Analytical Mechanics. Phys 601

Artem G. Abanov

Contents

LECTURE 1
Oscillations. Oscillations with friction.

- Oscillators:

$$m\ddot{x} = -kx, \qquad ml\ddot{\phi} = -mg\sin\phi \approx -mg\phi, \qquad -L\ddot{Q} = \frac{Q}{C},$$

All of these equation have the same form

$$\ddot{x} = -\omega_0^2 x, \qquad \omega_0^2 = \begin{cases} k/m \\ g/l \\ 1/LC \end{cases}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- The solution

$$x(t) = A\sin(\omega t) + B\cos(\omega t) = C\sin(\omega t + \phi), \qquad B = x_0, \qquad \omega A = v_0.$$

- Oscillates forever: $C = \sqrt{A^2 + B^2}$ — amplitude; $\phi = \tan^{-1}(A/B)$ — phase.
- Oscillations with friction:

$$m\ddot{x} = -kx - \gamma\dot{x}, \qquad -L\ddot{Q} = \frac{Q}{C} + R\dot{Q},$$

- Consider

$$\ddot{x} = -\omega_0^2 x - 2\gamma\dot{x}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

This is a linear equation with constant coefficients. We look for the solution in the form $x = \Re C e^{i\omega t}$, where ω and C are complex constants.

$$\omega^2 - 2i\gamma\omega - \omega_0^2 = 0, \qquad \omega = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

- Two solutions, two independent constants.
- Two cases: $\gamma < \omega_0$ and $\gamma > \omega_0$.
- In the first case (underdamping):

$$x = e^{-\gamma t}\Re\left[C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right] = Ce^{-\gamma t}\sin\left(\Omega t + \phi\right), \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}$$

Decaying oscillations. Shifted frequency.
- In the second case (overdamping):

$$x = Ae^{-\Gamma_- t} + Be^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} > 0$$

- For the initial conditions $x(t=0) = x_0$ and $v(t=0) = 0$ we find $A = x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}$, $B = -x_0\frac{\Gamma_-}{\Gamma_+ - \Gamma_-}$. For $t \to \infty$ the B term can be dropped as $\Gamma_+ > \Gamma_-$, then $x(t) \approx x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}e^{-\Gamma_- t}$.
- At $\gamma \to \infty$, $\Gamma_- \to \frac{\omega_0^2}{2\gamma} \to 0$. The motion is arrested. The example is an oscillator in honey.

LECTURE 2
Oscillations with external force. Resonance.

2.1. Comments on dissipation.

- Time reversibility. A need for a large subsystem.
- Locality in time.

2.2. Resonance

- Let's add an external force:
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$
- The full solution is the sum of the solution of the homogeneous equation with any solution of the inhomogeneous one. This full solution will depend on two arbitrary constants. These constants are determined by the initial conditions.
- Let's assume, that $f(t)$ is not decaying with time. The solution of the inhomogeneous equation also will not decay in time, while any solution of the homogeneous equation will decay. So in a long time $t \gg 1/\gamma$ The solution of the homogeneous equation can be neglected. In particular this means that the asymptotic of the solution does not depend on the initial conditions.
- Let's now assume that the force $f(t)$ is periodic. with some period. It then can be represented by a Fourier series. As the equation is linear the solution will also be a series, where each term corresponds to a force with a single frequency. So we need to solve
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$
where f is the force's amplitude.
- Let's look at the solution in the form $x = f\Im C e^{i\Omega_f t}$, and use $\sin(\Omega_f t) = \Im e^{i\Omega_f t}$. We then get
$$C = \frac{1}{\omega_0^2 - \Omega_f^2 + 2i\gamma\Omega_f} = |C|e^{-i\phi},$$
$$|C| = \frac{1}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}}, \quad \tan\phi = \frac{2\gamma\Omega_f}{\omega_0^2 - \Omega_f^2}$$
$$x(t) = f\Im|C|e^{i\Omega_f t + i\phi} = f|C|\sin\left(\Omega_f t - \phi\right),$$

- Resonance frequency:
$$\Omega_f^r = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\Omega^2 - \gamma^2},$$
where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ is the frequency of the damped oscillator.
- Phase changes sign at $\Omega_f^\phi = \omega_0 > \Omega_f^r$. Importance of the phase – phase shift.
- To analyze resonant response we analyze $|C|^2$.
- The most interesting case $\gamma \ll \omega_0$, then the response $|C|^2$ has a very sharp peak at $\Omega_f \approx \omega_0$:
$$|C|^2 = \frac{1}{(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2} \approx \frac{1}{4\omega_0^2}\frac{1}{(\Omega_f - \omega_0)^2 + \gamma^2},$$
so that the peak is very symmetric.
- $|C|^2_{\text{max}} \approx \frac{1}{4\gamma^2\omega_0^2}$.
- to find HWHM we need to solve $(\Omega_f - \omega_0)^2 + \gamma^2 = 2\gamma^2$, so HWHM $= \gamma$, and FWHM $= 2\gamma$.
- Q factor (quality factor). The good measure of the quality of an oscillator is $Q = \omega_0/\text{FWHM} = \omega_0/2\gamma$. (decay time) $= 1/\gamma$, period $= 2\pi/\omega_0$, so $Q = \pi\frac{\text{decay time}}{\text{period}}$.
- For a grandfather's wall clock $Q \approx 100$, for the quartz watch $Q \sim 10^4$.

2.3. Response.

- Response. The main quantity of interest. What is "property"?
- The equation
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t).$$
The LHS is time translation invariant!
- Multiply by $e^{i\omega t}$ and integrate over time. Denote
$$x_\omega = \int x(t)e^{i\omega t}dt.$$
Then we have
$$\left(-\omega^2 - 2i\gamma\omega + \omega_0^2\right)x_\omega = \int f(t)e^{i\omega t}dt, \qquad x_\omega = -\frac{\int f(t')e^{i\omega t'}dt'}{\omega^2 + 2i\gamma\omega - \omega_0^2}$$
- The inverse Fourier transform gives
$$x(t) = \int \frac{d\omega}{2\pi}e^{-i\omega t}x_\omega = -\int f(t')dt' \int \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \int \chi(t-t')f(t')dt'.$$
- Where the response function is $(\gamma < \omega_0)$
$$\chi(t) = -\int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \begin{cases} e^{-\gamma t}\frac{\sin(t\sqrt{\omega_0^2 - \gamma^2})}{\sqrt{\omega_0^2-\gamma^2}} & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases}, \qquad \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$
- Causality principle. Poles in the lower half of the complex ω plane. True for any (linear) response function. The importance of $\gamma > 0$ condition.

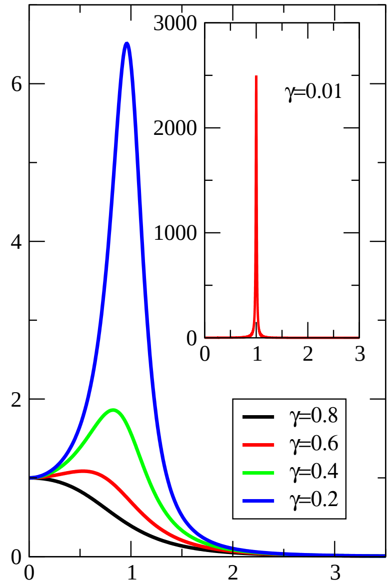


Figure 1. Resonant response. For insert $Q = 50$.

LECTURE 3
Work energy theorem. Energy conservation. Potential energy.

3.1. Mathematical preliminaries.

- Functions of many variables.
- Differential of a function of many variables.
- Examples.

3.2. Work.

- A work done by a force: $\delta W = \vec{F} \cdot d\vec{r}$.
- Superposition. If there are many forces, the total work is the sum of the works done by each.
- Finite displacement. Line integral.

3.3. Change of kinetic energy.

- If a body of mass m moves under the force $\vec{F}$, then.

$$m\frac{d\vec{v}}{dt} = \vec{F}, \qquad m d\vec{v} = \vec{F}dt, \qquad m\vec{v} \cdot d\vec{v} = \vec{F} \cdot \vec{v}dt = \vec{F} \cdot d\vec{r} = \delta W.$$

  So we have

$$d\frac{mv^2}{2} = \delta W$$

- The change of kinetic energy equals the total work done by all forces.

3.4. Conservative forces. Energy conservation.

- Fundamental forces. Depend on coordinate, do not depend on time.
- Work done by the forces over a closed loop is zero.
- Work is independent of the path.
- Consider two paths: first dx, then dy; first dy then dx

$$\delta W = F_x(x,y)dx + F_y(x+dx,y)dy = F_y(x,y)dy + F_x(x,y+dy)dx, \qquad \left.\frac{\partial F_y}{\partial x}\right|_y = \left.\frac{\partial F_x}{\partial y}\right|_x .$$

- So a small work done by a conservative force:
$$\delta W = F_x dx + F_y dy, \qquad \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$
is a full differential!
$$\delta W = -dU$$
- It means that there is such a function of the coordinates $U(x, y)$, that
$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad \text{or} \quad \vec{F} = -\text{grad}U \equiv -\vec{\nabla}U.$$
- So on a trajectory:
$$d\left(\frac{mv^2}{2} + U\right) = 0, \qquad K + U = \text{const.}$$
- If the force $\vec{F}(\vec{r})$ is known, then there is a test for if the force is conservative.
$$\nabla \times \vec{F} = 0.$$
In $1D$ the force that depends only on the coordinate is always conservative.
- Examples.
- In $1D$ in the case when the force depends only on coordinates the equation of motion can be solved in quadratures.
- The number of conservation laws is enough to solve the equations.
- If the force depends on the coordinate only $F(x)$, then there exists a function — potential energy — with the following property
$$F(x) = -\frac{\partial U}{\partial x}$$
Such function is not unique as one can always add an arbitrary constant to the potential energy.
- The total energy is then conserved
$$K + U = \text{const.}, \qquad \frac{m\dot{x}^2}{2} + U(x) = E$$
- Energy E can be calculated from the initial conditions: $E = \frac{mv_0^2}{2} + U(x_0)$
- The allowed areas where the particle can be are given by $E - U(x) > 0$.
- Turning points. Prohibited regions.
- Notice, that the equation of motion depends only on the difference $E - U(x) = \frac{mv_0^2}{2} + U(x_0) - U(x)$ of the potential energies in different points, so the zero of the potential energy (the arbitrary constant that was added to the function) does not play a role.
- We thus found that
$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$
- Energy conservation law cannot tell the direction of the velocity, as the kinetic energy depends only on absolute value of the velocity. In $1D$ it cannot tell which sign to use "+" or "−". You must not forget to figure it out by other means.

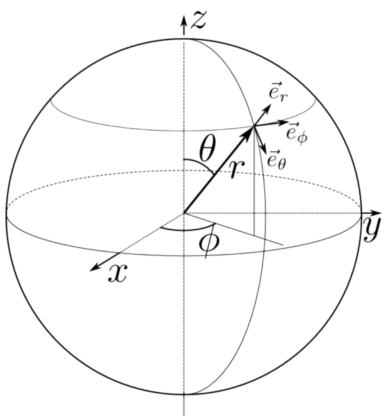
- We then can solve the equation

$$\pm\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{E-U(x)}} = dt, \qquad t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E-U(x')}}$$

- Examples:
  - Motion under a constant force.
  - Oscillator.
  - Pendulum.
- Divergence of the period close to the maximum of the potential energy.

LECTURE 4
Central forces. Effective potential.

4.1. Spherical coordinates.



- The spherical coordinates are given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \quad .$$

- The coordinates r, θ, and ϕ, the corresponding unit vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$.
- the vector $d\vec{r}$ is then

$$d\vec{r} = \vec{e}_r dr + \vec{e}_\theta r d\theta + \vec{e}_\phi r\sin\theta d\phi.$$
$$d\vec{r} = \vec{e}_x dx + \vec{e}_y dy + \vec{e}_z dz$$

- Imagine now a function of coordinates U. We want to find the components of a vector $\vec{\nabla}U$ in the spherical coordinates.
- Consider a function U as a function of Cartesian coordinates: $U(x,y,z)$. Then

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = \vec{\nabla}U\cdot d\vec{r}.$$
$$\vec{\nabla}U = \frac{\partial U}{\partial x}\vec{e}_x + \frac{\partial U}{\partial y}\vec{e}_y + \frac{\partial U}{\partial z}\vec{e}_z$$

- On the other hand, like any vector we can write the vector $\vec{\nabla}U$ in the spherical coordinates.
$$\vec{\nabla}U = (\vec{\nabla}U)_r\vec{e}_r + (\vec{\nabla}U)_\theta\vec{e}_\theta + (\vec{\nabla}U)_\phi\vec{e}_\phi,$$
where $(\vec{\nabla}U)_r$, $(\vec{\nabla}U)_\theta$, and $(\vec{\nabla}U)_\phi$ are the components of the vector $\vec{\nabla}U$ in the spherical coordinates. It is those components that we want to find
- Then
$$dU = \vec{\nabla}U \cdot d\vec{r} = (\vec{\nabla}U)_r dr + (\vec{\nabla}U)_\theta r d\theta + (\vec{\nabla}U)_\phi r\sin\theta d\phi$$
- On the other hand if we now consider U as a function of the spherical coordinates $U(r,\theta,\phi)$, then
$$dU = \frac{\partial U}{\partial r}dr + \frac{\partial U}{\partial \theta}d\theta + \frac{\partial U}{\partial \phi}d\phi$$
- Comparing the two expressions for dU we find
$$\begin{array}{l} (\vec{\nabla}U)_r = \frac{\partial U}{\partial r} \\ (\vec{\nabla}U)_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta} \\ (\vec{\nabla}U)_\phi = \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} \end{array}.$$
- In particular
$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\vec{e}_r - \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\vec{e}_\phi.$$

4.2. Central force

- Consider a motion of a body under central force. Take the origin in the center of force.
- A central force is given by
$$\vec{F} = F(r)\vec{e}_r.$$
- Such force is always conservative: $\vec{\nabla} \times \vec{F} = 0$, so there is a potential energy:
$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\vec{e}_r, \quad \frac{\partial U}{\partial \theta} = 0, \quad \frac{\partial U}{\partial \phi} = 0,$$
so that potential energy depends only on the distance r, $U(r)$.
- The torque of the central force $\tau = \vec{r}\times\vec{F} = 0$, so the angular momentum is conserved: $\vec{J} = \text{const}$.
- The motion is all in one plane! The plane which contains the vector of the initial velocity and the initial radius vector.
- We take this plane as $x - y$ plane.
- The angular momentum is $\vec{J} = J\vec{e}_z$, where $J = |\vec{J}| = \text{const}.$. This constant is given by initial conditions $J = m|\vec{r}_0 \times \vec{v}_0|$.
$$mr^2\dot{\phi} = J, \qquad \dot{\phi} = \frac{J}{mr^2}$$
- In the $x - y$ plane we can use the polar coordinates: r and ϕ.
- The velocity in these coordinates is
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi = \dot{r}\vec{e}_r + \frac{J}{mr}\vec{e}_\phi$$

- The kinetic energy then is

$$K = \frac{m\dot{\vec{v}}^2}{2} = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2}$$

- The total energy then is

$$E = K + U = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} + U(r).$$

- If we introduce the effective potential energy

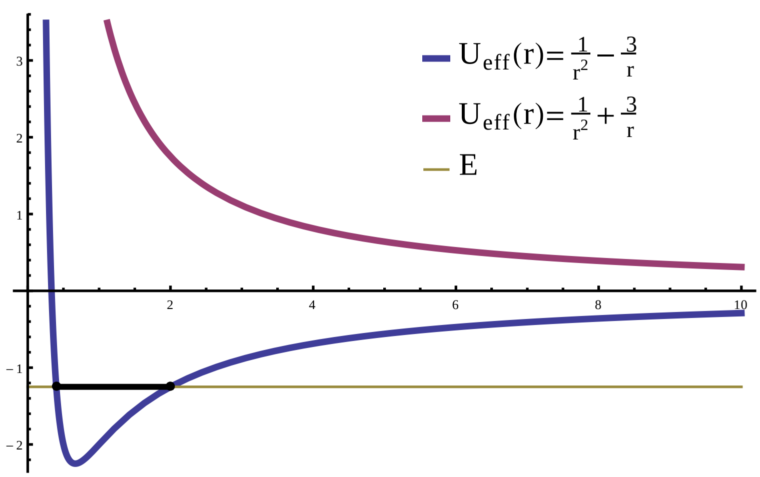
$$U_{eff}(r) = \frac{J^2}{2mr^2} + U(r),$$

then we have

$$\frac{m\dot{r}^2}{2} + U_{eff}(r) = E, \qquad m\ddot{r} = -\frac{\partial U_{eff}}{\partial r}$$

- This is a one dimensional motion which was solved before.

4.3. Kepler orbits.



Historically, the Kepler problem — the problem of motion of the bodies in the Newtonian gravitational field — is one of the most important problems in physics. It is the solution of the problems and experimental verification of the results that convinced the physics community in the power of Newton's new math and in the correctness of his mechanics. For the first time people could understand the observed motion of the celestial bodies and make accurate predictions. The whole theory turned out to be much simpler than what existed before.

