroscopic system which has an internal parameter ϕ. This parameter is not a conserved quantity (for example, magnetization). Assume that we know the free energy $F(\phi, T, V)$ of the system as a function of all variables, including the variable ϕ.

a. Show that

$$\left(\frac{\partial S}{\partial \phi}\right)_{E,V} = -\frac{1}{T}\left(\frac{\partial F}{\partial \phi}\right)_{T,V}$$

b. Find the average value of $\langle x \rangle$ for the string in the equilibrium in the problem **String**.

c. How will the formula of the part **a.** change if the pressure is constant, instead of volume?

Problem 5. 2^{nd} order.

Let's consider a macroscopic system, which has the free energy of the form

$$F(\phi, T) = F_0(T) + a(T - T_c)\phi^2 + b\phi^4,$$

where $a > 0$, $b > 0$, $T_c > 0$ are temperature independent constants and ϕ is a parameter. The value of this parameter is defined by an equilibrium condition.

a. Calculate the value of ϕ in equilibrium, for both $T > T_c$ and $T < T_c$.

b. Calculate the value of F in equilibrium, for both $T > T_c$ and $T < T_c$.

c. Calculate the value of C_V for both $T > T_c$ and $T < T_c$.

Homework 4. Due Feb. 24.

Problem 1. Atmosphere

Using the equation of state $PV = NT$ and the expression for the internal energy $E = c_V NT$ of an ideal gas:

a. Derive an expression for chemical potential as a function of temperature and pressure $\mu(P, T)$ for an ideal gas.

b. Derive the pressure of an ideal gas as a function of an altitude in constant gravitational field $g = 9.8m/s^2$ using the result of **a.** Assume that temperature of the gas is constant and does not depend on the altitude.

c. Derive the density of the gas as a function of an altitude using the result of **b.**

d. Estimate how much the density of N_2 molecules drops at the top of Everest (height about 9km), assuming $T = 300K$.

Problem 2. Phase transition.

For a vapor in equilibrium with its solid phase, find the relation between the pressure P and the temperature T, if the molar latent heat q is constant. The vapor is close to the ideal gas, and the molar volume of the gas is much larger than that of solid.

Problem 3. Mixture of gases

A thermally isolated container is divided by a partition into two compartments, the right hand compartment having a volume b times larger than the left one. The left compartment contains ν moles of an ideal gas at temperature T, and pressure P. The right compartment also contains ν moles of an ideal gas at temperature T. The partition is now removed. Calculate:

a. The final pressure

b. The total change in the entropy if the gases are different.

c. The total change in the entropy if the gases are the same.

d. What is the total change in the entropy if the gases are the same and $b = 1$? Can you explain the result?

Problem 4. EXTRA PROBLEM, due the end of the semester.

Consider a gas of electrons at temperature T. The concentration of electrons is small. There is no gravity.

a. The gas is s on top of infinitely large the positively charged plate with uniform charge σ per area. The number of electrons is N per area of the plate. Find the concentration as a function of distance to the plate.

b. The total number of electrons is N. Find the concentration of the electrons as a function of a distance from a uniformly and positively charged sphere of radius R and total charge Q

Homework 5. Due Mar. 2.

Problem 1. Distribution, average, and fluctuations

You flip a coin N times and count the number of heads. You repeat this activity many many times.

a. With what probability the number of heads is n?

b. What is the average number of heads?

c. What is the standard deviation (r.m.s. fluctuation)?

Problem 2. Distribution function of an oscillator.

A classical one dimensional oscillator ($V(x) = \frac{m\omega^2 x^2}{2}$) has a statistical distribution function $\varrho(p, q) = Ae^{-E(p,q)/T}$, where A is a normalization parameter, T is a parameter which is called temperature, and $E(p, q)$ is the oscillator's energy.

a. Find the normalization constant A.

b. Find the average coordinate of the particle.

c. Find the average momentum of the particle.

d. find the r.m.s. fluctuations of the particle' coordinate and momentum.

e. Find the average energy of the particle.

f. What is it's heat capacity?

g. Find the distribution function for a quantity $f = f(p, x)$, where $f(p, x) = p - \omega m x$.

Problem 3. Distribution function.

