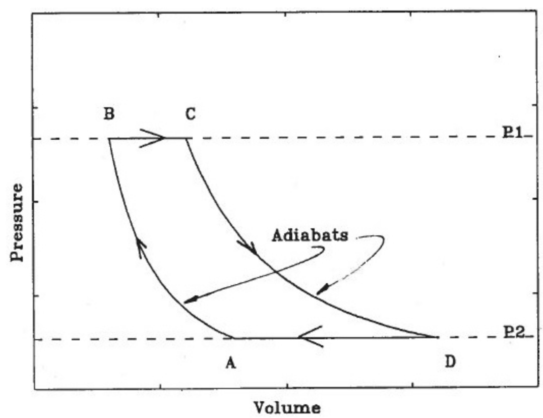
EXAM 1. Friday, Oct. 5, 2012

Problem 1. Name. 1pt

Write down your name. Clearly. In block letters!

Problem 2. 1993, Spring, graduate. 33pt.



A particular ideal gas is characterized by the following equations of state:

$$PV = Nk_BT \qquad \text{and} \qquad E = cNk_BT,$$

where c is a constant. The gas is used as a medium in a heat engine/refrigerator whose cycle an be represented on a $P-V$ diagram as shown in the figure.

a.*(11pt.)* Determine the change in the internal energy of the gas for one complete cycle ($A \to B \to C \to D \to A$).

b.*(11pt.)* Determine the equation of the adiabats.

c.*(11pt.)* Determine the thermodynamic efficiency of the above heat engine. Express your answer in terms of c, P_1, and P_2.

Problem 3. N bricks. 33pt.

You have N identical bricks with temperatures T_i, $i = 1 \dots N$.

a.*(13pt.)* What is the highest temperature you can give to one of the bricks.

b.*(20pt.)* If all of the initial temperatures are spread over a small interval of temperatures ΔT such that the average temperature is T_0, and $\Delta T \ll T_0$, find the highest temperature of part a. for $N \to \infty$.

Problem 4. Compressibility. 33pt.

a.*(16pt.)* Prove the following thermodynamic identity:

$$\kappa_T = \frac{1}{n^2}\left(\frac{\partial n}{\partial \mu}\right)_T$$

where κ_T is the isothermal compressibility and $n = N/V$ is the particle number density. Here P denotes the pressure, T the absolute temperature, and μ the chemical potential.

b.*(17pt.)* Prove the following for a quasistatic adiabatic process

$$\frac{dV}{V} = -\kappa_T \frac{C_V}{C_P} dP.$$

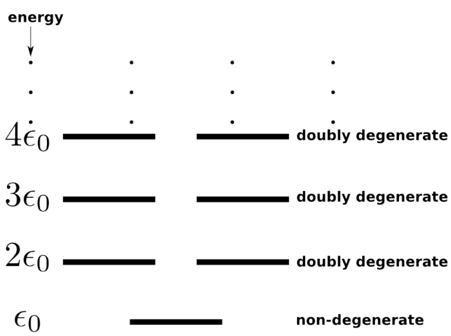
Show that for the ideal gas this equation gives $PV^\gamma = \text{const.}$, where $\gamma = C_P/C_V$.

EXAM 2. Friday, Nov. 2, 2012

Problem 1. Name. 1pt

Write down your name. Clearly. In block letters!

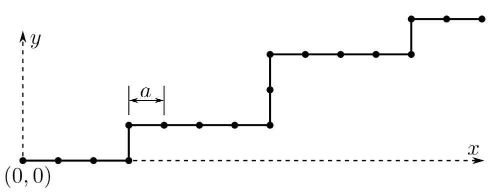
Problem 2. Qualifying Exam. 1993 Fall. Undergraduate level. 33pt



Shown in the figure is a set of available single-particle states and their energies. By properly filling these single-particle states, find the energies and degeneracies for a system of four identical non-interacting particles in the system's lowest energy level and in its first excited energy level, assuming that these particles are:

1. identical spinless bosons,
2. identical spinless fermions.
3. At a finite temperature T, calculate the ratio $r = P_1/P_0$, again for the two cases defined above. P_1 is the probability for finding the four-particle system in the first excited energy level, and P_0 is the probability for finding the same system in the lowest energy level.

Problem 3. Qualifying Exam. 1994 Fall. Graduate level. 33pt



A two dimensional fictitious "polymer" is constructed as follows: It consists of N identical monomers of length a each, joined into a chain. (See figure.) Starting from the left end of the "polymer", which corresponds to the coordinates $x = 0$, $y = 0$, each new monomer added prefers to extend the "polymer" to the right (i.e., to the $+x$ direction) with energy $\epsilon = 0$, or upward (i.e., to the $+y$ direction), with energy $\epsilon = J > 0$. It is given that no monomer added will extend the polymer to the left or downward.

1. A macro state of this "polymer" is characterized by N and m, where m denotes the number of monomers which extend the "polymer" upward. Find the entropy of this macro state as a function of N and m. Assume that both N and m are $\gg 1$, and use Stirling's formula $\ln N! \approx N(\ln N - 1)$ to simplify your result.
2. Write down the free energy of this macro-state of the "polymer", as a function of N, m, and the temperature T.
3. From minimizing this free energy, calculate the equilibrium value of m as a function of N and T.