- In the Kepler problem we want to consider the motion of a body of mass m in the gravitational central force due to much larger mass M.
- As $M \gg m$ we ignore the motion of the larger mass M and consider its position fixed in space (we will discuss what happens when this limit is not applicable later)
- The force that acts on the mass m is given by the Newton's law of gravity:

$$\vec{F} = -\frac{GmM}{r^3}\vec{r} = -\frac{GmM}{r^2}\vec{e}_r$$

where $\vec{e}_r$ is the direction from M to m.
- The potential energy is then given by

$$U(r) = -\frac{GMm}{r}, \qquad -\frac{\partial U}{\partial r} = -\frac{GmM}{r^2}, \qquad U(r \to \infty) \to 0$$

- The effective potential is
$$U_{eff}(r) = \frac{J^2}{2mr^2} - \frac{GMm}{r},$$
where J is the angular momentum.
- For the Coulomb potential we will have the same r dependence, but for the like charges the sign in front of the last term is different — repulsion.
- In case of attraction for $J \neq 0$ the function $U_{eff}(r)$ always has a minimum for some distance r_0. It has no minimum for the repulsive interaction.
- Looking at the graph of $U_{eff}(r)$ we see, that
  - for the repulsive interaction there can be no bounded orbits. The total energy E of the body is always positive. The minimal distance the body may have with the center is given by the solution of the equation $U_{eff}(r_{min}) = E$.
  - for the attractive interaction if $E > 0$, then the motion is not bounded. The minimal distance the body may have with the center is given by the solution of the equation $U_{eff}(r_{min}) = E$.
  - for the attractive for $U_{eff}(r_{min}) < E < 0$, the motion is bounded between the two real solutions of the equation $U_{eff}(r) = E$. One of the solution is larger than r_0, the other is smaller.
  - for the attractive for $U_{eff}(r_{min}) = E$, the only solution is $r = r_0$. So the motion is around the circle with fixed radius r_0. For such motion we must have
$$\frac{mv^2}{r_0} = \frac{GmM}{r_0^2}, \qquad \frac{J^2}{mr_0^3} = \frac{GmM}{r_0^2}, \qquad r_0 = \frac{J^2}{Gm^2M}$$
and
$$U_{eff}(r_0) = E = \frac{mv^2}{2} - \frac{GmM}{r_0} = -\frac{1}{2}\frac{GmM}{r_0}$$
these results can also be obtained from the equation on the minimum of the effective potential energy $\frac{\partial U_{eff}}{\partial r} = 0$.
- In the motion the angular momentum is conserved and all motion happens in one plane.
- In that plane we describe the motion by two time dependent polar coordinates $r(t)$ and $\phi(t)$. The dynamics is given by the angular momentum conservation and the effective equation of motion for the r coordinate
$$\dot{\phi} = \frac{J}{mr^2(t)}, \qquad m\ddot{r} = -\frac{\partial U_{eff}(r)}{\partial r}.$$
- For now I am not interested in the time evolution and only want to find the trajectory of the body. This trajectory is given by the function $r(\phi)$. In order to find it I will use the trick we used before
$$\frac{dr}{dt} = \frac{d\phi}{dt}\frac{dr}{d\phi} = \frac{J}{mr^2(t)}\frac{dr}{d\phi} = -\frac{J}{m}\frac{d(1/r)}{d\phi}, \qquad \frac{d^2r}{dt^2} = -\frac{J^2}{m^2r^2}\frac{d^2(1/r)}{d\phi^2}$$
- On the other hand
$$\frac{\partial U_{eff}}{\partial r} = -\frac{J^2}{m}(1/r)^3 + GMm\,(1/r)^2.$$

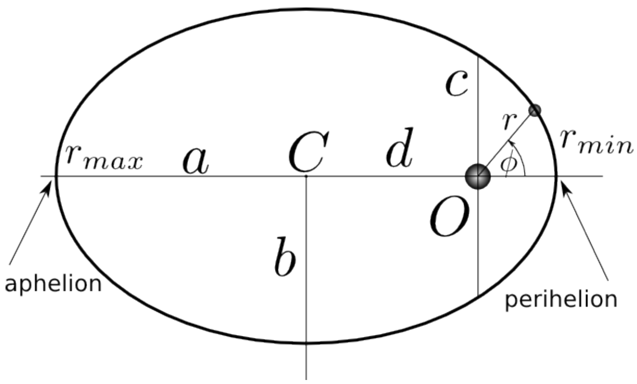
- Now I denote $u(\phi) = 1/r(\phi)$ and get

$$-\frac{J^2}{m}u^2\frac{d^2u}{d\phi^2} = \frac{J^2}{m}u^3 - GMmu^2$$

or

$$u'' = -u + \frac{GMm^2}{J^2}$$

LECTURE 5
Kepler orbits continued



- We stopped at the equation
$$u'' = -u + \frac{GMm^2}{J^2}$$
- The general solution of this equation is
$$u = \frac{GMm^2}{J^2} + A\cos(\phi - \phi_0)$$
- We can put $\phi_0 = 0$ by redefinition. So we have
$$\frac{1}{r} = \gamma + A\cos\phi, \qquad \gamma = \frac{GMm^2}{J^2}$$
If $\gamma = 0$ this is the equation of a straight line in the polar coordinates.
- A more conventional way to write the trajectory is
$$\frac{1}{r} = \frac{1}{c}\left(1 + \epsilon\cos\phi\right), \qquad c = \frac{J^2}{GMm^2} = \frac{1}{\gamma}$$
where $\epsilon > 0$ is dimentionless number – eccentricity of the ellipse, while c has a dimension of length
- We see that
  - If $\epsilon < 1$ the orbit is periodic.
  - If $\epsilon < 1$ the minimal and maximal distance to the center — the perihelion and aphelion are at $\phi = 0$ and $\phi = \pi$ respectively.
$$r_{min} = \frac{c}{1+\epsilon}, \qquad r_{max} = \frac{c}{1-\epsilon}$$

  – If $\epsilon > 1$, then the trajectory is unbounded.

- If we know c and ϵ we know the orbit, so we must be able to find out J and E from c and ϵ. By definition of c we find $J^2 = cGMm^2$. In order to find E, we notice, that at $r = r_{min}$, $\dot{r} = 0$, so at this moment $v = r_{min}\dot{\phi} = J/mr_{min}$, so the kinetic energy $K = mv^2/2 = J^2/2mr_{min}^2$, the potential energy is $U = -GmM/r_{min}$. So the total energy is

$$E = K + U = -\frac{1-\epsilon^2}{2}\frac{GmM}{c}, \qquad J^2 = cGMm^2,$$

  Indeed we see, that if $\epsilon < 1$, $E < 0$ and the orbit is bounded.
- The ellipse can be written as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

  with

$$a = \frac{c}{1-\epsilon^2}, \quad b = \frac{c}{\sqrt{1-\epsilon^2}}, \quad d = a\epsilon, \quad b^2 = ac.$$

- One can check, that the position of the large mass M is one of the focuses of the ellipse — NOT ITS CENTER!
- This is the **first Kepler's law**: all planets go around the ellipses with the sun at one of the foci.

5.1. Kepler's second law

The conservation of the angular momentum reads

$$\frac{1}{2}r^2\dot{\phi} = \frac{J}{2m}.$$

We see, that in the LHS rate at which a line from the sun to a comet or planet sweeps out area:

$$\frac{dA}{dt} = \frac{J}{2m}.$$

This rate is constant! So

- **Second Kepler's law:** A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

5.2. Kepler's third law

Consider now the closed orbits only. There is a period T of the rotation of a planet around the sun. We want to find this period.

The total area of an ellipse is $A = \pi ab$, so as the rate dA/dt is constant the period is

$$T = \frac{A}{dA/dt} = \frac{2\pi abm}{J},$$

Now we square the relation and use $b^2 = ac$ and $c = \frac{J^2}{GMm^2}$ to find

$$T^2 = 4\pi^2\frac{m^2}{J^2}a^3c = \frac{4\pi^2}{GM}a^3$$

Notice, that the mass of the planet and its angular momentum canceled out! so

- **Third Kepler's law:** For all bodies orbiting the sun the ration of the square of the period to the cube of the semimajor axis is the same.

This is one way to measure the mass of the sun. For all planets one plots the cube of the semimajor axes as x and the square of the period as y. One then draws a straight line through all points. The slope of that line is $GM/4\pi^2$.

5.3. Another way

- Another way to solve the problem is starting from the following equations:

$$\dot{\phi} = \frac{J}{mr^2(t)}, \qquad \frac{m\dot{r}^2}{2} + U_{eff}(r) = E$$

- For now I am not interested in the time evolution and only want to find the trajectory of the body. This trajectory is given by the function $r(\phi)$. In order to find it I will express $\dot{r}$ from the second equation and divide it by $\dot{\phi}$ from the first. I then find

$$\frac{\dot{r}}{\dot{\phi}} = \frac{dr}{d\phi} = r^2\sqrt{\frac{2m}{J^2}}\sqrt{E - U_{eff}(r)}$$

or

$$\frac{J}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E - U_{eff}(r)}} = d\phi, \qquad \frac{J}{\sqrt{2m}}\int^r \frac{dr'}{r'^2\sqrt{E - U_{eff}(r')}} = \int d\phi$$

The integral becomes a standard one after substitution $x = 1/r$.

5.4. Conserved Laplace-Runge-Lenz vector $\vec{A}$

The Kepler problem has an interesting additional symmetry. This symmetry leads to the conservation of the Laplace-Runge-Lenz vector $\vec{A}$. If the gravitational force is $\vec{F} = -\frac{k}{r^2}\vec{e}_r$, then we define:

$$\vec{A} = \vec{p} \times \vec{J} - mk\vec{e}_r,$$

where $\vec{J} = \vec{r} \times \vec{p}$ This vector can be defined for both gravitational and Coulomb forces: $k > 0$ for attraction and $k < 0$ for repulsion.

An important feature of the "inverse square force" is that this vector is conserved. Let's check it. First we notice, that $\dot{\vec{J}} = 0$, so we need to calculate:

$$\dot{\vec{A}} = \dot{\vec{p}} \times \vec{J} - mk\dot{\vec{e}}_r$$

Now using

$$\dot{\vec{p}} = \vec{F}, \qquad \dot{\vec{e}}_r = \vec{\omega} \times \vec{e}_r = \frac{1}{mr^2}\vec{J} \times \vec{e}_r$$

We then see

$$\dot{\vec{A}} = \vec{F} \times \vec{J} - \frac{k}{r^2}\vec{J} \times \vec{e}_r = \left(\vec{F} + \frac{k}{r^2}\vec{e}_r\right) \times \vec{J} = 0$$

So this vector is indeed conserved.

The question is: Is this conservation of vector $\vec{A}$ an independent conservation law? If it is the three components of the vector $\vec{A}$ are three new conservation laws. And the answer is that not all of it.

- As $\vec{J} = \vec{r} \times \vec{p}$ is orthogonal to $\vec{e}_r$, we see, that $\vec{J} \cdot \vec{A} = 0$. So the component of $\vec{A}$ perpendicular to the plane of the planet rotation is always zero.
- Now let's calculate the magnitude of this vector

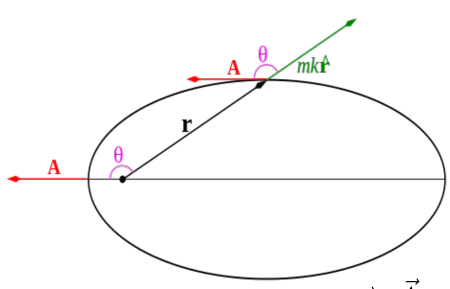
$$\vec{A} \cdot \vec{A} = \vec{p}^2 \vec{J}^2 - (\vec{p} \cdot \vec{J})^2 + m^2 k^2 - 2mk \vec{e}_r \cdot [\vec{p} \times \vec{J}] = \vec{p}^2 \vec{J}^2 + m^2 k^2 - \frac{2mk}{r} \vec{J} \cdot [\vec{r} \times \vec{p}]$$
$$= 2m \left(\frac{\vec{p}^2}{2m} - \frac{k}{r} \right) \vec{J}^2 + m^2 k^2 = 2mE\vec{J}^2 + m^2 k^2 = \epsilon^2 k^2 m^2.$$

  So we see, that the magnitude of $\vec{A}$ is not an independent conservation law.
- We are left with only the direction of $\vec{A}$ within the orbit plane. Let's check this direction. As the vector is conserved we can calculate it in any point of orbit. So let's consider the perihelion. At perihelion $\vec{p}_{per} \perp \vec{r}_{per} \perp \vec{J}$, where the subscript *per* means the value at perihelion. So simple examination shows that $\vec{p}_{per} \times \vec{J} = pJ\vec{e}_{per}$. So at this point $\vec{A} = (p_{per}J - mk)\vec{e}_{per}$. However, vector $\vec{A}$ is a constant of motion, so if it has this magnitude and direction in one point it will have the same magnitude and direction at all points! On the other hand $J = p_{per} r_{min}$, so $\vec{A} = mr_{min}(2\frac{p_{per}^2}{2m} - \frac{k}{r_{min}})\vec{e}_{per} = mr_{min}(2K_{per} + U_{per})$. We know that $r_{min} = \frac{c}{1+\epsilon}$, $K_{per} = \frac{1}{2}\frac{k}{c}(1+\epsilon)^2$ and $U_{per} = -\frac{k}{c}(1+\epsilon)$. So

$$\vec{A} = mk\epsilon\vec{e}_{per}.$$

  We see, that for Kepler orbits $\vec{A}$ points to the point of the trajectory where the planet or comet is the closest to the sun.
- So we see, that $\vec{A}$ provides us with only one new independent conserved quantity.

5.4.1. Kepler orbits from $\vec{A}$



The existence of an extra conservation law simplifies many calculations. For example we can derive equation for the trajectories without solving any differential equations. Let's do just that.

Let's derive the equation for Kepler orbits (trajectories) from our new knowledge of the conservation of the vector $\vec{A}$.

$$\vec{r} \cdot \vec{A} = \vec{r} \cdot [\vec{p} \times \vec{J}] - mkr = J^2 - mkr$$

On the other hand

$$\vec{r} \cdot \vec{A} = rA\cos\theta, \quad \text{so} \quad rA\cos\theta = J^2 - mkr$$

Or

$$\frac{1}{r} = \frac{mk}{J^2}\left(1 + \frac{A}{mk}\cos\theta\right), \qquad c = \frac{J^2}{mk}, \qquad \epsilon = \frac{A}{mk}.$$

5.5. Bertrand's theorem

Bertrand's theorem states that among central force potentials with bound orbits, there are only two types of central force potentials with the property that all bound orbits are also closed orbits:

(a) an inverse-square central force such as the gravitational or electrostatic potential

$$V(\mathbf{r}) = \frac{-k}{r},$$

(b) the radial harmonic oscillator potential

$$V(\mathbf{r}) = \frac{1}{2}kr^2.$$

The theorem was discovered by and named for Joseph Bertrand.

The proof can be found here: `https://en.wikipedia.org/wiki/Bertrand%27s_theorem`

LECTURE 6
Scattering cross-section.

- Set up of a scattering problem. Experiment, detector, etc.
- Energy. Impact parameter. The scattering angle. Impact parameter as a function of the scattering angle $\rho(\theta)$.
- Flux of particle. Same energy, different impact parameters, different scattering angles.
- The scattering problem, n — the flux, number of particles per unit area per unit time. dN the number of particles scattered between the angles θ and $\theta + d\theta$ per unit time. A suitable quantity do describe the scattering
$$d\sigma = \frac{dN}{n}.$$
It has the units of area and is called differential cross-section.
- If we know the function $\rho(\theta)$, then only the particles which are in between $\rho(\theta)$ and $\rho(\theta + d\theta)$ are scattered at the angle between θ and $\theta + d\theta$. So $dN = n2\pi\rho d\rho$, or
$$d\sigma = 2\pi\rho d\rho = 2\pi\rho \left|\frac{d\rho}{d\theta}\right| d\theta$$
(The absolute value is needed because the derivative is usually negative.)
- Often $d\sigma$ refers not to the scattering between θ and $\theta + d\theta$, but to the scattering to the solid angle $d\omega = 2\pi \sin\theta d\theta$. Then
$$d\sigma = \frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| d\omega$$

Examples

- Cross-section for scattering of particles from a perfectly rigid sphere of radius R.
  - The scattering angle $\theta = 2\phi$.
  - $R\sin\phi = \rho$, so $\rho = R\sin(\theta/2)$.
  - $$\sigma = \frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| d\omega = \frac{1}{4}R^2 d\omega$$
  - Independent of the incoming energy. The scattering does not probe what is inside.

- The total cross-section area is

$$\sigma = \int d\sigma = \frac{1}{4} R^2 2\pi \int_0^{\pi} \sin\theta d\theta = \pi R^2$$

- Cross-section for scattering of particles from a spherical potential well of depth U_0 and radius R.
  - Energy conservation

$$\frac{mv_0^2}{2} = \frac{mv^2}{2} - U_0, \qquad v = v_0\sqrt{1 + \frac{2U_0}{mv_0^2}} = v_0\sqrt{1 + U_0/E}$$

  - Angular momentum conservation

$$v_0 \sin\alpha = v \sin\beta, \qquad \sin\alpha = n(E)\sin\beta, \qquad n(E) = \sqrt{1 + U_0/E}$$

  - Scattering angle

$$\theta = 2(\alpha - \beta)$$

  - Impact parameter

$$\rho = R\sin\alpha$$

  - So we have

$$\frac{\rho}{R} = n\sin(\alpha - \theta/2) = n\sin\alpha\cos(\theta/2) - n\cos\alpha\sin(\theta/2) = n\frac{\rho}{R}\cos(\theta/2) - n\sqrt{1 - \rho/R}\sin(\theta/2)$$

$$\rho^2 = R^2 \frac{n^2 \sin^2(\theta/2)}{1 + n^2 - 2n\cos(\theta/2)}.$$

  - The differential cross-section is

$$d\sigma = \frac{R^2 n^2}{4\cos(\theta/2)} \frac{(n\cos(\theta/2) - 1)(n - \cos(\theta/2))}{(1 + n^2 - 2n\cos(\theta/2))^2} d\omega$$

  - Differential cross-section depends on E/U_0, where E is the energy of incoming particles. By measuring this dependence we can find U_0 from the scattering.
  - The scattering angle changes from 0 ($\rho = 0$) to θ_{max}, where $\cos(\theta_{max}) = 1/n$ (for $\rho = R$). The total cross-section is the integral

$$\sigma = \int_0^{\theta_{max}} d\sigma = \pi R^2.$$

    It does not depend on energy or U_0.
- Return to the rigid sphere but with U_0.

LECTURE 7
Rutherford's formula.

Consider the scattering of a particle of initial velocity v_∞ from the central force given by the potential energy $U(r)$.

- The energy is
$$E = \frac{mv_\infty^2}{2}.$$
- The angular momentum is given by
$$L_\phi = mv_\infty \rho,$$
where ρ is the impact parameter.
- The trajectory is given by
$$\pm(\phi - \phi_0) = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{r} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}, \qquad U_{eff}(r) = U(r) + \frac{L_\phi^2}{2mr^2}$$

where r_0 and ϕ_0 are some distance and angle on the trajectory.

At some point the particle is at the closest distance r_0 to the center. The angle at this point is ϕ_0 (the angle at the initial infinity is zero.) Let's find the distance r_0. As the energy and the angular momentum are conserved and at the closest point the velocity is perpendicular to the radius we have

$$E = \frac{mv_0^2}{2} + U(r_0), \qquad L_\phi = mr_0 v_0.$$

so we find that the equation for r_0 is

$$U_{eff}(r_0) = E.$$

This is, of course, obvious from the picture of motion in the central field as a one dimensional motion in the effective potential $U_{eff}(r)$.

The angle ϕ_0 is then given by

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}. \tag{7.1}$$

From geometry the scattering angle θ is given by the relation

$$\pi - \theta + 2\phi_0 = 2\pi. \tag{7.2}$$

So we see, that for a fixed v_0 the energy E is given, but the angular momentum L_ϕ depends on the impact parameter ρ. The equation (7.1) then gives the dependence of ϕ_0 on ρ. Then the equation (7.2) gives the dependence of the scattering angle θ on the impact parameter ρ. If we know that dependence, we can calculate the scattering cross-section.

$$d\sigma = \frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| d\omega$$

Example: Coulomb interaction. Let's say that we have a repulsive Coulomb interaction

$$U = \frac{\alpha}{r}, \qquad \alpha > 0$$

In this case the geometry gives

$$\theta = 2\phi_0 - \pi.$$

Let's calculate ϕ_0

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}}$$

where r_0 is the value of r, where the expression under the square root is zero.

Let's take the integral

$$\int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E - \alpha x - x^2 \frac{L_\phi^2}{2m}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E + \frac{\alpha^2 m}{2L_\phi^2} - \frac{L_\phi^2}{2m}(x + \frac{\alpha m}{L_\phi^2})^2}}$$

$$= \sqrt{\frac{2m}{L_\phi^2}} \int_0^{1/r_0} \frac{dx}{\sqrt{\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4} - (x + \frac{\alpha m}{L_\phi^2})^2}}$$

changing $\sqrt{\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4}} \sin\psi = x + \frac{\alpha m}{L_\phi^2}$ we find that the integral is

$$\sqrt{\frac{2m}{L_\phi^2}} \int_{\psi_1}^{\pi/2} d\psi,$$

where $\sin(\psi_1) = \frac{\alpha m}{L_\phi^2}\left(\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4}\right)^{-1/2}$ So we find that

$$\phi_0 = \pi/2 - \psi_1$$

or

$$\cos\phi_0 = \sin\psi_1 = \frac{\alpha m}{L_\phi^2}\left(\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4}\right)^{-1/2}.$$

Using $L_\phi = \rho\sqrt{2mE}$ this gives

$$\sin\frac{\theta}{2} = \frac{\alpha}{2E}\left(\rho^2 + \frac{\alpha^2}{4E^2}\right)^{-1/2}$$

or

$$\frac{\alpha^2}{4E^2}\cot^2\frac{\theta}{2} = \rho^2$$

The differential cross-section then is

$$d\sigma = \frac{d\rho^2}{d\theta}\frac{1}{2\sin\theta}d\omega = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}d\omega$$

- Notice, that the total cross-section diverges at small scattering angles.

Discussion.

- The beam. How do you characterize it?
- The statistics. What is measured?
- The beam again. Interactions.
- The forward scattering diverges.
- The cut off of the divergence is given by the size of the atom.
- Back scattering. Almost no dependence on θ.
- Energy dependence $1/E^2$.
- Plot $d\sigma$ as a function of $1/(4E)^2$, expect a straight line at large $1/(4E)^2$.
- The slope of the line gives α^2.
- What is the behavior at very large E? What is the crossing point?
- The crossing point tells us the size of the nucleus $d\sigma = \frac{R^2}{4}d\omega$.
- How much data we need to collect to get certainty of our results?