A classical one dimensional particle, confined to the region $y \geq y_0$ is in a potential

$$V(y) = V_0 \log{(y/y_0)}$$

The statistical distribution is given by $\varrho(p, q) = Ae^{-E(p,q)/T}$, where A is a normalization parameter, T is a parameter which is called temperature, and $E(p, q)$ is the particle's energy.

a. Find the normalization constant A. Determine the critical temperature T_c above which the particle escapes to infinity (You need to figure out what it means.).

b. Write down the normalized positional distribution function $f(y)$ (i.e. the probability per unit distance to find the particle between y and $y + dy$) for this particle for $T < T_c$.

c. Find the average distance $\langle y \rangle$ for the particle. What happens if $0 < T_c/2 - T \ll T_c/2$

Problem 4. EXTRA PROBLEM, due the end of the semester.

In a huge cavity of temperature T there two neutral atoms at large distance R from each other. Each atom has a magnetic susceptibility χ. Find the force between the two atoms. Neglect the retardation due to the finite speed of light.

Homework 6. Due Mar. 9.

Problem 1.

There is a system consisting of N independent particles. Each particle can have only one of the three energy levels $-\epsilon_0$, 0, $+\epsilon_0$. Find the thermodynamic weight W_M of a state with the total energy $E = M\epsilon_0$, $(M = -N, \ldots, N)$.

a. Calculate the entropy of the system as a function of its energy.

b. Find the relation between the temperature and the energy of the system.

c. How does the energy of the system depend on the temperature for $T \ll \epsilon_0$ and $T \gg \epsilon_0$?

Problem 2. An oscillator.

N classical particles with mass m put in the $3D$ harmonic oscillator trap potential $V = \frac{m\omega^2 r^2}{2}$.

a. Calculate the number of states (volume of phase space) with the total energy $E < E_0$.

b. Then using $w(E) = Ae^{-E/T}$, calculate the average energy of the system. (Do not forget normalization)

c. Calculate the heat capacity of the system.

d. Calculate the entropy of the system.

Problem 3. A spin 1/2

A spin 1/2 is in a magnetic field $\mathcal{H}$ pointing in z direction. The spin is at equilibrium with heat bath at temperature T.

a. Calculate the average components of the spin.

b. Calculate the average square of the components.

c. Calculate the r.m.s. fluctuation of the components.

Problem 4. N spins 1/2

N spins 1/2 are in a magnetic field $\mathcal{H}$ pointing in z direction. They are in equilibrium with heat bath at temperature T.

a. Calculate the free energy of the system?

b. What is the heat capacity of the system?

c. What is the magnetization $\mathcal{M}$ of the system?

d. What is the magnetic susceptibility of the system? (magnetic susceptibility is defined as $\chi = (\partial\mathcal{M}/\partial\mathcal{H})_T$)

Plot the answers for the last three questions.

Homework 7. Due the first class after Spring Break, It counts as the first exam!!

Problem 1. Fluctuations.

A brick of mass M is sitting on a table at temperature T.

a. How high the brick is hovering over the table?

b. Estimate the numbers.

Problem 2. Rotating gas.

The canonical partition function of a classical, monoatomic, ideal gas in a cylinder rotating with angular velocity ω is given by (see Kubo chapter 2, problem 4)

$$Z = \frac{1}{N!}\left[\pi R^2 L \left(\frac{2\pi m T}{h^2}\right)^{3/2} \frac{e^x - 1}{x}\right]^N$$

where $x = \frac{m\omega^2 R^2}{2T}$.

a. Find the angular momentum M of the rotating gas as a function of temperature and angular velocity.

b. Consider the limits of the obtained expression for M corresponding to very high and very low temperatures. Give the physical interpretation of obtained results. What is the criterion of high and low in this case?

c. How much energy one should supply to heat the gas from a very low temperature T_0 to a very high temperature T_f? Denote the initial angular velocity of the cylinder as ω_0 . Neglect the moment of inertia of the cylinder (vessel) itself.

Problem 3. Quantum oscillator.

A quantum particle of charge q is in the oscillator potential $V(x) = \frac{m\omega^2}{2}x^2$ in $1D$ at temperature T.

a. Find the heat capacity of this system.

b. Find the electric dipole susceptibility of the system. $(d = -(\partial F/\partial E)_{V,N,T}$, $\chi = (\partial d/\partial E)_{V,N,T}$

c. For the oscillator state ψ_n in the presence of electric field E calculate $x_n = \langle\psi_n|\hat{x}|\psi_n\rangle$, and then average displacement $\bar{x} = \sum_n x_n w_n$.