4. Calculate the equilibrium value of the free energy as a function of N and T, and from which, calculate the specific heat of this "polymer" as a function of N and T. (Note: Simplify your expression for the free energy to something quite simple before calculating specific heat! *Make sure your result has the right dimension, or else you get no credit for this point.*)

5. Show that in the limit $k_B T \gg J$, the specific heat of this "polymer" is inversely proportional to T^2, and obtain the coefficient of proportionality.

Problem 4. $3D$ quantum oscillator. 33pt

A $3D$ harmonic oscillator has mass m and the same spring constant $m\omega^2$ for all three directions. It is coupled to a heat bath of temperature T.

1. What are the energy E_0 and degeneracy g_0 of the ground sate? Of the first excited state? Of the second excited state?

2. What is the free energy of the oscillator?

3. What is the heat capacitance of the oscillator?

4. What is the rms fluctuation $\delta E \equiv (\overline{E^2} - (\bar{E})^2)^{1/2}$ in the energy?

5. (**extra 5 points**) What is the degeneracy of the Kth excited energy level?

EXAM 3. Final. Friday, Dec. 7, 2012

Problem 1. 1992-Fall-SM-G-5

In experiments on the absorption spectrum of gases at finite temperature, atoms are always moving either towards or away from the light source with distribution of velocities v_x. As a consequences the frequency of the photon seen by an electron in a Bohr atom is Doppler shifted to a value ν_D according to the classical formula

$$\nu_D = \nu_0(1 + v_x/c).$$

1. Write down the normalized Boltzmann distribution for v_x.
2. Determine the distribution function $g(\nu_D)$ for the fraction of gas atoms that will absorb light at frequency ν_D.
3. Determine the fractional linewidth

$$\left(\frac{\Delta\nu}{\nu_0}\right)_{rms} \equiv \sqrt{\left\langle \frac{(\nu_D - \nu_0)^2}{\nu_0^2} \right\rangle}.$$

4. Estimate the fractional linewidth for a gas of hydrogen atoms at room temperature.

Problem 2. 1996-Fall-SM-G-5

A system of N non-interacting Fermi particles with spin 1/2 and mass m, is confined to a volume V. It is at temperature $T = 0$. The particles have an energy-momentum dispersion given by $\epsilon = p^2/2m$.

1. Determine the chemical potential.
2. Determine the internal energy.
3. The original volume V adjoins a vacuum region with volume $\delta V \ll V$. Both chambers are thermally isolated from the environment. The Fermi gas now freely expands into the vacuum region. Assume that the internal energy as a function of the temperature can be parametrized at low temperatures by

$$U = \alpha(V) + \beta(V)T^2,$$

   where $\alpha(V)$ and $\beta(V)$ are functions of volume. Determine the final temperature of the gas after the expansion. Your answer should be expressed in terms of the functions $\alpha(V)$ and $\beta(V)$.
4. Determine $\alpha(V)$.
5. Determine $\beta(V)$.

[**Hint:** The following integral expressions may prove useful:

$$\int_0^\infty \frac{\phi(x)dx}{e^x z^{-1}+1} = \sum_{k=0}^{\infty}\left(\frac{2^{2k-1}-1}{2^{2k-2}}\right)\zeta(2k)\Phi^{(2k)}(\ln z) \approx \Phi(\ln z)+\zeta(2)\Phi^{(2)}(\ln z)+\frac{7}{4}\zeta(4)\Phi^{(4)}(\ln z),$$

where $\Phi(\xi) = \int_0^\xi \phi(x)dx$, and $\Phi^{(2n)}(\xi) = \left.\frac{\partial^{2n}\Phi(x)}{\partial x^{2n}}\right|_{x=\xi}$.

$$\int_0^\infty \frac{x^{n-1}dx}{e^x z^{-1}+1} = \Gamma(n)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k^n}, \qquad \int_0^\infty \frac{x^{n-1}dx}{e^x z^{-1}-1} = \Gamma(n)\sum_{k=1}^{\infty}\frac{z^k}{k^n}$$

$$\int_0^\infty \frac{x^{n-1}dx}{e^x z^{-1}-1} = \Gamma(n)\left[\Gamma(1-n)[-\ln z]^{n-1} + \sum_{k=0}^{\infty}(-1)^k\xi(n-k)\frac{[-\ln z]^k}{\Gamma(k+1)}\frac{z^k}{k^n}\right]$$

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Problem 3. 1997-Spring-SM-G-5

Consider N particles of a non-interacting spin-1/2 Fermi gas of mass m confined to a two dimensional plane of area A. Take $\epsilon = \epsilon_0\,(p/p_0)^{3/2}$.

1. Determine the $T=0$ energy of this Fermi gas.
2. Determine the $T=0$ surface pressure, σ, of the Fermi gas. This surface pressure is the force per unit length acting on the confining boundary of the gas.

Problem 4. 1998-Spring-SM-G-5

Consider a cyclic engine operating with one mole of an ideal gas in the following cycle:

$a \to b$: expansion at constant pressure, the temperature going from T_a to $3T_a$.

$b \to c$: expansion at constant temperature $3T_a$, the volume going to $4V_a$.

$c \to d$: cooling at constant volume $4V_a$, the temperature going to T_a.

$b \to a$: compression at constant temperature, the volume going from $4V_a$ to V_a.

1. Sketch the cycle on a $P-V$ diagram.
2. Find the entropy change and the change in internal energy of the gas for each part of the cycle. Find the net entropy change and the net change in internal energy over the full cycle.
3. Calculate the thermodynamic efficiency of this engine and compare it to the ideal efficiency of an engine operating between T_a and $3T_a$. Assume no irreversible process occur.

THE END!