LECTURE 8
Functionals.

8.1. Difference between functions and functionals.

8.2. Examples of functionals.

- Area under the graph.
- Length of a path. Invariance under reparametrization.

It is important to specify the space of functions.

- Energy of a horizontal sting in the gravitational field.
- General form $\int_{x_1}^{x_2} L(x, y, y', y'', \dots)dx$. Important: In function L the y, y', y'' and so on are independent variables. It means that we consider a function $L(x, z_1, z_2, z_3, \dots)$ of normal variables x, z_1, z_2, z_3, …and for any function $y(x)$ at some point x we calculate $y(x)$, $y'(x)$, $y''(x)$, …and plug x and these values instead of z_1, z_2, z_3, …in $L(x, z_1, z_2, z_3, \dots)$. We do that for all points x, and then do the integration.
- Value at a point as functional. The functional which for any function returns the value of the function at a given point.
- Functions of many variables. Area of a surface. Invariance under reparametrization.

8.3. Discretization. Fanctionals as functions.

8.4. Minimization problem

- Minimal distance between two points.
- Minimal time of travel. Ferma Principe.
- Minimal potential energy of a string.
- etc.

8.5. The Euler-Lagrange equations

- The functional $A[y(x)] = \int_{x_1}^{x_2} L(y(x), y'(x), x)dx$ with the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.
- The problem is to find a function $y(x)$ which is the stationary "point" of the functional $A[y(x)]$.
- Derivation of the Euler-Lagrange equation.

- The Euler-Lagrange equation reads
$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}.$$

8.6. Examples

- Shortest path $\int_{x_1}^{x_2}\sqrt{1+(y')^2}dx$, $y(x_1)=y_1$, and $y(x_2)=y_2$.
$$L(y(x),y'(x),x)=\sqrt{1+(y')^2}, \qquad \frac{\partial L}{\partial y}=0, \qquad \frac{\partial L}{\partial y'}=\frac{y'}{\sqrt{1+(y')^2}}.$$
the Euler-Lagrange equation is
$$\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}}=0, \qquad \frac{y'}{\sqrt{1+(y')^2}}=\text{const.}, \qquad y'(x)=\text{const.}, \qquad y=ax+b.$$
The constants a and b should be computed from the boundary conditions $y(x_1)=y_1$ and $y(x_2)=y_2$.
- Shortest time to fall – Brachistochrone.
  - What path the rail should be in order for the car to take the least amount of time to go from point A to point B under gravity if it starts with zero velocity.
  - Lets take the coordinate x to go straight down and y to be horizontal, with the origin in point A.
  - The boundary conditions: for point A: $y(0)=0$; for point B: $y(x_B)=y_B$.
  - The time of travel is
$$T=\int\frac{ds}{v}=\int_0^{x_B}\frac{\sqrt{1+(y')^2}}{\sqrt{2gx}}dx$$
.
  - We have
$$L(y,y',x)=\frac{\sqrt{1+(y')^2}}{\sqrt{2gx}}, \qquad \frac{\partial L}{\partial y}=0, \qquad \frac{\partial L}{\partial y'}=\frac{1}{\sqrt{2gx}}\frac{y'}{\sqrt{1+(y')^2}}.$$
  - The Euler-Lagrange equation is
$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+(y')^2}}\right)=0, \qquad \frac{1}{x}\frac{(y')^2}{1+(y')^2}=\frac{1}{2a}, \qquad y'(x)=\sqrt{\frac{x}{2a-x}}$$
  - So the path is given by
$$y(x)=\int_0^x\sqrt{\frac{x'}{2a-x'}}dx'$$
  - The integral is taken by substitution $x=a(1-\cos\theta)$. It then becomes $a\int(1-\cos\theta)d\theta=a(\theta-\sin\theta)$. So the path is given by the parametric equations
$$x=a(1-\cos\theta), \qquad y=a(\theta-\sin\theta).$$
the constant a must be chosen such, that the point x_B, y_B is on the path.

LECTURE 9
Euler-Lagrange equation continued.

9.1. Reparametrization

The *form* of the Euler-Lagrange equation does not change under the reparametrization.

Consider a functional and corresponding E-L equation

$$A = \int_{x_1}^{x_2} L(y(x), y'_x(x), x)dx, \qquad \frac{d}{dx}\frac{\partial L}{\partial y'_x} = \frac{\partial L}{\partial y(x)}$$

Let's consider a new parameter ξ and the function $x(\xi)$ converts one old parameter x to another ξ. The functional

$$A = \int_{x_1}^{x_2} L(y(x), y'_x(x), x)dx = \int_{\xi_1}^{\xi_2} L\left(y(\xi), y'_\xi\frac{d\xi}{dx}, x\right)\frac{dx}{d\xi}d\xi,$$

where $y(\xi) \equiv y(x(\xi))$. So that

$$L_\xi = L\left(y(\xi), y'_\xi\frac{d\xi}{dx}, x\right)\frac{dx}{d\xi}$$

The E-L equation then is

$$\frac{d}{d\xi}\frac{\partial L_\xi}{\partial y'_\xi} = \frac{\partial L_\xi}{\partial y(\xi)}$$

Using

$$\frac{\partial L_\xi}{\partial y'_\xi} = \frac{dx}{d\xi}\frac{\partial L}{\partial y'_x}\frac{d\xi}{dx} = \frac{\partial L}{\partial y'_x}, \qquad \frac{\partial L_\xi}{\partial y(\xi)} = \frac{dx}{d\xi}\frac{\partial L}{\partial y(x)}$$

we see that E-L equation reads

$$\frac{d}{d\xi}\frac{\partial L}{\partial y'_x} = \frac{dx}{d\xi}\frac{\partial L}{\partial y(x)}, \qquad \frac{d}{dx}\frac{\partial L}{\partial y'_x} = \frac{\partial L}{\partial y(x)}.$$

So we return back to the original form of the E-L equation.

What we found is that E-L equations are invariant under the parameter change.

9.2. The Euler-Lagrange equations, for many variables.

9.3. Problems of Newton laws.

- Not invariant when we change the coordinate system:

$$\text{Cartesian:} \begin{cases} m\ddot{x} = F_x \\ m\ddot{y} = F_y \end{cases}, \qquad \text{Cylindrical:} \begin{cases} m\left(\ddot{r} - r\dot{\phi}^2\right) = F_r \\ m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = F_\phi \end{cases}.$$

- Too complicated, too tedious. Consider two pendulums.
- Difficult to find conservation laws.
- Symmetries are not obvious.

9.4. Newton second law as Euler-Lagrange equations

9.5. Hamilton's Principle. Action. Only minimum!

9.6. Lagrangian. Generalized coordinates.

LECTURE 10
Lagrangian mechanics.

10.1. Hamilton's Principle. Action.

For each conservative mechanical system there exists a functional, called action, which is minimal on the solution of the equation of motion

10.2. Lagrangian.

Lagrangian is not energy. We do not minimize energy. We minimize action.

10.3. Examples.

- Free fall.
- A mass on a stationary wedge. No friction.
- A mass on a moving wedge. No friction.
- A pendulum.
- A bead on a vertical rotating hoop.
  - Lagrangian.
  $$L = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta - mrR(1-\cos\theta).$$
  - Equation of motion.
  $$R\ddot{\theta} = (\Omega^2 R\cos\theta - g)\sin\theta.$$
  There are four equilibrium points
  $$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R}$$
  - Critical Ω_c. The second two equilibriums are possible only if
  $$\frac{g}{\Omega^2 R} < 1, \qquad \Omega > \Omega_c = \sqrt{g/R}.$$
  - Effective potential energy for $\Omega \sim \Omega_c$. From the Lagrangian we can read the effective potential energy:
  $$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mrR(1-\cos\theta).$$

Assuming $\Omega \sim \Omega_c$ we are interested only in small θ. So

$$U_{eff}(\theta) \approx \frac{1}{2} mR^2(\Omega_c^2 - \Omega^2)\theta^2 + \frac{3}{4!} mR^2\Omega_c^2\theta^4$$

$$U_{eff}(\theta) \approx mR^2\Omega_c(\Omega_c - \Omega)\theta^2 + \frac{3}{4!} mR^2\Omega_c^2\theta^4$$

- Spontaneous symmetry breaking. Plot the function $U_{eff}(\theta)$ for $\Omega < \Omega_c$, $\Omega = \Omega_c$, and $\Omega > \Omega_c$. Discuss universality.
- Small oscillations around $\theta = 0$, $\Omega < \Omega_c$

$$mR^2\ddot{\theta} = -mR^2(\Omega_c^2 - \Omega^2)\theta, \qquad \omega = \sqrt{\Omega_c^2 - \Omega^2}.$$

- Small oscillations around θ_0, $\Omega > \Omega_c$.

$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mrR(1 - \cos\theta),$$

$$\frac{\partial U_{eff}}{\partial\theta} = -mR(\Omega^2 R\cos\theta - g)\sin\theta, \qquad \frac{\partial^2 U_{eff}}{\partial\theta^2} = mR^2\Omega^2\sin^2\theta - mR\cos\theta(\Omega^2 R\cos\theta - g)$$

$$\left.\frac{\partial U_{eff}}{\partial\theta}\right|_{\theta=\theta_0} = 0, \qquad \left.\frac{\partial^2 U_{eff}}{\partial\theta^2}\right|_{\theta=\theta_0} = mR^2(\Omega^2 - \Omega_c^2)$$

So the Tylor expansion gives

$$U_{eff}(\theta \sim \theta_0) \approx \text{const} + \frac{1}{2} mR^2(\Omega^2 - \Omega_c^2)(\theta - \theta_0)^2$$

The frequency of small oscillations then is

$$\omega = \sqrt{\Omega^2 - \Omega_c^2}.$$

- The effective potential energy for small θ and $|\Omega - \Omega_c|$

$$U_{eff}(\theta) = \frac{1}{2} a(\Omega_c - \Omega)\theta^2 + \frac{1}{4} b\theta^4.$$

- θ_0 for the stable equilibrium is given by $\partial U_{eff}/\partial\theta = 0$

$$\theta_0 = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \sqrt{\frac{a}{b}(\Omega - \Omega_c)} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

Plot $\theta_0(\Omega)$. Non-analytic behavior at Ω_c.

- Response: how θ_0 responses to a small change in Ω.

$$\frac{\partial\theta_0}{\partial\Omega} = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \frac{1}{2}\sqrt{\frac{a}{b}}\frac{1}{\sqrt{(\Omega - \Omega_c)}} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

Plot $\frac{\partial\theta_0}{\partial\Omega}$ vs Ω. The response *diverges* at Ω_c.

- A double pendulum.
  - Choosing the coordinates.
  - Potential energy.
  - Kinetic energy. Normally, most trouble for students.

LECTURE 11
Lagrangian mechanics.

11.1. Non uniqueness of the Lagrangian.

11.2. Generalized momentum.

- For a coordinate q the generalized momentum is defined as

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

- For a particle in a potential field $L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r})$ we have

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

- For a rotation around a fixed axis $L = \frac{I\dot{\phi}^2}{2} - U(\phi)$, then

$$p = \frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi} = J.$$

  The generalized momentum is just an angular momentum.

11.3. Ignorable coordinates. Conservation laws.

If one chooses the coordinates in such a way, that the Lagrangian does not depend on say one of the coordinates q_1 (but it still depends on $\dot{q}_1$, then the corresponding generalized momentum $p_1 = \frac{\partial L}{\partial \dot{q}_1}$ is conserved as

$$\frac{d}{dt}p_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1} = 0$$

- Problem of a freely horizontally moving cart of mass M with hanged pendulum of mass m and length l.

11.4. Momentum conservation. Translation invariance

Let's consider a translationally invariant problem. For example all interactions depend only on the distance between the particles. The Lagrangian for this problem is $L(\vec{r}_1, \ldots \vec{r}_i, \dot{\vec{r}}_1, \ldots \dot{\vec{r}}_i)$. Then we add a constant vector ϵ to all coordinate vectors and define

$$L_\epsilon(\vec{r}_1, \ldots \vec{r}_i, \dot{\vec{r}}_1, \ldots \dot{\vec{r}}_i, \vec{\epsilon}) \equiv L(\vec{r}_1 + \vec{\epsilon}, \ldots \vec{r}_i + \vec{\epsilon}, \dot{\vec{r}}_1, \ldots \dot{\vec{r}}_i)$$

It is clear, that in the translationally invariant system the Lagrangian will not change under such a transformation. So we find

$$\frac{\partial L_\epsilon}{\partial \vec{\epsilon}} = 0.$$

But according to the definition

$$\frac{\partial L_\epsilon}{\partial \vec{\epsilon}} = \sum_i \frac{\partial L}{\partial \vec{r}_i}.$$

On the other hand the Lagrange equations tell us that

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = \frac{d}{dt} \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} = \frac{d}{dt} \sum_i \vec{p}_i,$$

so

$$\frac{d}{dt} \sum_i \vec{p}_i = 0, \qquad \sum_i \vec{p}_i = \text{const}.$$

We see, that the total momentum of the system is conserved!

11.5. Noether's theorem

Let's assume that the Lagrangian has a one parameter continuous symmetry. Namely $L(\mathbf{q}, \dot{\mathbf{q}}, t) = L(h_\epsilon \mathbf{q}, \dot{h_\epsilon \mathbf{q}}, t)$, where h_ϵ is some symmetry transformation which depends on the parameter ϵ. Then using notations $\mathbf{Q}(\epsilon, t) = h_\epsilon \mathbf{q}(t)$ we find $\partial_\epsilon L(\mathbf{Q}, \dot{\mathbf{Q}}, t) = 0$. On the other hand

$$\partial_\epsilon L(\mathbf{Q}, \dot{\mathbf{Q}}, t) = \frac{\partial L}{\partial \mathbf{Q}} \frac{\partial \mathbf{Q}}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{Q}}} \frac{\partial \dot{\mathbf{Q}}}{\partial \epsilon} = \frac{\partial L}{\partial \mathbf{Q}} \frac{\partial \mathbf{Q}}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{Q}}} \frac{d}{dt} \frac{\partial \mathbf{Q}}{\partial \epsilon} = \left(\frac{\partial L}{\partial \mathbf{Q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{Q}}} \right) \frac{\partial \mathbf{Q}}{\partial \epsilon} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{Q}}} \frac{\partial \mathbf{Q}}{\partial \epsilon} \right)$$

We see, that if $\mathbf{Q}$ is a solution of the Lagrange equation, then we find the

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{Q}}} \frac{\partial \mathbf{Q}}{\partial \epsilon} \right) = 0$$

Or that

$$\frac{\partial L}{\partial \dot{\mathbf{Q}}} \frac{\partial \mathbf{Q}}{\partial \epsilon} = \text{const}.$$

during the motion.

So the message is that for every symmetry of the Lagrangian there is a conserved quantity. Examples:

- Momentum conservation: $\vec{r} \to \vec{r} + \epsilon \vec{e}_\epsilon$. The Noether's theorem gives

$$\frac{\partial L}{\partial \dot{\vec{r}}} \vec{e}_\epsilon = \vec{p} \cdot \vec{e}_\epsilon = \text{const}.$$

- Angular momentum: $\vec{r} \to \vec{r} + d\phi \vec{e}_\phi \times \vec{r}$. The Noether's theorem gives

$$\frac{\partial L}{\partial \dot{\vec{r}}} \vec{e}_\phi \times \vec{r} = \vec{p} \cdot (\vec{e}_\phi \times \vec{r}) = \vec{e}_\phi \cdot (\vec{r} \times \vec{p})$$

11.6. Energy conservation.

Consider a Lagrangian, which does not depend on time explicitly: $L(\mathbf{q}, \dot{\mathbf{q}})$. Let's compare the value of the action

$$\mathcal{A} = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})dt, \qquad \mathbf{q}(t_1) = \mathbf{q}_1, \quad \mathbf{q}(t_2) = \mathbf{q}_2$$

with the value of the action

$$\mathcal{A}_\epsilon = \int_{t_1+\epsilon}^{t_2+\epsilon} L(\mathbf{Q}, \dot{\mathbf{Q}})dt, \qquad \mathbf{Q}(t_1+\epsilon) = \mathbf{q}_1, \quad \mathbf{Q}(t_2+\epsilon) = \mathbf{q}_2$$

on the functions $\mathbf{q}(t)$ and $\mathbf{Q}(t) = \mathbf{q}(t-\epsilon)$. It is clear, that if $\mathbf{q}$ satisfies the boundary conditions, then so does $\mathbf{Q}(t)$. Then by changing the variables of integration we find, that the value of the action is the same for both functions and does not depend on ϵ. So in this case $\partial_\epsilon \mathcal{A}_\epsilon|_{\epsilon=0} = 0$. On the other hand

$$\partial_\epsilon \mathcal{A}_\epsilon|_{\epsilon=0} = L|_{t_2} - L|_{t_1} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{Q}}\frac{\partial \mathbf{Q}}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{Q}}}\frac{\partial \dot{\mathbf{Q}}}{\partial \epsilon}\right)_{\epsilon=0} dt =$$

$$L|_{t_2} - L|_{t_1} + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{Q}}}\frac{\partial \mathbf{Q}}{\partial \epsilon}\right)_{\epsilon=0} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{Q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{Q}}}\right)_{\epsilon=0} \left.\frac{\partial \mathbf{Q}}{\partial \epsilon}\right|_{\epsilon=0} dt$$

If we now consider the value of the action on the solutions of the Lagrange equations, then we see, that the last term is zero. We also can substitute $\mathbf{q}$ and $\dot{\mathbf{q}}$ instead of $\mathbf{Q}$ and $\dot{\mathbf{Q}}$, and $-\dot{\mathbf{q}} = \left.\frac{\partial \mathbf{Q}}{\partial \epsilon}\right|_{\epsilon=0}$. We then find:

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} - L\right)_{t_2} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} - L\right)_{t_1}.$$

As times t_1 and t_2 are arbitrary, then we conclude, that

$$E = \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} - L$$

is a conserved quantity. It's called energy.

Example:

- $L = \frac{m\dot{\vec{v}}^2}{2} - U(\vec{r})$.
- A particle on a circle.
- A pendulum.
- A cart with a pendulum.
- A string with tension and gravity.

LECTURE 12
Dependence of Action on time and coordinate. Lagrangian's equations for magnetic forces.