Problem 4. N bricks

You have N identical bricks with temperatures T_i, $i = 1 \ldots N$.

a. What is the highest temperature you can give to one of the bricks.

b. If all of the initial temperatures are in a small interval $T_0 < T_i < T_0 + \Delta T$, where $\Delta T \ll T_0$, find the highest temperature of part a. for $N \to \infty$.

Problem 5. String

A $3D$ string of length L has a temperature dependent tension $f(T)$.

a. What is the average value of the amplitude a_k^α of the kth harmonic of a small deformation of the string?

b. What is $\langle a_k^\alpha a_{k'}^\beta \rangle$ for two harmonics k and k'?

c. Calculate $\left\langle e^{\vec{n}\vec{a}_k} \right\rangle$.

Homework 8. Fluctuations. Due Mar. 23

Problem 1. $\langle(\Delta W)^2\rangle$

Find $\langle(\Delta W)^2\rangle$.

Problem 2. $\langle\Delta P\Delta T\rangle$

Find $\langle\Delta P\Delta T\rangle$.

Problem 3. $\langle\Delta P\Delta V\rangle$

Find $\langle\Delta P\Delta V\rangle$.

Problem 4. $\langle\Delta S\Delta V\rangle$

Find $\langle\Delta S\Delta T\rangle$.

Problem 5. Two masses

In $1D$ two equal masses m are connected with the walls and with each other by springs with spring coefficients k. The springs are unstretched. The coordinate of the first mass is x_1, the coordinate of the second is x_2. The temperature is T. Find

a. $\langle(\Delta x_1)^2\rangle$

b. $\langle(\Delta x_2)^2\rangle$

c. $\langle\Delta x_1\Delta x_2\rangle$

d. What will happen if we allow the masses to fluctuate in $3D$?

Problem 6. N masses

In $1D$ N equal masses m are connected with the walls and with each other by springs with spring coefficients k. The springs are unstretched. The coordinates of the masses are x_i, $i = 1 \ldots N$. The temperature is T. Find

a. $\langle(\Delta x_i)^2\rangle$

b. $\langle\Delta x_i\Delta x_j\rangle$

Homework 9. Maxwell and Ising. Due March 30.

Problem 1. $\langle 1/v \rangle$.

Find $\langle 1/v \rangle$, where v is the magnitude of the particle's velocity. Express the result through $\langle v \rangle$.

Problem 2. Small hole.

A vessel with an ideal gas is held at constant temperature T. There is a small whole in the wall of the vessel. Calculate how the density of the gas is changing with time if outside of the vessel is vacuum.

Problem 3. Ising chain.

Ising chain, or one dimensional Ising model is the following. There is a $1D$ chain with sites enumerated by $i = 1 \ldots N$. in each site there is Ising variable $\sigma_i = \pm 1$. The Hamiltonian of the chain is given by

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}.$$

Calculate specific heat of this chain. Is there a phase transition?

Homework 10. Due Apr. 06

Problem 1. Oscillator. An exercise.

For one $1D$ particle of charge q at temperature T in the potential $V(x) = \frac{m\omega^2}{2}x^2$ calculate the zero field electric dipole susceptibility using perturbation theory.

a. In classical case.

b. In quantum case.

Problem 2. Oscillator. Anharmonicity.

For the classical $1D$ harmonic oscillator $V(x) = \frac{m\omega^2}{2}x^2$, temperature T charge q and small anharmonicity term $V_a = \beta x^4$:

a. Calculate the correction to the heat capacity.

b. Calculate the correction to the zero field electric dipole susceptibility.

Problem 3. Occupation numbers.

Calculate the fluctuations of the occupation numbers for

a. Fermi gas.

b. Bose gas.

c. Classical gas.

Homework 11. Due Apr. 13

Problem 1. Landau levels.

N spin 1/2 fermions with positive charge $+e$ placed in a magnetic field with field strength B in the $\hat{z}$-direction. The single particle energy levels are Landau levels, are characterized by two quantum numbers and can be written as

$$\epsilon_n(p_z) = (n + 1/2)\hbar\omega_c + \frac{p_z^2}{2m}$$

where p_z ($-\infty \le p_z \le \infty$) is the continuous projection of the momentum on the $\hat{z}$ direction and non-negative integer $n = 0, 1, 2 \ldots$ is associated with the motion in $x - y$ plane. Here m is the mass of the particle and $\omega_c = eB/mc$ is the cyclotron frequency.