12.1. Dependence of Action on time and coordinate.

12.1.1. Dependence on coordinate.

Let's assume, that we have the action

$$\mathcal{S}[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt,$$

which we minimize with the boundary conditions $q(t_1) = q_1$, $q(t_2) = q_2$. We need to solve the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \qquad q(t_1) = q_1, \quad q(t_2) = q_2.$$

Let's denote this solution as $q(t)$. We then can calculate the value of the action on the function $q(t)$.

$$S = \mathcal{S}[q(t)].$$

Now let's take the same action, but minimize it on the functions satisfying different boundary conditions $q(t_1) = q_1$, $q(t_2) = q_2 + \Delta q$. Let's denote the solution $\tilde{q}(t)$. We then calculate the value of the Action on $\tilde{q}(t)$.

$$\tilde{S} = \mathcal{S}[\tilde{q}(t)].$$

This way we can say that the value of the Action on solutions is a function of the boundary condition. We then want to compute

$$\frac{\partial S}{\partial q} \equiv \lim_{\Delta q \to 0} \frac{\tilde{S} - S}{\Delta q}.$$

It is clear, that when $\Delta q \to 0$, $q(t)$ and $\tilde{q}(t)$ are close to each other, so we say $\tilde{q}(t) = q(t) + \delta q(t)$ and write

$$\tilde{S} = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)dt = S + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right) dt$$

$$= S + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q dt + \left.\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_2} - \left.\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_1} = S + \left.\frac{\partial L}{\partial \dot{q}}\right|_{t_2} \Delta q,$$

So we see, that

$$\frac{\partial S}{\partial q} = p.$$

12.1.2. Dependence on time.

Let's now consider almost the same problem as before, but the boundary conditions now are

$$q(t_1) = q_1, \quad q(t_2) = q_2, \qquad \text{and} \qquad q(t_1) = q_1, \quad q(t_1 + \Delta t) = q_2.$$

Again, we denote the solution of the first problem as $q(t)$, and solution of the second problem as $\tilde{q}(t)$. Correspondingly, the Action on the solutions has values

$$S = \mathcal{S}[q(t)], \qquad \tilde{S} = \mathcal{S}[\tilde{q}(t)].$$

We want to calculate

$$\frac{\partial S}{\partial t} \equiv \lim_{\Delta t \to 0} \frac{\tilde{S} - S}{\Delta t}.$$

Again on the interval from t_1 to t_2 we can write $\tilde{q}(t) = q(t) + \delta q(t)$, so

$$\tilde{S} = \int_{t_1}^{t_2+\Delta t} L(\tilde{q}, \dot{\tilde{q}}, t)dt = \Delta t\; L|_{t_2} + \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)dt$$

$$= S + \Delta t\; L|_{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right) dt = S + \Delta t\; L|_{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q dt + \left.\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_2}$$

So we have

$$\frac{\tilde{S} - S}{\Delta t} = L|_{t_2} + \left.\frac{\partial L}{\partial \dot{q}}\right|_{t_2} \left.\frac{\delta q}{\Delta t}\right|_{t_2}.$$

Examining the picture it is easy to see, that $\left.\frac{\delta q}{\Delta t}\right|_{t_2} = -\left.\dot{q}\right|_{t_2}$, so

$$\frac{\partial S}{\partial t} = -\left(p\dot{q} - L\right)_{t_2} = -E_{t_2}.$$

(Energy does not have to be conserved.)

12.2. Lagrangian's equations for magnetic forces.

The equation of motion is

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

The question is what Lagrangian gives such equation of motion?

Consider the magnetic field. As there is no magnetic charges one of the Maxwell equations reads

$$\nabla \cdot \vec{B} = 0$$

This equation is satisfied by the following solution

$$\vec{B} = \nabla \times \vec{A},$$

for any vector field $\vec{A}(\vec{r}, t)$.

For the electric field another Maxwell equation reads

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

we see that then

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t},$$

where ϕ is the electric potential.

The vector potential $\vec{A}$ and the potential ϕ are not uniquely defined. One can always choose another potential

$$\vec{A}' = \vec{A} + \nabla F, \qquad \phi' = \phi - \frac{\partial F}{\partial t}$$

Such fields are called gauge fields, and the transformation above is called gauge transformation. Such fields cannot be measured.

Notice, that if $\vec{B}$ and $\vec{E}$ are zero, the gauge fields do not have to be zero. For example if $\vec{A}$ and ϕ are constants, $\vec{B} = 0$, $\vec{E} = 0$.

Now we can write the Lagrangian:

$$L = \frac{m\dot{\vec{r}}^2}{2} - q(\phi - \dot{\vec{r}} \cdot \vec{A})$$

- It is impossible to write the Lagrangian in terms of the physical fields $\vec{B}$ and $\vec{E}$!
- The expression

  $$\phi dt - d\vec{r} \cdot \vec{A}$$

  is a full differential if and only if

  $$-\nabla\phi - \frac{\partial \vec{A}}{\partial t} = 0, \qquad \nabla \times \vec{A} = 0,$$

  which means that the it is full differential, and hence can be thrown out, only if the physical fields are zero!

The generalized momenta are

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}$$

The Lagrange equations are :

$$\frac{d}{dt}\vec{p} = \frac{\partial L}{\partial \vec{r}}$$

Let's consider the x component

$$\frac{d}{dt}p_x = \frac{\partial L}{\partial x},$$

$$m\ddot{x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_x}{\partial y} + q\dot{z}\frac{\partial A_x}{\partial z} + q\frac{\partial A_x}{\partial t} = -q\frac{\partial \phi}{\partial x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_y}{\partial x} + q\dot{z}\frac{\partial A_z}{\partial x}$$

$$m\ddot{x} = q\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \dot{z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\right)$$

$$m\ddot{x} = q\left(E_x + \dot{y}B_z - \dot{z}B_y\right)$$

LECTURE 13
Hamiltonian and Hamiltonian equations.

13.1. Hamiltonian.

Given a Lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ the energy

$$E = \sum_i p_i \dot{q}_i - L, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

is a number defined on a trajectory! One can say that it is a function of initial conditions.

We can construct a function **function** in the following way: we first solve the set of equations

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

with respect to $\dot{q}_i$, we then have these functions

$$\dot{q}_i = \dot{q}_i(\{q_j\}, \{p_j\})$$

and define a function $H(\{q_i\}, \{p_i\})$

$$H(\{q_i\}, \{p_i\}) = \sum_i p_i \dot{q}_i(\{q_j\}, \{p_j\}) - L(\{q_i\}, \{\dot{q}_i(\{q_j\}, \{p_j\})\}),$$

This function is called a Hamiltonian!

The importance of variables:

- A Lagrangian is a function of generalized coordinates and velocities: q and $\dot{q}$.
- A Hamiltonian is a function of the generalized coordinates and **momenta**: q and p.

Here are the steps to get a Hamiltonian from a Lagrangian

(a) Write down a Lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ – it is a function of generalized coordinates and velocities q_i, $\dot{q}_i$

(b) Find generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

(c) Treat the above definitions as equations and solve them for all $\dot{q}_i$, so for each velocity $\dot{q}_i$ you have an expression $\dot{q}_i = \dot{q}_i(\{q_j\}, \{p_j\})$.

(d) Substitute these function $\dot{q}_i = \dot{q}_i(\{q_j\}, \{p_j\})$ into the expression

$$\sum_i p_i \dot{q}_i - L(\{q_i\}, \{\dot{q}_i\}).$$

The resulting function $H(\{q_i\}, \{p_i\})$ of generalized coordinates and momenta is called a Hamiltonian.

13.2. Examples.

- A particle in a potential field.
- Kepler problem.
- Motion in electromagnetic field.
- Rotation around a fixed axis.
- A pendulum.
- A cart and a pendulum.
- New notation for the partial derivatives. What do we keep fixed?
- Derivation of the Hamiltonian equations.

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

- Energy conservation.
- Velocity.
- $H(p, x) = \sqrt{m^2c^4 + p^2c^2} + U(x)$.

13.3. Phase space. Hamiltonian field. Phase trajectories.

- Motion in the phase space.
- Trajectories do not intersect. (Singular points)
- Harmonic oscillator.
- Pendulum.

13.4. From Hamiltonian to Lagrangian.

LECTURE 14
Liouville's theorem. Poisson brackets.

14.1. Liouville's theorem.

Lets consider a more general problem. Lets say that the dynamics of n variables is given by n equations

$$\dot{\vec{x}} = \vec{f}(\vec{x}).$$

These equations provide a map from any point $\vec{x}(t = 0)$ to some other point $\vec{x}(t)$ in our space in a latter time. This way we say, that there is a map $g^t : \vec{x}(0) \to \vec{x}(t)$. We can use this map, to map an original region $D(0)$ in $\vec{x}$ space to another region $D(t)$ at a later time $D(t) = g^t D(0)$. The original region $D(0)$ had a volume $v(0)$, the region $D(t)$ has a volume $v(t)$. We want to find how this volume depends on t. To do that we consider a small time increment dt. The map g^{dt} is given by (I keep only terms linear in dt)

$$g^{dt}(\vec{x}) = \vec{x} + \vec{f}(\vec{x})dt.$$

The volume $v(dt)$ is given by

$$v(dt) = \int_{D(dt)} d^n x$$

We now consider our map as a change of variables, from $\vec{x}(0)$ to $\vec{x}(dt)$. Then

$$v(dt) = \int_{D(0)} \det \frac{\partial g^{dt}(x_i)}{\partial x_j} d^n x.$$

Using our map we find that the matrix

$$\frac{\partial g^{dt}(x_i)}{\partial x_j} = \delta_{ij} + \frac{\partial f_i}{\partial x_j} dt = \hat{E} + dt\hat{A}.$$

We need the determinant of this matrix only in the linear order in dt. We use the following formula $\log \det \hat{M} = \operatorname{tr} \log \hat{M}$ to find

$$\det\left(\hat{E} + dt\hat{A}\right) = e^{\operatorname{tr}\log\left(\hat{E}+dt\hat{A}\right)} \approx e^{dt\operatorname{tr}\hat{A}} \approx 1 + dt\operatorname{tr}\hat{A},$$

and find

$$v(dt) = v(0) + dt \int_{D(0)} \sum_i \frac{\partial f_i(\vec{x})}{\partial x_i} d^n x,$$

or

$$\frac{dv}{dt} = \int_{D(t)} \sum_i \frac{\partial f_i(\vec{x})}{\partial x_i} d^n x.$$

For the Hamiltonian mechanics we take n to be even, half of xs are the coordinates q_i, and the other half are momenta p_i. Then we fave

$$\sum_{i=1}^{n} \frac{\partial f_i(\vec{x})}{\partial x_i} = \sum_{i=1}^{n/2} \left[\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right] = 0.$$

So the Hamiltonian mechanics conserves a volume of the phase space region. **Minus sign is very important.**

14.2. Poisson brackets.

Consider a function of time, coordinates and momenta: $f(t, q, p)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \frac{\partial f}{\partial t} + \{H, f\}$$

where we defined the Poisson brackets for any two functions g and f

$$\{g, f\} = \sum_i \left(\frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$$

In particular we see, that

$$\{p_i, q_k\} = \delta_{i,k}.$$

Poisson brackets are

- Antisymmetric.
- Bilinear.
- For a constant c, $\{f, c\} = 0$.
- $\{f_i f_2, g\} = f_1 \{f_2, g\} + f_2 \{f_1, g\}$.

Let's consider an arbitrary transformation of variables: $P_i = P_i(\{p\}, \{q\})$, and $Q_i = Q_i(\{p\}, \{q\})$. We then have

$$\dot{P}_i = \{H, P_i\}, \qquad \dot{Q}_i = \{H, Q_i\}.$$

or

$$\begin{aligned} \dot{P}_i &= \sum_k \left(\frac{\partial H}{\partial p_k} \frac{\partial P_i}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial P_i}{\partial p_k} \right) \\ &= \sum_{k,\alpha} \left(\left(\frac{\partial H}{\partial P_\alpha} \frac{\partial P_\alpha}{\partial p_k} + \frac{\partial H}{\partial Q_\alpha} \frac{\partial Q_\alpha}{\partial p_k} \right) \frac{\partial P_i}{\partial q_k} - \left(\frac{\partial H}{\partial P_\alpha} \frac{\partial P_\alpha}{\partial q_k} + \frac{\partial H}{\partial Q_\alpha} \frac{\partial Q_\alpha}{\partial q_k} \right) \frac{\partial P_i}{\partial p_k} \right) \\ &= -\sum_\alpha \left(\frac{\partial H}{\partial P_\alpha} \{P_i, P_\alpha\} + \frac{\partial H}{\partial Q_\alpha} \{P_i, Q_\alpha\} \right) \end{aligned}$$

Analogously,

$$\dot{Q}_i = -\sum_\alpha \left(\frac{\partial H}{\partial Q_\alpha} \{Q_i, Q_\alpha\} + \frac{\partial H}{\partial P_\alpha} \{Q_i, P_\alpha\} \right)$$

We see, that the Hamiltonian equations keep their form if

$$\{P_i, Q_\alpha\} = \delta_{i,\alpha}, \qquad \{P_i, P_\alpha\} = \{Q_i, Q_\alpha\} = 0$$

The variables that have such Poisson brackets are called the *canonical variables*, they are *canonically conjugated*. Transformations that keep the canonical Poisson brackets are called *canonical transformations*.

LECTURE 15

Hamiltonian equations. Jacobi's identity. Integrals of motion. Angular momentum.

- Evaluations.

15.1. Hamiltonian mechanics

- The Poisson brackets are property of the phase space and have nothing to do with the Hamiltonian.
- The Hamiltonian is just a function on the phase space.
- Given the phase space p_i, q_i, the Poisson brackets and the Hamiltonian. We can construct the equations of the Hamiltonian mechanics:
$$\dot{p}_i = \{H, p_i\}, \qquad \dot{q}_i = \{H, q_i\}.$$
- In this formulation there is no need to distinguish between the coordinates and momenta. So we can use $\xi_1 \dots \xi_{2N}$ instead of $q_1 \dots q_N$ and $p_1 \dots p_N$, with given Poisson brackets $\{\xi_i, \xi_j\}$.
- The Poisson brackets $\{\xi_i, \xi_j\}$ do not have to be constants. Generally, they are a set of functions with two indexes i and j of all the coordinates $\{\xi\}$.
- The equations of motion are then
$$\dot{\xi}_i = \{H, \xi_i\}.$$
- Time evolution of any function $f(\{\xi\}, t)$ is given by the equation
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}.$$
difference between the full and the partial derivatives!

15.2. New formulation of the Hamiltonian mechanics.

Here is the new formulation of mechanics:

- We have a phase space with coordinates ξ_i, where $i = 1 \dots 2N$ for N degrees of freedom.
- This phase space is equipped with Poisson brackets: $\{\xi_i, \xi_j\}$. What it means is that for any two coordinates ξ_i and ξ_j we know a function $\{\xi_i, \xi_j\}$ which depends on two indexes i and j and all the coordinates.
- Poisson brackets are defined axiomatically as

  - Antisymmetric.
  - Bilinear.
  - For a constant c and any function f, $\{f, c\} = 0$.
  - For any three functions f_1, f_2, and g: $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$.
  - Jacobi's identity. For any three functions f, g, and h: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- Any function on the phase space $H(\{\xi\}, t)$ can be a Hamiltonian (which function you take as a Hamiltonian depends on the problem you are solving.)
- Time evolution of any function $f(\{\xi\}, t)$ is given by the equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}.$$

In this formulation the phase space and the Poisson brackets play the major role. They are independent of a Hamiltonian (the are defined before the Hamiltonian even introduced) If we know the Hamiltonian we can also construct the Hamiltonian equations of time evolution of any function.

In particular the time evolution of the Hamiltonian itself is given by

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t}$$

as $\{H, H\} = 0$ due to antisymmetry of the Poisson brackets. So if the Hamiltonian does not explicitly depend on time, then it is conserved on the trajectories.

- In this formulation we separated the properties of the phase space (the Poisson bracket structure) from the Hamiltonian itself.
- Canonical Poisson brackets is just one example of the possible Poisson bracket structure. (in some sense, this is analogous to the statement that the Euclidean geometry is just one example of all possible geometries.)
- The Jacobi identity puts a very strong restriction on all possible Poisson brackets structure.

15.3. How to compute Poisson brackets for any two functions.

In order to use our new formulation we need a way to compute the Poisson bracket between any two functions f and g if we know all $\{\xi_i, \xi_j\}$. In general the Poisson bracket $\{\xi_i, \xi_j\}$ is the function of all the phase space coordinates. We only require that all the properties listed in definition hold for the Poisson brackets of coordinates $\{\xi_i, \xi_j\}$.

The answer is:

$$\{f, g\} = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}.$$

(summation over the repeated indexes is implied.) Notice the order of indexes. It is important.

Let's prove this formula.

- We start with the Poisson bracket of $\{\xi_j, g\}$, where g is an arbitrary function on the phase space (for simplicity we take that g does not depend on time explicitly).
- In order to compute it we consider ξ_j as our Hamiltonian. This Hamiltonian then gives the time evolution

$$\frac{dg}{dt} = \{\xi_j, g\}.$$

- On the other hand, by the chain rule
$$\frac{dg}{dt} = \frac{\partial g}{\partial \xi_i}\frac{d\xi_i}{dt} = \frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}.$$
- Comparing the two results we see, that
$$\{\xi_j, g\} = \frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}$$
- To compute the Poisson bracket $\{g, f\}$ we consider the function g as the Hamiltonian, then
$$\frac{df}{dt} = \{g, f\}.$$
- On the other hand, by the chain rule
$$\frac{df}{dt} = \frac{\partial f}{\partial \xi_j}\frac{d\xi_j}{dt} = \frac{\partial f}{\partial \xi_j}\{g, \xi_j\} = -\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}$$
so that
$$\{f, g\} = \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}.$$
(take notice of the order of indexes, it is important as the Poisson brackets are antisymmetric.)
- Using this rule we see, that if all the requirements for the Poisson brackets are satisfied for all $\{\xi_i, \xi_j\}$, then these requirements are satisfied for any functions f and g.

There is one more identity the Poisson brackets must satisfy – the Jacobi's identity.

Given the phase space equipped with the Poisson brackets with above properties any function on the phase space can be considered as a Hamiltonian. The Hamiltonian dynamics is then fully defined.

15.4. The Jacobi's identity.

Using the definition of the Poisson brackets in the canonical coordinates it is easy, but lengthy to prove, that for any three functions f, g, and h:
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$
As it holds for any three functions this is the property of the phase space and the Poisson brackets.

This property is called the Jacobi's identity. It is a part of the axiomatic definition for Poisson brackets in general.

15.4.1. Proof of the Jacobi identity.

First we need to establish what we want to prove, after all the Jacobi identity is the part of the axiomatic definition of the Poisson brackets. The statement we want to prove is the following:

- If the Jacobi identity is satisfied by the Poisson brackets of the phase space coordinates, then it is satisfied for any three arbitrary functions.

We will prove this statement in two steps. First we consider the situation when $\{\xi_i, \xi_j\} =$ const — independent of the phase space coordinates. (This case covers the canonical Poisson brackets.) Then we consider the general case.

15.4.2. The case of $\{\xi_i, \xi_j\} =$ const.

We have the phase space coordinates $\xi_1 \dots \xi_{2N}$, and $\{\xi_i, \xi_j\} =$ const. This constants will depend on i and j, but they do not depend on the coordinates $\{\xi\}$.

Consider the first term in the Jacobi's identity $\{f, \{g, h\}\}$. According to our rule of computing the Poisson brackets we have

$$\{g, h\} = \frac{\partial g}{\partial \xi_j}\frac{\partial h}{\partial \xi_i}\{\xi_j, \xi_i\}.$$

- Remember, we are using the Einstein notations!

Using the same rule again we have

$$\{f, \{g, h\}\} = \frac{\partial f}{\partial \xi_p}\frac{\partial}{\partial \xi_l}\left(\frac{\partial g}{\partial \xi_i}\frac{\partial h}{\partial \xi_j}\{\xi_i, \xi_j\}\right)\{\xi_p, \xi_l\} = \frac{\partial f}{\partial \xi_p}\frac{\partial}{\partial \xi_l}\left(\frac{\partial g}{\partial \xi_i}\frac{\partial h}{\partial \xi_j}\right)\{\xi_i, \xi_j\}\{\xi_p, \xi_l\},$$

where we used $\xi_1 \dots \xi_{2N}$, and $\{\xi_i, \xi_j\} =$ const to pull $\{\xi_i, \xi_j\}$ from under the differentiation.

Now we take the derivative and find

$$\{f, \{g, h\}\} = \frac{\partial f}{\partial \xi_p}\frac{\partial^2 g}{\partial \xi_i \partial \xi_l}\frac{\partial h}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial f}{\partial \xi_p}\frac{\partial g}{\partial \xi_i}\frac{\partial^2 h}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}.$$

Cycle permutations of the functions f, g, and h gives the other two terms

$$\{g, \{h, f\}\} = \frac{\partial g}{\partial \xi_p}\frac{\partial^2 h}{\partial \xi_i \partial \xi_l}\frac{\partial f}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial g}{\partial \xi_p}\frac{\partial h}{\partial \xi_i}\frac{\partial^2 f}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\},$$

$$\{h, \{f, g\}\} = \frac{\partial h}{\partial \xi_p}\frac{\partial^2 f}{\partial \xi_i \partial \xi_l}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial h}{\partial \xi_p}\frac{\partial f}{\partial \xi_i}\frac{\partial^2 g}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}$$

We want to show that the sum of all these terms is zero for *arbitrary* functions f, g, and h. Each term has on second derivative of one of the functions. In order for the sum of the terms to be zero the terms with the same second derivative must cancel each other. Consider the terms which have the second derivative of the function f, these terms originate from $\{g, \{h, f\}\}$ and $\{h, \{f, g\}\}$. Let's isolate them and take a closer look at their structure

$$\frac{\partial g}{\partial \xi_p}\frac{\partial h}{\partial \xi_i}\frac{\partial^2 f}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial h}{\partial \xi_p}\frac{\partial^2 f}{\partial \xi_i \partial \xi_l}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}$$

- Remember, we are using the Einstein notations!
- The summations of all the indexes is assumed.
- It does not matter which latter we use to label the indexes, as we sum over all their possible values anyway.
- Then take the first term above an relabel the indexes according to the scheme

$$p \to j, j \to l, l \to i, i \to p$$

.

We get

$$\frac{\partial g}{\partial \xi_j}\frac{\partial h}{\partial \xi_p}\frac{\partial^2 f}{\partial \xi_l \partial \xi_i}\{\xi_p, \xi_l\}\{\xi_j, \xi_i\} + \frac{\partial h}{\partial \xi_p}\frac{\partial^2 f}{\partial \xi_i \partial \xi_l}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}$$

We see, that now the only difference between these two terms is that the first one has $\{\xi_j, \xi_i\}$, while the second has $\{\xi_i, \xi_j\}$. As the Poisson brackets are antisymmetric the two terms are the same, but with the opposite sign. So the sum of the two terms is zero.

The other terms are obtained by a simple cyclic permutation of the functions f, g, and h. So if the second derivative terms of f cancel each other the other terms will also cancel each other. Then the total sum of all therms is zero, as it should be by Jacobi identity.

So we proved, that in the case $\{\xi_i, \xi_j\} = \text{const}$ the Jacobi identity is satisfied.

15.4.3. The case of arbitrary $\{\xi_i, \xi_j\}$.

This case is different from the previous one only in one point. We cannot pull $\{\xi_i, \xi_j\}$ from under the differentiation. We then have

$$\{f, \{g, h\}\} = \frac{\partial f}{\partial \xi_p}\frac{\partial^2 g}{\partial \xi_i \partial \xi_l}\frac{\partial h}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial f}{\partial \xi_p}\frac{\partial g}{\partial \xi_i}\frac{\partial^2 h}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial f}{\partial \xi_p}\frac{\partial g}{\partial \xi_i}\frac{\partial h}{\partial \xi_j}\frac{\partial \{\xi_i, \xi_j\}}{\partial \xi_l}\{\xi_p, \xi_l\}$$

The first two terms are exactly the same as before, so as before they will cancel each other. We then need to concentrate only on the last term. Let's write all three of these terms obtained by cyclic permutation of the functions f, g, and h.

$$\frac{\partial f}{\partial \xi_p}\frac{\partial g}{\partial \xi_i}\frac{\partial h}{\partial \xi_j}\frac{\partial \{\xi_i, \xi_j\}}{\partial \xi_l}\{\xi_p, \xi_l\} + \frac{\partial g}{\partial \xi_p}\frac{\partial h}{\partial \xi_i}\frac{\partial f}{\partial \xi_j}\frac{\partial \{\xi_i, \xi_j\}}{\partial \xi_l}\{\xi_p, \xi_l\} + \frac{\partial h}{\partial \xi_p}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\frac{\partial \{\xi_i, \xi_j\}}{\partial \xi_l}\{\xi_p, \xi_l\}.$$

- We relabel the indexes in the last two terms in such a way, that all terms have the same derivatives.
- It means, that we relabel

  $$j \to p, i \to j, p \to i$$

  in the second term.
- We relabel

  $$i \to p, j \to i, p \to j$$

  in the third term.

We get

$$\frac{\partial f}{\partial \xi_p}\frac{\partial g}{\partial \xi_i}\frac{\partial h}{\partial \xi_j}\left(\frac{\partial \{\xi_i, \xi_j\}}{\partial \xi_l}\{\xi_p, \xi_l\} + \frac{\partial \{\xi_j, \xi_p\}}{\partial \xi_l}\{\xi_i, \xi_l\} + \frac{\partial \{\xi_p, \xi_i\}}{\partial \xi_l}\{\xi_j, \xi_l\}\right).$$

As the functions f, g, and h are arbitrary in order for the above to be zero the expression in the brackets must be zero. Let's then concentrate on the term in the brackets. To start with we take a hard look at the first term in the brackets

$$\frac{\partial \{\xi_i, \xi_j\}}{\partial \xi_l}\{\xi_p, \xi_l\}.$$

Remember, that the Poisson bracket $\{\xi_p, F\} = \frac{\partial F}{\partial \xi_l}\{\xi_p, \xi_l\}$ for ANY function F on the phase space. So treating $\{\xi_i, \xi_j\}$ as some function on the phase space we get

$$\frac{\partial \{\xi_i, \xi_j\}}{\partial \xi_l}\{\xi_p, \xi_l\} = \{\xi_p, \{\xi_i, \xi_j\}\}.$$

Applying this trick to every term in the brackets we get

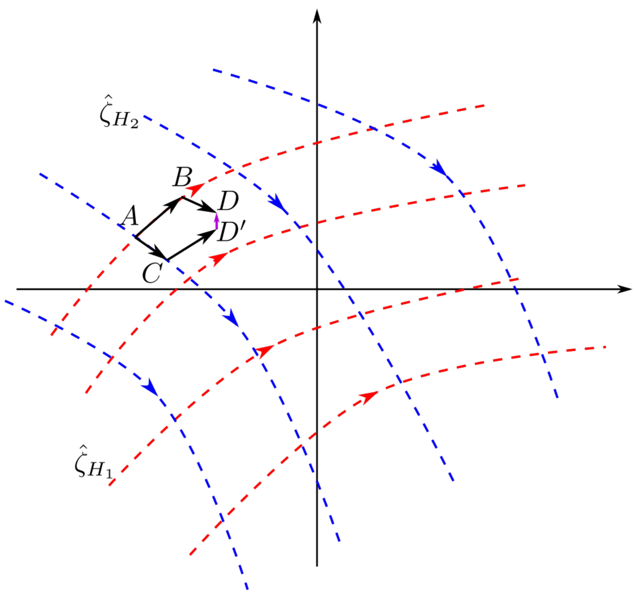
$$\{\xi_p, \{\xi_i, \xi_j\}\} + \{\xi_i, \{\xi_j, \xi_p\}\} + \{\xi_j, \{\xi_p, \xi_i\}\}.$$

But this is zero by the Jacobi identity for the Poisson brackets!

So we proved, that if the Jacobi identity is satisfied by the Poisson brackets of the phase space coordinates, then it is satisfied for any three arbitrary functions.

- As it holds for any functions this is the property of the phase space and the Poisson brackets themselves.

15.5. Commutation of Hamiltonian flows.



For a Hamiltonian H we can introduce the operator $\hat{\zeta}_H$ of the Hamiltonian flow by the following definition: for any function g

$$\hat{\zeta}_H g \equiv \{H, g\}$$

Using this definition for one of the coordinates (plugging ξ_i instead of g) we see, that $\hat{\zeta}_H \xi_i = \{H, \xi_i\} = \frac{d\xi_i}{dt}$. So this Hamiltonian flow induces the Hamiltonian vector field we considered earlier.

Let's now consider two Hamiltonians H_1 and H_2 and compute the commutator of their flows. Namely, for any function g we have (using Jacobi's identity)

$$\hat{\zeta}_{H_1}\hat{\zeta}_{H_2} g - \hat{\zeta}_{H_2}\hat{\zeta}_{H_1} g = \{H_1, \{H_2, g\}\} - \{H_2, \{H_1, g\}\} = \{\{H_1, H_2\}, g\} = \hat{\zeta}_{\{H_1,H_2\}} g.$$

As this is true for *any* function g we have

$$\hat{\zeta}_{H_1}\hat{\zeta}_{H_2} - \hat{\zeta}_{H_2}\hat{\zeta}_{H_1} = \hat{\zeta}_{\{H_1,H_2\}}.$$

So the commutator of the Hamiltonian flows is also a Hamiltonian flow.

On the figure

- The red dashed lines show the flow $\hat{\zeta}_{H_1}$, the blue dashed lines show the flow $\hat{\zeta}_{H_2}$.
- The operator $\hat{\zeta}_{H_2}\hat{\zeta}_{H_1}$ shifts the point A along ABD path.
- The operator $\hat{\zeta}_{H_1}\hat{\zeta}_{H_2}$ shifts the point A along ACD' path.
- So the operator $\hat{\zeta}_{H_1}\hat{\zeta}_{H_2} - \hat{\zeta}_{H_2}\hat{\zeta}_{H_1}$ shifts point D' to point D.
- This shift can be described by another Hamiltonian flow $\hat{\zeta}_{\{H_1,H_2\}}$.

15.6. Time evolution of Poisson brackets.

Consider two arbitrary functions $f(\{\xi\}, t)$ and $g(\{\xi\}, t)$. We want to compute the full time derivative of their Poisson bracket

$$\frac{d}{dt}\{f, g\}.$$

It means, that we have a phase space with Poisson brackets. We also have a Hamiltonian. We solve the Hamiltonian equations of motion $\dot{\xi}_i = \{H, \xi_i\}$ and find $\xi_i(t)$ for all is if we do not distinguish between coordinates and momenta) We compute the Poisson bracket $\{f, g\}$ – it will be some function of all ξ. We substitute the solutions $\xi(t)$ in this function and then take the time derivative.

Our general procedure allows us to do it much simpler, but before we do that I want to compute

$$\frac{\partial}{\partial t}\{f, g\}.$$

This is partial derivative. So we just consider the explicit dependence of $\{f, g\}$ on time. We keep fixed all other variables except t, so I will leave them out to shorten the formulas

$$\begin{aligned}
&\{f(t+\Delta t), g(t+\Delta t)\} - \{f(t), g(t)\} \\
&= \{f(t+\Delta t), g(t+\Delta t)\} - \{f(t), g(t+\Delta t)\} + \{f(t), g(t+\Delta t)\} - \{f(t), g(t)\} \\
&= \{f(t+\Delta t) - f(t), g(t+\Delta t)\} + \{f(t), g(t+\Delta t) - g(t)\}
\end{aligned}$$

so, dividing by Δt and taking the limit $\Delta t \to 0$ we get

$$\frac{\partial}{\partial t}\{f, g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\}.$$

Notice

- The only property of the Poisson brackets which we used is its bi-linearity.

Now Let's compute the full time evolution of the Poisson bracket $\{f, g\}$ under the Hamiltonian H.

$$\begin{aligned}
&\frac{d}{dt}\{f, g\} = \frac{\partial}{\partial t}\{f, g\} + \{H, \{f, g\}\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} + \{\{H, f\}, g\} + \{f, \{H, g\}\} \\
&= \left\{\frac{\partial f}{\partial t} + \{H, f\}, g\right\} + \left\{f, \frac{\partial g}{\partial t} + \{H, g\}\right\}
\end{aligned}$$

Notice, that in this derivation we used

- the Jacobi's identity,
- the antisymmetry,
- and the bi-linearity

of the Poisson brackets.

So we get

$$\frac{d}{dt}\{f, g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}.$$

- Notice, that these are the full derivatives, not partial!!

15.7. Integrals of motion.

A conserved quantity is such a function $f(\{\xi\},t)$, that $\frac{df}{dt}=0$ under the evolution induced by a Hamiltonian H. So if we have two conserved quantities $f(\{\xi\},t)$ and $g(\{\xi\},t)$, then

$$\frac{d}{dt}\{f,g\}=\left\{\frac{df}{dt},g\right\}+\left\{f,\frac{dg}{dt}\right\}=0$$

So if we have two conserved quantities we can construct a new conserved quantity! Sometimes it will turn out to be an independent conservation law!

15.8. Angular momentum.

This is an example of a case where the Poisson brackets do not have a global canonical form.

15.8.1. Poisson Brackets.

Let's calculate the Poisson brackets for the components of angular momentum: $\vec{M}=\vec{r}\times\vec{p}$.

Coordinate $\vec{r}$ and momentum $\vec{p}$ are canonically conjugated so

$$\{p^i,r^j\}=\delta_{ij},\qquad \{p^i,p^j\}=\{r^i,r^j\}=0.$$

As our coordinates and momenta are canonical, we can use the definition of the Poisson brackets through derivatives — the way they were introduced from the very beginning. However, I will show that we can compute the Poisson brackets between the angular momentum components algebraically — using only the properties of the Poisson brackets.

Using $M^i=\epsilon^{ilk}r^lp^k$ we write

$$\{M^i,M^j\}=\epsilon^{ilk}\epsilon^{jmn}\{r^lp^k,r^mp^n\}=\epsilon^{ilk}\epsilon^{jmn}\left(r^l\{p^k,r^mp^n\}+p^k\{r^l,r^mp^n\}\right)=$$
$$\epsilon^{ilk}\epsilon^{jmn}\left(r^lp^n\{p^k,r^m\}+r^lr^m\{p^k,p^n\}+p^kp^n\{r^l,r^m\}+p^kr^m\{r^l,p^n\}\right)=$$
$$\epsilon^{ilk}\epsilon^{jmn}\left(r^lp^n\delta_{km}-p^kr^m\delta_{ln}\right)=\left(\epsilon^{ilk}\epsilon^{jkn}-\epsilon^{ikn}\epsilon^{jlk}\right)p^nr^l=p^ir^j-r^ip^j=-\epsilon^{ijk}M^k$$

(I used $\epsilon^{ilk}\epsilon^{jnk}=\delta^{ij}\delta^{ln}-\delta^{in}\delta^{lj}$). In short the result is

$$\{M^i,M^j\}=-\epsilon^{ijk}M^k$$

Notice:

- The components of the angular momentum construct their own phase space closed under the Poisson brackets!
- Unlike the usual phase space this phase space looks odd (3) dimensional!
- This puzzle is resolved by the following observation:

$$\{M^i,\vec{M}^2\}=\{M^i,M^kM^k\}=2\{M^i,M^k\}M^k=-2\epsilon^{ikj}M^jM^k=0.$$

- So for any Hamiltonian which depends on $\vec{M}$ only, the $\vec{M}^2$ will be conserved!

$$\frac{d\vec{M}^2}{dt}=\{H,\vec{M}^2\}=\frac{\partial H}{\partial M^i}\{M^i,\vec{M}^2\}=0.$$

- So in $3D$ space of $\vec{M}$ all motion will happen on the spheres $\vec{M}^2=$ cons..
- The sphere is $2D$ – even dimension.
- There is no way to construct global canonical coordinates on this space.

15.8.2. Spin in magnetic field.

We can now consider a Hamiltonian mechanics, say for the Hamiltonian

$$H = \vec{h} \cdot \vec{M}.$$

In this case

$$\dot{M}^i = \{H, M^i\} = h_j\{M^j, M^i\} = -h_j \epsilon^{jik} M^k,$$

or

$$\dot{\vec{M}} = \vec{h} \times \vec{M}.$$

Notice:

- $\dot{M^2} = \vec{M} \cdot \dot{\vec{M}} = \vec{M} \cdot \left[\vec{h} \times \vec{M}\right] = 0.$
- Energy is conserved, so $\vec{h} \cdot \vec{M} = \text{const}$. The projection of $\vec{M}$ on the direction of $\vec{h}$ does not change with time.
- This equation (Bloch equation) describes a vector $\vec{M}$ rotating with constant angular velocity around the direction of $\vec{h}$.

15.8.3. Euler equations

Consider a free rigid body with tensor of inertia $\hat{I}$. The Hamiltonian is just the kinetic energy.

$$H = \frac{1}{2} M^i (\hat{I}^{-1})_{ij} M^j.$$

The equations of motion then is

$$\dot{M}^k = \{H, M^k\} = \frac{1}{2}\{M^i, M^k\}(\hat{I}^{-1})_{ij} M^j + \frac{1}{2} M^i (\hat{I}^{-1})_{ij}\{M^j, M^k\} = \epsilon^{kil} M^l (\hat{I}^{-1})_{ij} M^j.$$

Let's write this equation in the system of coordinates of the principal axes of the body. Then the tensor of inertia is diagonal, and for x component we get

$$\dot{M}^x = M^z I^{-1}_{yy} M^y - M^y I^{-1}_{zz} M^z.$$

or, using that $M^x = I_{xx}\Omega^x$, etc we get

$$I_{xx}\dot{\Omega}^x = (I_{zz} - I_{yy})\Omega^z \Omega^y,$$

and two more equations under the cyclic permutations. These are Euler equations that we have derived before!

- Three degrees of freedom. We must have three second order differential equations for complete description. We have only three first order equations. Three more equations are missing.
- The equations are written for the components of $\vec{\Omega}$ in the non-internal system of coordinates which is rotating with $\vec{\Omega}$.
- In order to find the orientation of the rigid body as a function of time we need to write and solve three more first order differential equations.
- These are equations which express the vector Ω through the time derivatives of the Euler angles.

LECTURE 16
Oscillations.

16.1. Small oscillations.

Problem with one degree of freedom: $U(x)$. The Lagrangian is

$$L = \frac{m\dot{x}^2}{2} - U(x).$$

The equation of motion is

$$m\ddot{x} = -\frac{\partial U}{\partial x}$$

If the function $U(x)$ has an extremum at $x = x_0$, then $\left.\frac{\partial U}{\partial x}\right|_{x=x_0} = 0$. Then $x = x_0$ is a (time independent) solution of the equation of motion.

Consider a small deviation from the solution $x = x_0 + \delta x$. Assuming that δx stays small during the motion we have

$$U(x) = U(x_0 + \delta x) \approx U(x_0) + U'(x_0)\delta x + \frac{1}{2}U''(x_0)\delta x^2 = U(x_0) + \frac{1}{2}U''(x_0)\delta x^2$$

The equation of motion becomes

$$m\ddot{\delta x} = -U''(x_0)\delta x$$

- If $U''(x_0) > 0$, then we have small oscillations with the frequency

$$\omega^2 = \frac{U''(x_0)}{m}$$

  This is a stable equilibrium.
- If $U''(x_0) < 0$, then the solution grows exponentially, and at some point our approximation becomes invalid. The equilibrium is unstable.

Look at what it means graphically.

Generality: consider a system with infinitesimally small dissipation and external perturbations. The perturbations will kick it out of any unstable equilibrium. The dissipation will bring it down to a stable equilibrium. It may take a very long time.

After that the response of the system to small enough perturbations will be defined by the small oscillations around the equilibrium

16.2. Many degrees of freedom.

Consider two equal masses in $1D$ connected by springs of constant k to each other and to the walls.

There are two coordinates: x_1 and x_2.

There are two modes $x_1 - x_2$ and $x_1 + x_2$.

The potential energy of the system is

$$U(x_1, x_2) = \frac{kx_1^2}{2} + \frac{k(x_1 - x_2)^2}{2} + \frac{kx_2^2}{2}$$

The Lagrangian

$$L = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{kx_1^2}{2} - \frac{k(x_1 - x_2)^2}{2} - \frac{kx_2^2}{2}$$

The equations of motion are

$$m\ddot{x}_1 = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -2kx_2 + kx_1$$

These are two second order differential equations. Total they must have four solutions. Let's look for the solutions in the form

$$x_1 = A_1 e^{i\omega t}, \qquad x_2 = A_2 e^{i\omega t}$$

then

$$-\omega^2 m A_1 = -2kA_1 + kA_2$$
$$-\omega^2 m A_2 = -2kA_2 + kA_1$$

or

$$(2k - m\omega^2)A_1 - kA_2 = 0$$
$$(2k - m\omega^2)A_2 - kA_1 = 0$$

or

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

In order for this set of equations to have a non trivial solution we must have

$$\det \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} = 0, \qquad (2k - m\omega^2)^2 - k^2 = 0, \qquad (k - m\omega^2)(3k - m\omega^2) = 0$$

There are two modes with the frequencies

$$\omega_a^2 = k/m, \qquad \omega_b^2 = 3k/m$$

and corresponding eigen vectors

$$\begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix} = A^a \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix} = A^b \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The general solution then is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_a t + \phi_a) + A^b \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_b t + \phi_b)$$

What will happen if the masses and springs constants are different?