The degeneracy of each level is given by

$$g\left[\epsilon_n(p_z)\right] = \frac{m\omega_c}{2\pi\hbar}A,$$

where $A = V/L$ is the area of the system in the $x - y$ plane, V is the volume of the system, and L is the length of the system in the $\hat{z}$ direction.

Assuming the BOLTZMANN statistics in canonical ensemble is valid, determine

a. The equation of state.

b. The magnetization of the gas.

c. What condition on the chemical potential must be true so that the use of the Boltzmann statistics is justified?

Problem 2. Dielectric constant of ideal gas.

Consider an ideal classical gas of rigid dipolar molecules in an electric field E. The dipole moment of each molecule is μ. Calculate the linear dielectric constant ϵ of the gas as a function of temperature T and density $\rho = N/V$.

Problem 3. Sticky molecules.

A box of volume V contains N molecules of a classical gas. The molecules can either be bound to the walls of the box with binding energy ϵ_b and dispersion relation of a $2D$ gas $\epsilon(p) = (p_x^2 + p_y^2)/2m$, or move freely within the volume of the box with dispersion relation of a $3D$ gas $\epsilon(p) = (p_x^2 + p_y^2 + p_z^2)/2m$.

Find what portion of the molecules are stuck to the surface. Obtain an explicit results in two limiting cases of low and high temperature.

Homework 12. Due Apr. 20

Problem 1. Degenerate electron gas.

a. Find the temperature dependence of the chemical potential of the D dimensional gas of fermions at small temperatures.

b. What is the condition for the temperature to be small enough?

c. Estimate this temperature for an electron gas in a typical metal.

Problem 2.

The identical particles of the D-dimensional non interacting gas have dispersion relation $\epsilon_p = A|p|^\alpha$.

a. Calculate the density of states $\nu_D(\epsilon)$.

b. Calculate how PV depends on the Energy of the gas.

c. In case the particles are Fermions calculate the Fermi momentum p_F and Fermi energy ϵ_F of the gas.

Problem 3. Mean speed.

a. Find the mean speed of the fermions in an ideal D-dimensional fermi gas at $T = 0$ (the dispersion is $\epsilon_p = p^2/2m$) in terms of v_F – Fermi velocity (velocity at ϵ_F).

b. Find the average kinetic energy of a fermion in terms of v_F.

Problem 4. Electrons in metals.

Find how the heat capacity of the D dimensional Fermi gas with $\epsilon(p) = \vec{p}^2/2m$ depends on the density at small temperatures.

Homework 13. Due Apr. 27

Problem 1. Two particles.

There are three quantum states of energies 0, ϵ, and 2ϵ. Consider a system of two indistinguishable non-interacting particles which can occupy these states. The system is coupled with the heat bath at temperature T.

a. Calculate the free energy of the system in case the particles are Bosons.

b. Calculate the free energy of the system in case the particles are Fermions.

c. What is the ration of occupation probability of the highest energy state of the system to the lowest energy state in each of these cases?

Problem 2. Chemical potential of the Bose gas.

Find how the chemical potential of a D-dimensional Bose gas depends on temperature when the chemical potential is small.

Problem 3. He^4.

Consider a non interacting gas of He^4 (spin$=0$) confined to a film (i.e., confined in $\hat{z}$ direction, but infinite in the x and y directions). Because of the confinement, the p_z component of the momentum is quantized into discrete levels, $n = 0, 1, \ldots$, so that the energy spectrum is

$$\epsilon_n(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \epsilon_n,$$

where $\mathbf{p}$ is the two-dimensional (p_x, p_y) momentum, and $0 = \epsilon_0 < \epsilon_1 < \ldots \epsilon_n < \ldots$.

a. Write down the density of states for the system.

b. Write down the expression which determines the density of He^4 atoms n as a function of a chemical potential μ and temperature T. Evaluate integrals explicitly.

c. As $T \to 0$, i.e. $T \ll \epsilon_1$ find the limiting value of μ and how the specific heat depends on T in this limit.

d. Determine $\mu(N, V, T)$ in the limit $|\mu| \gg T$ (classical limit) assuming $\epsilon_n = \epsilon n$.