Repeat the previous calculation for arbitrary m_1, m_2, k_1, k_2, k_3.

General scheme.

16.3. Oscillations. Many degrees of freedom. General case.

Let's consider a general situation in detail. We start from an arbitrary Lagrangian

$$L = K(\{\dot{q}_i\}, \{q_i\}) - U(\{q_i\})$$

Very generally the kinetic energy is zero if all velocities are zero. It will also increase if any of the velocities increase.

It is assumed that the potential energy has a minimum at some values of the coordinates $q_i = q_{i0}$. Let's first change the definition of the coordinates $x_i = q_i - q_{0i}$. We rewrite the Lagrangian in these new coordinates.

$$L = K(\{\dot{x}_i\}, \{x_i\}) - U(\{x_i\})$$

We can take the potential energy to be zero at $x_i = 0$, also as $x_i = 0$ is a minimum we must have $\partial U/\partial x_i = 0$.

Let's now assume, that the motion has very small amplitude. We then can use Taylor expansion in both $\{\dot{x}_i\}$ and $\{x_i\}$ up to the second order.

The time reversal invariance demands that only even powers of velocities can be present in the expansion. Also as the kinetic energy is zero if all velocities are zero, we have $K(0, \{x_i\})$, so we have

$$K(\{\dot{x}_i\}, \{x_i\}) \approx \frac{1}{2}\sum_{i,j} \left.\frac{\partial K}{\partial \dot{x}_i \partial \dot{x}_j}\right|_{\dot{x}=0, x=0} \dot{x}_i \dot{x}_j = \frac{1}{2}k_{ij}\dot{x}_i\dot{x}_j,$$

where the constant matrix k_{ij} is symmetric and positive definite.

For the potential energy we have

$$U(\{x_i\}) \approx \frac{1}{2}\sum_{i,j} \left.\frac{\partial U}{\partial x_i \partial x_j}\right|_{x=0} x_i x_j = \frac{1}{2}u_{ij}x_i x_j,$$

where the constant matrix u_{ij} is symmetric. If $x = 0$ is indeed a minimum, then the matrix u_{ij} is also positive definite.

The Lagrangian is then

$$L = \frac{1}{2}k_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}u_{ij}x_i x_j$$

where k_{ij} and u_{ij} are just constant matrices. The Lagrange equations are

$$k_{ij}\ddot{x}_j = -u_{ij}x_j$$

We are looking for the solutions in the form

$$x_j^a = A_j^a e^{i\omega_a t},$$

then

$$\left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a = 0 \tag{16.1}$$

In order for this linear equation to have a nontrivial solution we must have

$$\det\left(\omega_a^2 k_{ij} - u_{ij}\right) = 0$$

After solving this equation we can find all N of eigen/normal frequencies ω_a and the eigen/normal modes of the small oscillations A_i^a.

We can prove, that all ω_a^2 are positive (if U is at minimum.) Let's substitute the solutions ω_a and A_j^a into equation (16.1), multiply it by $(A_i^a)^*$ and sum over the index i.

$$\sum_{ij} \left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a A_i^* = 0.$$

From here we see

$$\omega_a^2 = \frac{\sum_{ij} u_{ij} A_j^a A_i^*}{\sum_{ij} k_{ij} A_j^a A_i^*}$$

As both matrices k_{ij} and u_{ij} are symmetric and positive definite, we have the ration of to positive real numbers in the RHS. So ω_a^2 must be positive and real.

LECTURE 17
Oscillations. Zero modes. Oscillations of an infinite series of springs. Oscillations of a rope. Phonons.

17.1. Oscillations. Zero modes.

Examples

- Problem with three masses on a ring. Symmetries. Zero mode.
- Two masses, splitting of symmetric and anitsymmetric modes.
- Double pendulum with middle mass $m = 0$ and $m \to 0$.

17.2. Series of springs.

Consider one dimension string of N masses m connected with identical springs of spring constants k. The first and the last masses are connected by the same springs to a wall. The question is what are the normal modes of such system?

- The difference between the infinite number of masses and finite, but large — zero mode.

This system has N degrees of freedom, so we must find N modes. We call x_i the displacement of the ith mass from its equilibrium position. The Lagrangian is:

$$L = \sum_{i=1}^{N} \frac{m\dot{x}_i^2}{2} - \frac{k}{2} \sum_{i=0}^{N+1} (x_i - x_{i+1})^2, \qquad x_0 = x_{N+1} = 0.$$

17.2.1. First solution

The matrix $-\omega^2 k_{ij} + u_{ij}$ is

$$-\omega^2 k_{ij} + u_{ij} = \begin{pmatrix} -m\omega^2 + 2k & -k & 0 & \dots & \dots \\ -k & -m\omega^2 + 2k & -k & 0 & \dots \\ 0 & -k & -m\omega^2 + 2k & -k & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

This is $N \times N$ matrix. Let's call its determinant D_N. We then see

$$D_N = (-m\omega^2 + 2k)D_{N-1} - k^2 D_{N-2}, \qquad D_1 = -m\omega^2 + 2k, \qquad D_2 = (-m\omega^2 + 2k)^2 - k^2$$

This is a linear difference equation with constant coefficients. The solution should be of the form $D_N = a^N$. Then we have

$$a^2 = (-m\omega^2 + 2k)a - k^2, \qquad a = \frac{-m\omega^2 + 2k \pm i\sqrt{m\omega^2(4k - m\omega^2)}}{2}.$$

So the general solution and initial conditions are

$$D_N = Aa^{N-1} + \bar{A}\bar{a}^{N-1}, \qquad A + \bar{A} = -m\omega^2 + 2k, \qquad Aa + \bar{A}\bar{a} = (-m\omega^2 + 2k)^2 - k^2.$$

The solution is $A = \frac{a^2}{a-\bar{a}}$. Now in order to find the normal frequencies we need to solve the following equation for ω.

$$D_N = \frac{a^2}{a - \bar{a}} a^{N-1} - \frac{\bar{a}^2}{a - \bar{a}} \bar{a}^{N-1} = 0, \qquad \text{or} \qquad \left(\frac{a}{\bar{a}}\right)^{N+1} = 1.$$

We now say that $a = ke^{i\phi}$, ($|a|^2 = k^2$) where $\cos\phi = \frac{-m\omega^2 - 2k}{2k}$ then

$$e^{2i\phi(N+1)} = 1, \qquad 2\phi(N+1) = 2\pi n, \qquad \text{where} \quad n = 1 \dots N.$$

So we have

$$\cos\phi = \cos\frac{\pi n}{N+1} = \frac{-m\omega^2 - 2k}{2k}, \qquad \omega_n^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N+1)}.$$

17.2.2. Second solution.

From the Lagrangian we find the equations of motion

$$\ddot{x}_i = -\frac{k}{m}(2x_i - x_{i+1} - x_{i-1}), \qquad x_0 = x_{N+1} = 0.$$

We look for the solution in the form

$$x_i = \sin(\beta i)\sin(\omega t), \qquad \sin\beta(N+1) = 0.$$

Substituting this guess into the equation we get

$$\begin{aligned}
&-\omega^2\sin(\beta j) = -\frac{k}{m}\left(2\sin(\beta j) - \sin\beta(j+1) - \sin\beta(j-1)\right) \\
&= -\frac{k}{m}\Im\left(2e^{ij\beta} - e^{i(j+1)\beta} - e^{i(j-1)\beta}\right) = -\frac{k}{m}\Im e^{ij\beta}\left(2 - e^{i\beta} - e^{-i\beta}\right) = \frac{k}{m}\Im e^{ij\beta}\left(e^{i\beta/2} - e^{-i\beta/2}\right)^2 \\
&= -4\frac{k}{m}\Im e^{ij\beta}\sin^2(\beta/2) = -4\frac{k}{m}\sin(j\beta)\sin^2(\beta/2).
\end{aligned}$$

So we have

$$\omega^2 = 4\frac{k}{m}\sin^2(\beta/2),$$

but β must be such that $\sin\beta(N+1) = 0$, so $\beta = \frac{\pi n}{N+1}$, and we have

$$\omega^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N+1)}, \qquad n = 1, \dots, N$$

17.3. A rope.

The potential energy of a (2D) rope of shape $y(x)$ is $T\int_0^L \sqrt{1+y'^2}dx \approx \frac{T}{2}\int_0^L y'^2 dx$. The kinetic energy is $\int_0^L \frac{\rho}{2}\dot{y}^2 dx$, so the Lagrangian is

$$L = \int_0^L \left(\frac{\rho}{2}\dot{y}^2 - \frac{T}{2}y'^2\right) dx, \qquad y(0) = y(L) = 0.$$

In order to find the normal modes we need to decide on the coordinates in our space of functions $y(x,t)$. We will use a standard Fourier basis $\sin kx$ and write any function as

$$y(x,t) = \sum_k A_{k,t} \sin kx, \qquad \sin kL = 0$$

The constants $A_{k,t}$ are the coordinates in the Fourier basis. We then have

$$L = \frac{L}{2}\sum_k \left(\frac{\rho}{2}\dot{A}_k^2 - \frac{T}{2}k^2 A_k^2\right)$$

We see, that it is just a set of decoupled harmonic oscillators and k just enumerates them. The normal frequencies are

$$\omega_k^2 = \frac{T}{\rho}k^2, \qquad \omega = \sqrt{\frac{T}{\rho}}k.$$

LECTURE 18
Oscillations with parameters depending on time. Kapitza pendulum.

18.1. Oscillations with time-dependent parameters.

- Oscillations with parameters depending on time.

$$L = \frac{1}{2}m(t)\dot{x}^2 - \frac{1}{2}k(t)x^2.$$

The Lagrange equation

$$\frac{d}{dt}m(t)\frac{d}{dt}x = -k(t)x.$$

We change the definition of time

$$m(t)\frac{d}{dt} = \frac{d}{d\tau}, \qquad \frac{d\tau}{dt} = \frac{1}{m(t)}$$

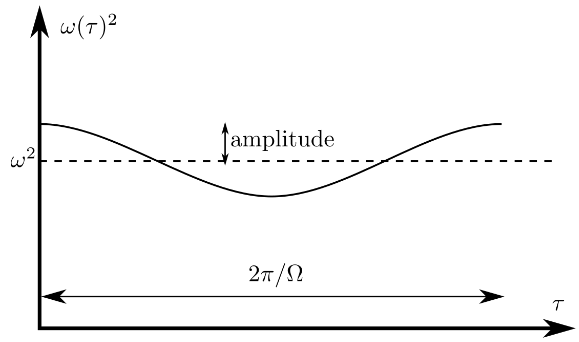
then the equation of motion is

$$\frac{d^2x}{d\tau^2} = -mkx.$$

So without loss of generality we can consider an equation

$$\ddot{x} = -\omega^2(\tau)x$$

- The most interesting is the situation when $\omega(\tau)$ is by itself a periodic function with frequency Ω. In this case the system returns back to where it was after time $2\pi/\Omega$.
- We call Ω the frequency of *change* of ω.
- Different time scales. Three different cases: $\Omega \gg \omega$, $\Omega \ll \omega$, and $\Omega \sim \omega$.

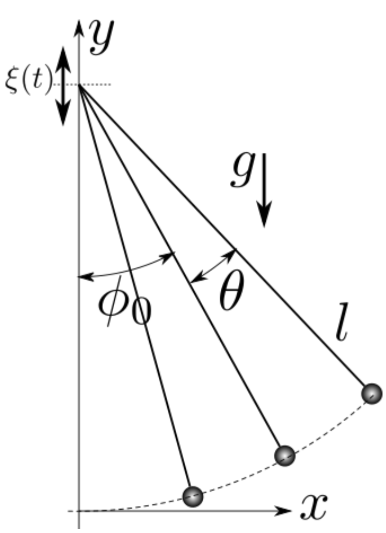


- $\Omega \gg \omega$ — Kapitza pendulum. (demo) Criteria: $\overline{\left(\dot{\xi}\right)^2} > gl$.
  * Importance of the time scale separation.
  * Averaging out fast processes – a natural thing to do.
  * Importance of non-linearity.
  * Universal mechanism – averaging over fast degrees of freedom leads to the change of the dynamics of the slow degree of freedom through non-linearity.
- $\Omega \ll \omega$ — Foucault pendulum as an example of slow change of the parameter $\Delta\phi =$solid angle of the path. (quantum: Berry phase 1984; classical: Hannay angle 1985.)
  * Topological in nature.
  * Universal.
- $\Omega \sim \omega$ — parametric resonance ($\omega_p = \frac{2}{n}\omega_0$)

$$\ddot{x} = -\omega^2(t)x, \qquad \omega^2(t) = \omega_0^2(1 + h\cos(\omega_p t)), \quad h \ll 1$$

  Different from the usual resonance:
  * If the initial conditions $x(t=0) = 0$, $\dot{x}(t=0) = 0$, then $x(t) = 0$.
  * Frequency ω_p is a fraction of ω_0.
  * At finite dissipation one must have a finite amplitude h in order to get to the resonance regime.

18.2. Kapitza pendulum $\Omega \gg \omega$



18.3. Set up of the problem.

- We have a standard rigid pendulum of length l in the gravitation field g. Somebody/something is shaking the pivot point of the pendulum vertically or horizontally very fast, but with a small amplitude. So the position of the pivot point with respect to its average is given by $\xi(t)$, where $|\xi(t)| \ll l$.
- Time scales difference. The shaking is very fast. So the frequency of the function $\xi(t)$, which we call Ω is much larger than the natural frequency of the pendulum $\omega = \sqrt{g/l}$, or $\Omega \gg \omega$.
- Expected results. We assume that the function $\xi(t)$ is so fast, that we do not actually see the motion of the pivot point. The pendulum then will perform some very complicated motion, but this motion can be split into a very fast one, which we do not see, and a slow "average" one. We are interested in that slow motion only. We will be able to clearly define these two motion because there is large separation of the time scales. We then expect, that the fast motion will change — "renormalize" — the slow motion in comparison with the motion of a normal pendulum of length l.

We will consider two cases — vertical and horizontal shaking — separately. One can also consider a general case, when both are present.

18.4. Vertical displacement.

18.4.1. Exact Lagrangian and exact equation of motion.

The coordinates

$$\begin{array}{ll} x = l\sin\phi & \dot{x} = l\dot{\phi}\cos\phi \\ y = l(1-\cos\phi) + \xi \ , & \dot{y} = l\dot{\phi}\sin\phi + \dot{\xi} \end{array}$$

The Lagrangian is

$$L = \frac{ml^2}{2}\dot{\phi}^2 + ml\dot{\phi}\dot{\xi}\sin\phi + mgl\cos\phi,$$

where I dropped the terms that are full derivatives over time. The equation of motion is

$$\ddot{\phi} + \frac{\ddot{\xi}}{l}\sin\phi = -\omega^2\sin\phi, \qquad \omega^2 = g/l.$$

This is a non-linear second order differential equation with time dependent coefficients. We cannot find the exact solution of it, but we do not want to. The exact solution will have both slow and fast motions. What we want is to "average" over the fast motion and to find an effective equation for the slow motion only.

18.4.2. Averaging over fast motion.

Look for the solution

$$\phi(t) = \phi_0(t) + \theta(t), \qquad \overline{\theta(t)} = 0,$$

where $\phi_0(t)$ is slow and $\theta(t)$ is fast. We are interested only in the function $\phi_0(t)$. The function $\theta(t)$ is so fast, that we cannot observe it. What we would like to do is to "average" the exact equation of motion so that $\theta(t)$ is "averaged out" and only $\phi_0(t)$ is left. In order to do that we need to define what do we mean by "average". **What does "averaging" mean?**

- Separation of the time scales $2\pi/\Omega \ll 2\pi/\omega$. Consider the time T such that $2\pi/\Omega \ll T \ll 2\pi/\omega$. During this time the fast motion goes over many cycles, while the slow motion almost does not happen.
- The averaging of any function, say $f(t)$, then means
$$\overline{f(t)} = \frac{1}{T}\int_0^T f(t)dt.$$
Notice, that this is not full averaging. The full averaging would require taking a limit $T \to \infty$. We do not do that.
- Notice, that with this averaging procedure we write
$$\overline{\theta(t)} = \frac{1}{T}\int_0^T \theta(t)dt = 0, \quad \text{while} \quad \overline{\phi_0(t)} = \frac{1}{T}\int_0^T \phi_0(t)dt = \phi_0(t),$$
as $\phi_0(t)$ can be considered as a constant during time T. This also means that
$$\overline{\phi(t)} = \overline{\phi_0(t) + \theta(t)} = \overline{\phi_0(t)} + \overline{\theta(t)} = \phi_0(t).$$
- Such procedure only makes sense if the result does not depend on T as long as $2\pi/\Omega \ll T \ll 2\pi/\omega$. The validity of separating the scales must also be checked afterwards.

We expect θ to be small, but $\dot\theta$ and $\ddot\theta$ are **NOT** small. θ is a very fast function, so its time derivatives are large!

Using the smallness of θ and the Taylor expansion we have
$$\ddot\phi_0 + \ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 = -\omega^2\sin\phi_0 - \omega^2\theta\cos\phi_0$$
The frequency of the function ϕ_0 is small, so the functions ϕ_0 and $\ddot\phi_0$ are slow, while the functions θ, $\ddot\theta$, and $\ddot\xi$ are fast. Let's group the fast terms to the left hand side of the equation and the slow term to the right hand side of the equation
$$\ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 + \omega^2\theta\cos\phi_0 = -\omega^2\sin\phi_0 - \ddot\phi_0 \tag{18.1}$$
The slow functions cannot cancel out the fast ones, so the fast oscillating functions must (almost) cancel each other:
$$\ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 + \omega^2\theta\cos\phi_0 = 0.$$
In this equation we notice, that θ is small, while $\ddot\theta$ and $\ddot\xi$ are not small. So the last two terms are much cmaller than the first two terms. Small terms cannot cancel the large ones, so the large terms must (almost) cancel each other. So we get
$$\ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 = 0$$

- Notice, that for times less then T we can treat ϕ_0 as a constant.

So the solution of the above equation is
$$\theta = -\frac{\xi}{l}\sin\phi_0.$$
As $\bar\xi = 0$, the requirement $\bar\theta = 0$ fixes the other terms coming from the integration of the second order differential equation.

Now we take the equation (18.1) and average it over the time T. In this procedure we can treat ϕ_0 as a constant. In addition for any bounded small function $f(t)$ the average of the time derivative is very small

$$\overline{\dot{f}(t)} = \frac{1}{T}\int_0^T \dot{f}(t)dt = \frac{f(T) - f(0)}{T}$$

The result is small, as it has large T in the denominator. So the averaging of the equation (18.1) gives.

$$\overline{\theta\ddot{\xi}}\frac{1}{l}\cos\phi_0 = -\omega^2\sin\phi_0 - \ddot{\phi}_0 \tag{18.2}$$

We now we use our result $\theta = -\frac{\xi}{l}\sin\phi_0$ in the left hand side

$$\overline{\theta\ddot{\xi}} = -\overline{\xi\ddot{\xi}}\frac{1}{l}\sin\phi_0.$$

To simplify it a bit we use

$$\overline{\xi\ddot{\xi}} = \frac{1}{T}\int_0^T \xi\ddot{\xi}dt = -\frac{1}{T}\int_0^T (\dot{\xi})^2 dt = -\overline{(\dot{\xi})^2}.$$

(One important technical note. The expression $\overline{(\dot{\xi})^2}$ means that you first take the derivative, then square, then average. The order of operations is important. Any other order of the same operations will give you zero.)

Our averaged equation (18.2) then becomes

$$\ddot{\phi}_0 = -\left(\omega^2\sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0\right) = -\frac{\partial}{\partial\phi_0}\left(-\omega^2\cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0\right)$$

18.4.3. Effective potential.

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2\cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0.$$

Notice:

- the first term in the effective potential energy is simply the standard gravitational term;
- the second term, however, comes from the "averaging" procedure;
- the second term is **NOT** small in comparison to the first, as $\dot{\xi}$ is not small.

Now we analyze this effective potential energy the usual way.

18.4.3.1. *The equilibrium.* The equilibrium positions are given by

$$0 = \left.\frac{\partial U}{\partial\phi_0}\right|_{\phi_0=\phi_{eq}} = \omega^2\sin\phi_{eq} + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_{eq} = 0, \qquad \left(\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_{eq}\right)\sin\phi_{eq} = 0$$

- We see, that a pair of solutions $\phi_{eq} = 0$ and $\phi_{eq} = \pi$ (these are the solutions of $\sin\phi_{eq} = 0$) always exists.

- We see, that if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$, a pair of new solutions appears. These are the solutions of the equation

$$\cos\phi_{eq} = -\frac{l^2\omega^2}{\overline{(\dot{\xi})^2}}$$

18.4.3.2. *The stability of equilibrium points.* The stability is defined by the sign of

$$\left.\frac{\partial^2 U}{\partial\phi_0^2}\right|_{\phi_0=\phi_{eq}} = \omega^2\cos\phi_{eq} + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos 2\phi_{eq}$$

One sees, that

- $\phi_{eq} = 0$ is always a stable solution.
- $\phi_{eq} = \pi$ is unstable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} > 1$, but becomes stable if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$ are unstable.

18.4.3.3. *The oscillations around the equilibrium.* For ϕ_0 close to the equilibrium point $\phi_{eq} = \pi$ we can introduce $\phi_0 = \pi + \tilde{\phi}$, where $\tilde{\phi}$ is small.

$$\ddot{\tilde{\phi}} = -\omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)\tilde{\phi}$$

We see, that for $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$, the oscillations in the upper point have the frequency

$$\tilde{\omega}^2 = \omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)$$

Remember, that above calculation is correct if Ω of the ξ is much larger than ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot{\xi})^2} \approx \Omega^2\xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime ($\overline{(\dot{\xi})^2}/l^2\omega^2 \sim 1$) is at

$$\Omega^2 \sim \omega^2\frac{l^2}{\xi_0^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

18.5. Horizontal displacement.

The derivation is analogous to the vertical case (see also the next lecture).

If ξ is horizontal, then it is convenient to redefine the angle $\phi_0 \to \pi/2 + \phi_0$, then the shake contribution changes sign and we get

$$U_{eff} = -\omega^2\cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0$$

The equilibrium position is found by

$$0 = \left.\frac{\partial U_{eff}}{\partial\phi_0}\right|_{\phi_0=\phi_{eq}} = \sin\phi_{eq}\left(\omega^2 - \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_{eq}\right).$$

Again we have find that

- $\phi_{eq} = 0$ and $\phi_{eq} = \pi$ are always the solutions.

- If $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$ a pair of new solutions appear, as before. They are given by

$$\cos\phi_{eq} = \frac{l^2\omega^2}{\overline{(\dot{\xi})^2}}.$$

- This is almost the same as for the vertical case, but with the + sign.
- However, this + sign changes the analysis of the stability of the equilibrium positions.
  - The position $\phi_{eq} = \pi$ is always unstable.
  - The position $\phi_{eq} = 0$ is STABLE for $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} < 1$, but becomes UNSTABLE for $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} > 1$.
  - The new solutions that appear for $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} > 1$ are STABLE.
- It means, that as $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}$ increases and crosses 1 the old Stable position $\phi_{eq} = 0$ becomes unstable and two new STABLE equilibrium positions $\cos\phi_{eq} = \frac{l^2\omega^2}{\overline{(\dot{\xi})^2}}$ appear.
- If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$ the stable equilibrium positions ϕ_{eq} are small, irrespective of if $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}$ is a bit smaller or a bit larger than 1.

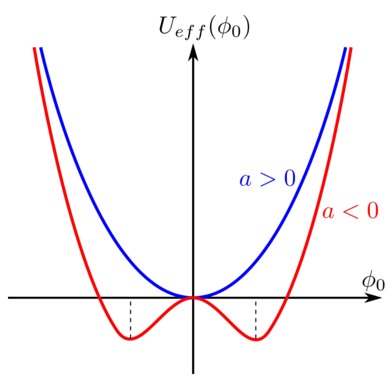
We want to analyze the regime when $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} \approx 1$ in more detail. In this regime the interesting ϕ are very small. Let's write U_{eff} for small angles up to the terms of the order of ϕ_0^4. We get (dropping the constant.)

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{24}\left(4\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} - 1\right)\phi_0^4$$

If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$, then

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{8}\phi_0^4.$$

18.5.1. Spontaneous symmetry braking.



Let's write our Effective energy for small ϕ_0 and $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$ in the following form

$$U_{eff}(\phi_0) = \frac{a}{2}\phi_0^2 + \frac{b}{4}\phi_0^4, \qquad a \equiv \omega^2\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right), \qquad b \equiv \frac{\omega^2}{2}.$$

What is important

- a is positive for $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} < 1$, but it is negative for $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} > 1$.
- b is always positive.
- Effective potential energy is symmetric with respect to $\phi_0 \to -\phi_0$.

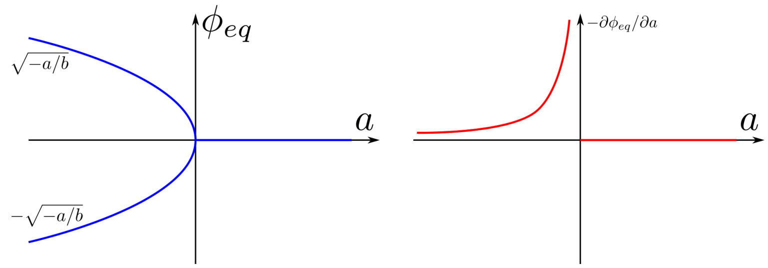
We can compute the equilibrium positions

$$\left.\frac{\partial U_{eff}}{\partial \phi_0}\right|_{\phi_0=\phi_{eq}} = (a + b\phi_{eq}^2)\phi_{eq} = 0$$

There solution $\phi_{eq} = 0$ always exists. The solutions $\phi_{eq} = \pm\sqrt{-a/b}$ exist only if $a < 0$. For $a > 0$ the position $\phi_{eq} = 0$ is stable. For $a < 0$ the position $\phi_{eq} = 0$ is unstable, but BOTH $\phi_{eq} = \pm\sqrt{-a/b}$ are stable!

BOTH new equilibrium points are stable, but the pendulum will be in only ONE of them, either in the left or in the right. So the symmetry of the obeseved "state" is less than the symmetry of the effective potential energy! The symmetry $\phi_0 \to -\phi_0$ is spontaneously broken!

- Notice, that for $a < 0$, if we substitute $\phi_0 = \phi_{eq} = \pm\sqrt{-a/b}$ into the effective potential energy, then both terms will be of the same order $\sim a^2/b$.
- This is why analyzing the spontaneous appearance of the new equilibrium positions we MUST keep the first two non-constant terms in the Taylor expansion of the potential energy.
- The higher order terms in the Taylor expansion will be smaller than the first two as a is small.



Notice, that

- at the "transition" $a = 0$, the function $\phi_{eq}(a)$ is non-analytic!
- at the "transition" $a = 0$, the "suseptibility" $-\partial\phi_{eq}/\partial a$ diverges! The system is very susceptible to noise!
- It is this noise which makes the pendulum go to the left or to the right.
- Hence the word "spontaneous".

LECTURE 19
Oscillations with parameters depending on time. Kapitza pendulum. Horizontal case.

Let's consider a shaken pendulum without the gravitation force acting on it. The fast shaking is given by a fast time dependent vector $\vec{\xi}(t)$. This vector defines a direction in space. I will call this direction $\hat{z}$, so $\vec{\xi}(t) = \hat{z}\xi(t)$.

The amplitude ξ is small $\xi \ll l$, where l is the length of the pendulum, but the shaking is very fast $\Omega \gg \omega$, the frequency of the pendulum motion (without gravity it is not well defined, but we will keep in mind that we are going to include gravity later.)

Let's now use a non inertial frame of reference connected to the point of attachment of the pendulum. In this frame of reference there is a artificial force which acts on the pendulum. This force is

$$\vec{f} = -\ddot{\xi} m \hat{z}.$$

If the pendulum makes an angle ϕ with respect to the axis $\hat{z}$, then the torque of the force $\vec{f}$ is $\tau = -lf \sin\phi$. So the equation of motion

$$ml^2\ddot{\phi} = lm\ddot{\xi}\sin\phi, \qquad \ddot{\phi} = \frac{\ddot{\xi}}{l}\sin\phi$$

Now we split the angle onto slow motion described by ϕ_0 – a slow function of time, and fast motion $\theta(t)$ a fast oscillating function of time such that $\bar{\theta} = 0$.

We then have

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0 + \theta)$$

Notice the non linearity of the RHS.

As $\theta \ll \phi_0$, we can use the Taylor expansion

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0) + \frac{\ddot{\xi}\theta}{l}\cos(\phi_0) \tag{19.1}$$

Double derivatives of θ and ξ are very large, so in the zeroth order we can write

$$\ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0), \qquad \theta = \frac{\xi}{l}\sin(\phi_0).$$

Now averaging the equation (19.1) in the way described in the previous lecture we get

$$\ddot{\phi}_0 = \frac{\overline{\ddot{\xi}\theta}}{l}\cos(\phi_0) = \frac{\overline{\ddot{\xi}\xi}}{l}\sin(\phi_0)\cos(\phi_0)$$

or

$$\ddot{\phi}_0 = \frac{\overline{\ddot{\xi}\theta}}{l}\cos(\phi_0) = -\frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0)$$

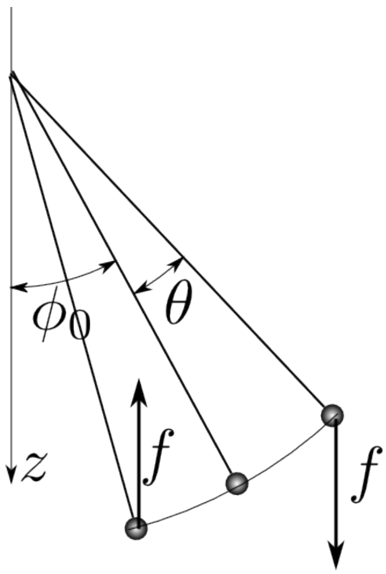


Figure 1. The Kapitza pendulum.

What is happening is illustrated on the figure. If ξ is positive, then $\ddot{\xi}$ is negative, so the torque is negative and is larger, because the angle $\phi = \phi_0 + \theta$ is larger. So the net torque is negative!

19.0.1. Vertical.

Now we can get the result from the previous lecture. We just need to add the gravitational term $-\omega^2 \sin\phi_0$.

$$\ddot{\phi}_0 = -\omega^2 \sin\phi_0 - \frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0).$$

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2 \cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0$$

The equilibrium positions are given by

$$\frac{\partial U}{\partial \phi_0} = \omega^2 \sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0 = 0, \qquad \sin\phi_0\left(\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_0\right) = 0$$

We see, that if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$, a pair of new solutions appears.

The stability is defined by the sign of

$$\frac{\partial^2 U}{\partial \phi_0^2} = \omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos 2\phi_0$$

One see, that

- $\phi_0 = 0$ is always a stable solution.
- $\phi_0 = \pi$ is unstable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} > 1$, but becomes stable if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$ are unstable.

For ϕ_0 close to π we can introduce $\phi_0 = \pi + \tilde{\phi}$

$$\ddot{\tilde{\phi}} = -\omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)\tilde{\phi}$$

We see, that for $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$ the frequency of the oscillations in the upper point have the frequency

$$\tilde{\omega}^2 = \omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)$$

Remember, that above calculation is correct if Ω of the ξ is much larger then ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot{\xi})^2} \approx \Omega^2\xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime is at

$$\Omega^2 > \omega^2 \frac{l^2}{\xi^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

19.0.2. Horizontal.

If ξ is horizontal, the it is convenient to redefine the angle $\phi_0 \to \pi/2 + \phi_0$, then the shake contribution changes sign and we get

$$U_{eff} = -\omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{4l^2} \cos 2\phi_0$$

The equilibrium position is found by

$$\frac{\partial U_{eff}}{\partial \phi_0} = \sin\phi_0 \left(\omega^2 - \frac{\overline{(\dot{\xi})^2}}{l^2} \cos\phi_0 \right).$$

Let's write U_{eff} for small angles, then (dropping the constant.)

$$U_{eff} \approx \frac{\omega^2}{2} \left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \right) \phi_0^2 + \frac{\omega^2}{24} \left(4\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} - 1 \right) \phi_0^4$$

If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$, then

$$U_{eff} \approx \frac{\omega^2}{2} \left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \right) \phi_0^2 + \frac{\omega^2}{8} \phi_0^4.$$

- Spontaneous symmetry braking.

LECTURE 20
Oscillations with parameters depending on time. Foucault pendulum.

The opposite situation, when the change of parameters is very slow – adiabatic approximation.

In rotation

$$\dot{\vec{r}} = \vec{\Omega} \times \vec{r}.$$

In our local system of coordinate (not inertial) a radius-vector is

$$\vec{r} = x\vec{e}_x + y\vec{e}_y.$$

So

$$\dot{\vec{r}} = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + x\vec{\Omega} \times \vec{e}_x + y\vec{\Omega} \times \vec{e}_y$$

I chose the system of coordinate such that $e_x \perp \vec{\Omega}$. Then

$$\vec{v}^2 = \dot{x}^2 + \dot{y}^2 + y^2\Omega^2 \cos^2\theta + \Omega^2 x^2 + 2\Omega(x\dot{y} - y\dot{x})\cos\theta$$

For a pendulum we have

$$x = l\phi\cos\psi, \quad y = l\phi\sin\psi$$

so

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= l^2\dot{\phi}^2 + l^2\phi^2\dot{\psi}^2 \\ x\dot{y} - y\dot{x} &= l^2\phi^2\dot{\psi} \end{aligned}$$

and

$$v^2 = l^2\dot{\phi}^2 + l^2\phi^2\dot{\psi}^2 + 2\Omega l^2\phi^2\dot{\psi}\cos\theta + \Omega^2 l^2\phi^2(\cos^2\psi + \sin^2\psi\cos^2\theta)$$

The Lagrangian then is

$$L = \frac{mv^2}{2} + mgl\cos\phi = \frac{mv^2}{2} - \frac{1}{2}mgl\phi^2$$

- In fact it is not exact as the centripetal force is missing. However, this force is of the order of Ω^2 and we will see, that the terms of that order can be ignored.

and the Lagrangian equations

$$\begin{aligned} \ddot{\phi} &= -\omega^2\phi + \phi\dot{\psi}^2 + 2\Omega\dot{\psi}\cos\theta + \Omega^2\phi(\sin^2\psi\cos^2\theta + \cos^2\psi) \\ 2\phi\dot{\phi}\dot{\psi} + \phi^2\ddot{\psi} + 2\phi\dot{\phi}\Omega\cos\theta &= -\frac{1}{2}\Omega^2\phi^2\sin 2\psi\sin^2\theta \end{aligned}$$

We will see, that $\dot{\psi} \sim \Omega$. Then neglecting all terms of the order of Ω^2 we find

$$\ddot{\phi} = -\omega^2\phi$$
$$\dot{\psi} = -\Omega\cos\theta$$

The total change of the angle ψ for the period is

$$\Delta\psi = \Omega T\cos\theta = 2\pi\cos\theta.$$

- Geometrical meaning.

20.1. General case.

We want to move a pendulum around the world along some closed trajectory. The question is what angle the plane of oscillations will turn after we return back to the original point?

We assume that the earth is not rotating.

We assume that we are moving the pendulum slowly.

First of all we need to decide on the system of coordinates. For our the simple case we can do it in the following way.

(a) We choose a global unit vector $\hat{z}$. The only requirement is that the z line does not intersect our trajectory.
(b) After that we can introduce the angles θ and ϕ in the usual way. (strictly speaking in order to introduce ϕ we also need o introduce a global vector $\hat{x}$, thus introducing a full global system of coordinates.)
(c) In each point on the sphere we introduce it's own system/vectors of coordinates $\hat{e}_\phi$, $\hat{e}_\theta$, and $\hat{n}$, where $\hat{n}$ is along the radius, $\hat{e}_\phi$ is orthogonal to both $\hat{n}$ and $\hat{z}$, and $\hat{e}_\theta = \hat{n}\times\hat{e}_\phi$.

We then have

$$\hat{e}_\theta^2 = \hat{e}_\phi^2 = \hat{n}^2 = 1, \qquad \hat{e}_\theta\cdot\hat{e}_\phi = \hat{e}_\theta\cdot\hat{n} = \hat{e}_\phi\cdot\hat{n} = 0.$$

Let's look how the coordinate vectors change when we change a point where we siting. So let as change our position by a small vector $d\vec{r}$. The coordinate vectors then change by $\hat{e}_\theta \to \hat{e}_\theta + d\hat{e}_\theta$, etc. We then see that

$$\hat{e}_\theta\cdot d\hat{e}_\theta = \hat{e}_\phi\cdot d\hat{e}_\phi = \hat{n}\cdot d\hat{n} = 0, \qquad \hat{e}_\theta\cdot d\hat{e}_\phi + d\hat{e}_\theta\cdot\hat{e}_\phi = \hat{e}_\theta\cdot d\hat{n} + d\hat{e}_\theta\cdot\hat{n} = \hat{e}_\phi\cdot d\hat{n} + d\hat{e}_\phi\cdot\hat{n} = 0.$$

or

$$d\hat{e}_\theta = a\hat{e}_\phi + b\hat{n}$$
$$d\hat{e}_\phi = -a\hat{e}_\theta + c\hat{n}$$
$$d\hat{n} = -b\hat{e}_\theta - c\hat{e}_\phi$$

Where coefficients a, b, and c are linear in $d\vec{r}$.

Let's now assume, that our $d\vec{r}$ is along the vector $\hat{e}_\phi$. Then it is clear, that $d\hat{n} = \sin(\theta)\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi$, and $d\hat{e}_\theta = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi$.

If $d\vec{r}$ is along the vector $\hat{e}_\theta$, then $d\hat{e}_\phi = 0$, and $d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta$.

Collecting it all together we have

$$d\hat{e}_\theta = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi - \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{n}$$
$$d\hat{e}_\phi = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\theta - \sin(\theta)\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{n}$$
$$d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta + \sin(\theta)\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi$$

Notice, that these are purely geometrical formulas.

Now let's consider a pendulum. In our local system of coordinates it's radius vector is

$$\vec{\xi} = x\hat{e}_\theta + y\hat{e}_\phi = \xi\cos\psi\hat{e}_\theta + \xi\sin\psi\hat{e}_\phi.$$

The velocity is then

$$\dot{\vec{\xi}} = \dot{\xi}(\cos\psi\hat{e}_\theta + \sin\psi\hat{e}_\phi) + \xi\dot{\psi}(-\sin\psi\hat{e}_\theta + \cos\psi\hat{e}_\phi) + \xi(\cos\psi\frac{\partial\hat{e}_\theta}{\partial\vec{r}} + \sin\psi\frac{\partial\hat{e}_\phi}{d\vec{r}})\frac{d\vec{r}}{dt}.$$

When we calculate $\dot{\vec{\xi}}^2$ we only keep terms no more than first order in $d\vec{r}/dt$

$$\dot{\vec{\xi}}^2 \approx \dot{\xi}^2 + \xi^2\dot{\psi}^2 + 2\xi^2\dot{\psi}\frac{\hat{e}_\phi\cdot\partial\hat{e}_\theta}{\partial\vec{r}}\frac{d\vec{r}}{dt} = \dot{\xi}^2 + \xi^2\dot{\psi}^2 + 2\xi^2\dot{\psi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt}$$

The potential energy does not depend on ψ, so the Lagrange equation for ψ is simply $\frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}}$. Moreover, as ξ is fast when we take the derivative $\frac{d}{dt}$ we differentiate only ξ. Then

$$4\xi\dot{\xi}\dot{\psi} + 4\xi\dot{\xi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt} = 0$$

so

$$\dot{\psi} = -\frac{1}{R\tan\theta}\frac{R\sin\theta d\phi}{dt} = -\cos\theta\frac{d\phi}{dt}$$

Finally,

$$d\psi = -\cos\theta d\phi.$$

LECTURE 21
Oscillations with parameters depending on time. Parametric resonance.

21.1. Generalities

Now we consider a situation when the parameters of the oscillator depend on time and the frequency of this dependence is comparable to the frequency of the oscillator. We start from the equation

$$\ddot{x} = -\omega^2(t)x,$$

where $\omega(t)$ is a periodic function of time. The interesting case is when $\omega(t)$ is almost a constant ω_0 with a small correction which is periodic in time with period T. Then the case which we are interested in is when $2\pi/T$ is of the same order as ω_0. We are going to find the resonance conditions. Such resonance is called "parametric resonance".

First we notice, that if the initial conditions are such that $x(t=0)=0$, and $\dot{x}(t=0)=0$, then $x(t)=0$ is the solution and no resonance happens. This is very different from the case of the usual resonance.

Let's assume, that we found two linearly independent solutions $x_1(t)$ and $x_2(t)$ of the equation. All the solutions are just linear combinations of $x_1(t)$ and $x_2(t)$.

If a function $x_1(t)$ a solution, then function $x_1(t+T)$ must also be a solution, as T is a period of $\omega(t)$. It means, that the function $x_1(t+T)$ is a linear combination of functions $x_1(t)$ and $x_2(t)$. The same is true for the function $x_2(t+T)$. So we have

$$\begin{pmatrix} x_1(t+T) \\ x_2(t+T) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

We can always choose such $x_1(t)$ and $x_2(t)$ that the matrix is diagonal. In this case

$$x_1(t+T) = \mu_1 x_1(t), \qquad x_2(t+T) = \mu_2 x_2(t)$$

so the functions are multiplied by constants under the translation on one period. The most general functions that have this property are

$$x_1(t) = \mu_1^{t/T} X_1(t), \qquad x_2(t) = \mu_2^{t/T} X_2(t),$$

where $X_1(t)$, and $X_2(t)$ are periodic functions of time.

The numbers μ_1 and μ_2 cannot be arbitrary. The functions x_1 and x_2 satisfy the Wronskian equation

$$W(t) = \dot{x}_1 x_2 - \dot{x}_2 x_1 = \text{const}$$

So on one hand $W(t+T) = \mu_1\mu_2 W(t)$, on the other hand $W(t)$ must be constant. So

$$\mu_1\mu_2 = 1.$$

Now, if x_1 is a solution so must be x_1^*. It means that either both μ_1 and μ_2 are real, or $\mu_1^* = \mu_2$. In the later case we have $|\mu_1| = |\mu_2| = 1$ and no resonance happens. In the former case we have $\mu_2 = 1/\mu_1$ (either both are positive, or both are negative). Then we have

$$x_1(t) = \mu^{t/T} X_1(t), \qquad x_2(t) = \mu^{-t/T} X_2(t).$$

We see, that one of the solutions is unstable, it increases exponentially with time. This means, that a small initial deviation from the equilibrium will exponentially grow with time. This is the parametric resonance.

21.2. Resonance.

Let's now consider the following dependence of ω on time

$$\omega^2 = \omega_0^2(1 + h\cos\gamma t)$$

where $h \ll 1$.

- The most interesting case is when $\gamma \sim 2\omega_0$. Explain.

So I will take $\gamma = 2\omega_0 + \epsilon$, where $\epsilon \ll \omega_0$. The equation of motion is

$$\ddot{x} + \omega_0^2[1 + h\cos(2\omega_0 + \epsilon)t]x = 0$$

(Mathieu's equation)

We seek the solution in the form

$$x = a(t)\cos(\omega_0 + \epsilon/2)t + b(t)\sin(\omega_0 + \epsilon/2)t$$

and retain only the terms first order in ϵ assuming that $\dot{a} \sim \epsilon a$ and $\dot{b} \sim \epsilon b$. We then substitute this solution into the equation use the identity

$$\cos(\omega_0 + \epsilon/2)t\cos(2\omega_0 + \epsilon)t = \frac{1}{2}\cos 3(\omega_0 + \epsilon/2)t + \frac{1}{2}\cos(\omega_0 + \epsilon/2)t$$

and neglect the terms with frequency $\sim 3\omega_0$ as they are off the resonance. The result is

$$-\omega_0(2\dot{a} + b\epsilon + \frac{1}{2}h\omega_0 b)\sin(\omega + \epsilon/2)t + \omega_0(2\dot{b} - a\epsilon + \frac{1}{2}h\omega_0 a)\cos(\omega + \epsilon/2)t = 0$$

So we have a pair of equations

$$2\dot{a} + b\epsilon + \frac{1}{2}h\omega_0 b = 0$$
$$2\dot{b} - a\epsilon + \frac{1}{2}h\omega_0 a = 0$$

We look for the solution in the form $a, b \sim a_0, b_0 e^{st}$, then

$$2sa_0 + b_0\epsilon + \frac{1}{2}h\omega_0 b = 0, \qquad 2sb_0 - a_0\epsilon + \frac{1}{2}h\omega_0 a_0 = 0.$$

The compatibility condition gives

$$s^2 = \frac{1}{4}\left[(h\omega_0/2)^2 - \epsilon^2\right].$$

Notice, that e^s is what we called μ before. The condition for the resonance is that s is real. It means that the resonance happens for

$$-\frac{1}{2}h\omega_0 < \epsilon < \frac{1}{2}h\omega_0$$

- The range of frequencies for the resonance depends on the amplitude h.
- The amplification s, also depends on the amplitude h.
- In case of dissipation the solution acquires a decaying factor $e^{-\lambda t}$, so s should be substituted by $s - \lambda$, so the range of instability is given by

$$-\sqrt{(h\omega_0/2)^2 - 4\lambda^2} < \epsilon < \sqrt{(h\omega_0/2)^2 - 4\lambda^2}$$

- At finite dissipation the parametric resonance requires finite amplitude $h = 4\lambda/\omega_0$.

LECTURE 22

Motion of a rigid body. Kinematics. Kinetic energy. Momentum. Tensor of inertia.

22.1. Kinematics.

We will use two different system of coordinates XYZ — fixed, or external inertial system of coordinates, and xyz the moving, or internal system of coordinates which is attached to the body itself and moves with it.

Let's $\vec{R}$ be radius vector of the center of mass O of a body with respect to the external frame of reference, $\vec{r}$ be the radius vector of any point P of the body with respect to the center of mass O, and $\vec{\mathbf{r}}$ the radius vector of the point P with respect to the external frame of reference: $\vec{\mathbf{r}} = \vec{R} + \vec{r}$. For any infinitesimal displacement $d\vec{\mathbf{r}}$ of the point P we have

$$d\vec{\mathbf{r}} = d\vec{R} + d\vec{r} = d\vec{R} + d\vec{\phi} \times \vec{r}.$$

Or dividing by dt we find the velocity of the point P as

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}, \qquad \vec{v} = \frac{d\vec{\mathbf{r}}}{dt}, \quad \vec{V} = \frac{d\vec{R}}{dt}, \quad \vec{\Omega} = \frac{d\vec{\phi}}{dt}.$$

Notice, that ϕ is not a vector, while $d\vec{\phi}$ is.

In the previous calculation the fact that O is a center of mass has not been used, so for any point O' with a radius vector $\vec{R}' = \vec{R} + \vec{a}$ we find the radius vector of the point P to be $\vec{r}' = \vec{r} - \vec{a}$, and we must have $\vec{v} = \vec{V}' + \vec{\Omega}' \times \vec{r}'$. Now substituting $\vec{r} = \vec{r}' + \vec{a}$ into $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$ we get $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \vec{r}'$. So we conclude that

$$\vec{V}' = \vec{V} + \vec{\Omega} \times \vec{a}, \qquad \vec{\Omega}' = \vec{\Omega}.$$

The last equation shows, that the vector of angular velocity is the same and does not depend on the particular moving system of coordinates. So $\vec{\Omega}$ can be called the angular velocity of the body.

If at some instant the vectors $\vec{V}$ and $\vec{\Omega}$ are perpendicular for some choice of O, then they will be perpendicular for any other O': $\vec{\Omega} \cdot \vec{V} = \vec{\Omega} \cdot \vec{V}'$. Then it is possible to solve the equation $\vec{V} + \vec{\Omega} \times \vec{a} = 0$. So in this case there exist a point (it may be outside of the body) with respect to which the whole motion is just a rotation. The line parallel to $\vec{\Omega}$ which goes through this point is called "instantaneous axis of rotation". (In the general case the instantaneous axis can be made parallel to $\vec{V}$.)

- In general both the magnitude and the direction of $\vec{\Omega}$ are changing with time, so is the "instantaneous axis of rotation".

22.2. Kinetic energy.

The total kinetic energy of a body is the sum of the kinetic energies of its parts. Lets take the origin of the moving system of coordinates to be in the center of mass. Then

$$K = \frac{1}{2}\sum m_\alpha v_\alpha^2 = \frac{1}{2}\sum m_\alpha \left(\vec{V} + \vec{\Omega}\times\vec{r}_\alpha\right)^2 = \frac{1}{2}\sum m_\alpha \vec{V}^2 + \sum m_\alpha \vec{V}\cdot\vec{\Omega}\times\vec{r}_\alpha + \frac{1}{2}\sum m_\alpha\left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$
$$= \frac{MV^2}{2} + \vec{V}\cdot\vec{\Omega}\times\sum m_\alpha\vec{r}_\alpha + \frac{1}{2}\sum m_\alpha\left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$

For the center of mass $\sum m_\alpha \vec{r}_\alpha = 0$ and we have

$$K = \frac{MV^2}{2} + \frac{1}{2}\sum m_\alpha\left(\vec{\Omega}^2\vec{r}_\alpha^2 - (\vec{\Omega}\cdot\vec{r}_\alpha)^2\right) = \frac{MV^2}{2} + \frac{I_{ij}\Omega^i\Omega^j}{2},$$

where

$$I_{ij} = \sum m_\alpha\left(\delta_{ij}\vec{r}_\alpha^2 - r_\alpha^i r_\alpha^j\right).$$

I_{ij} is the tensor of inertia. This tensor is symmetric and positive definite. The diagonal components of the tensor are called moments of inertia.

22.3. Angular momentum

The origin is at the center of mass. So we have

$$\vec{M} = \sum m_\alpha\vec{r}_\alpha\times\vec{v}_\alpha = \sum m_\alpha\vec{r}_\alpha\times(\vec{\Omega}\times\vec{r}_\alpha) = \sum m_\alpha\left(r_\alpha^2\vec{\Omega} - \vec{r}_\alpha(\vec{r}_\alpha\cdot\vec{\Omega})\right)$$

Writing this in components we have

$$M_i = I_{ij}\Omega^j.$$

- In general the direction of angular momentum $\vec{M}$ and the direction of the angular velocity $\vec{\Omega}$ do not coincide.

22.4. Tensor of inertia.

Tensor of inertia is a symmetric tensor of rank two. As any such tensor it can be reduced to a diagonal form by an appropriate choice of the moving axes. Such axes are called the principal axes of inertia. The diagonal components I_1, I_2, and I_3 are called the principal moments of inertia. In this axes the kinetic energy is simply

$$K = \frac{I_1\Omega_1^2}{2} + \frac{I_2\Omega_2^2}{2} + \frac{I_3\Omega_3^2}{2}.$$

(a) If all three principal moments of inertia are different, then the body is called "asymmetrical top".
(b) If two of the moments coincide and the third is different, then it is called "symmetrical top".
(c) If all three coincide, then it is "spherical top".

For any plane figure if z is perpendicular to the plane, then $I_1 = \sum m_\alpha y_\alpha^2$, $I_2 = \sum m_\alpha x_\alpha^2$, and $I_3 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = I_1 + I_2$. If symmetry demands that $I_1 = I_2$, then $\frac{1}{2} I_3 = I_1$. Example: a disk, a square.

If the body is a line, then (if z is along the line) $I_1 = I_2$, and $I_3 = 0$. Such system is called "rotator".

LECTURE 23
Motion of a rigid body. Rotation of a symmetric top. Euler angles.

Spherical top.

Arbitrary top rotating around one of its principal axes.

Consider a free rotation of a symmetric top $I_x = I_y \neq I_z$, where x, y, and z are the principal axes. The direction of the angular momentum does not coincide with the direction of any principle axes. Let's say, that the angle between $\vec{M}$ and the moving axes z at some instant is θ. We chose as the axis x the one that is in plane with the two vectors $\vec{M}$ and $\hat{z}$.

During the motion the total angular momentum is conserved.

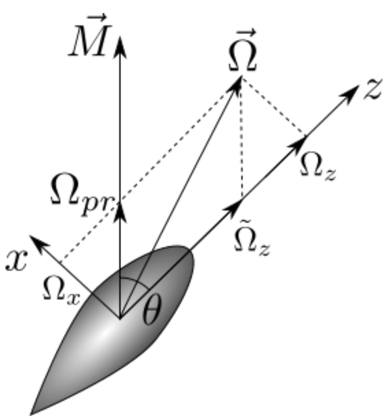


Figure 1

The whole motion can be thought as two rotations one the rotation of the body around the axes z and the other, called precession, is the rotation of the axis z around the direction of the vector $\vec{M}$.

At the instant the projection of the angular momentum on the z axis is $M\cos\theta$. This must be equal to $I_z\Omega_z$. So we have

$$\Omega_z = \frac{M}{I_z}\cos\theta.$$

In order to find the angular velocity of precession we write

$$\vec{\Omega} = \frac{\vec{M}}{M}\Omega_{pr} + \tilde{\Omega}_z\hat{z}$$

(Notice, that $\tilde{\Omega}_z \neq \Omega_z$.) and multiply this equation by $\hat{x}$. We find

$$\Omega_x = \Omega_{pr}\sin\theta.$$

On the other hand

$$\vec{M} = \Omega_x I_x\hat{x} + \Omega_z I_z\hat{z},$$

multiplying this again by $\hat{x}$ we find

$$M\sin\theta = \Omega_x I_x, \qquad \Omega_x = \frac{M}{I_x}\sin\theta.$$

hence

$$\Omega_{pr} = \frac{M}{I_x}.$$

We can also find $\tilde{\Omega}_z$, by multiplying the vector $\vec{\Omega}$ by $\hat{z}$. We get $\Omega_z = \frac{M_z}{M}\Omega_{pr} + \tilde{\Omega}_z$. Using Ω_{pr} and Ω_z we find

$$\tilde{\Omega}_z = M\left(\frac{1}{I_z} - \frac{1}{I_x}\right)\cos\theta.$$

23.1. Euler's angles

The total rotation of a rigid body is described by three angles. There are different ways to parametrize rotations. Here we consider what is called Euler's angles.

The fixed coordinates are XYZ, the moving coordinates xyz. The plane xy intersects the plane XY along the line ON called the line of nodes.

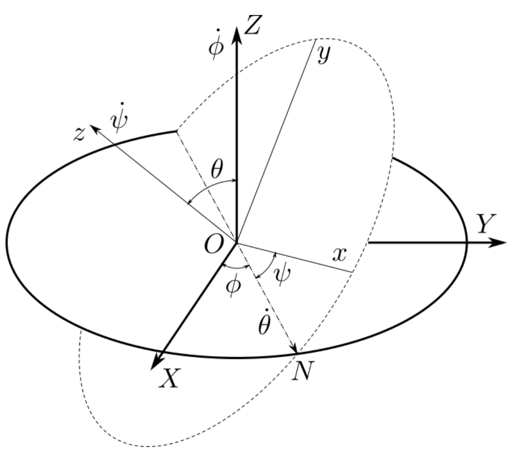


Figure 2

The angle θ is the angle between the Z and z axes. The angle ϕ is the angle between the X axes and the line of nodes, and the angle ψ is the angle between the x axes and the line of nodes. The angle θ is from 0 to π, the ϕ and ψ angles are from 0 to 2π.

I need to find the components of the angular velocity $\vec{\Omega}$ of in the moving frame and the time derivative of the angles $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$.

(a) The **vector** $\vec{\dot{\theta}}$ is along the line of nodes, so its components along x, y, and z are $\dot{\theta}_x = \dot{\theta}\cos\psi$, $\dot{\theta}_y = -\dot{\theta}\sin\psi$, and $\dot{\theta}_z = 0$.

(b) The **vector** $\vec{\dot{\phi}}$ is along the Z direction, so its component along z is $\dot{\phi}_z = \dot{\phi}\cos\theta$. Its components along x and y are $\dot{\phi}_y = \dot{\phi}\sin\theta\cos\psi$, and $\dot{\phi}_x = \dot{\phi}\sin\theta\sin\psi$.

(c) The **vector** $\vec{\dot{\psi}}$ is along the z direction, so $\dot{\psi}_z = \dot{\psi}$, and $\dot{\psi}_x = \dot{\psi}_y = 0$.

We now collect all angular velocities along each axis as $\Omega_x = \dot{\theta}_x + \dot{\phi}_x + \dot{\psi}_x$ etc. and find

$$\Omega_x = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$
$$\Omega_y = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$
$$\Omega_z = \dot{\phi}\cos\theta + \dot{\psi}$$

These equations allow us to first solve problem in the moving system of coordinates, find Ω_x, Ω_y, and Ω_z, and then calculate $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$.

Consider the symmetric top again $I_y = I_x$. We take Z to be the direction of the angular momentum. We can take the axis x coincide with the line of nodes. Then $\psi = 0$, and we have $\Omega_x = \dot{\theta}$, $\Omega_y = \dot{\phi}\sin\theta$, and $\Omega_z = \dot{\phi}\cos\theta + \dot{\psi}$.

The components of the angular momentum are $M_x = I_x\Omega_x = I_x\dot{\theta}$, $M_y = I_y\Omega_y = I_x\dot{\phi}\sin\theta$, and $M_z = I_z\Omega_z$. On the other hand $M_z = M\cos\theta$, $M_x = 0$, and $M_y = M\sin\theta$. Comparing those we find

$$\dot{\theta} = 0, \qquad \Omega_{pr} = \dot{\phi} = \frac{M}{I_x}, \qquad \Omega_z = \frac{M}{I_z}\cos\theta.$$

LECTURE 24
Symmetric top in gravitational field.

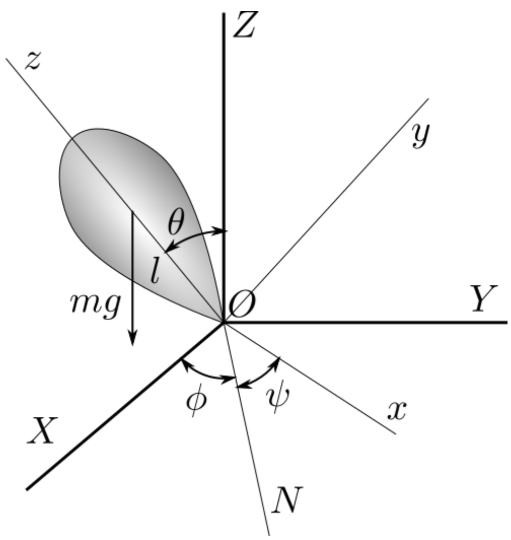


Figure 1

The angles are unconstrained and change $0 < \theta < \pi$, $0 < \psi, \phi < 2\pi$.

We want to consider the motion of the symmetric top ($I_x = I_y$) whose lowest point is fixed. We call this point O. The line ON is the line of nodes. The Euler angles θ, ϕ, and ψ fully describe the orientation of the top.

Instead of defining the tensor of inertia with respect to the center of mass, we will define it with respect to the point O. The principal axes which go through this point are parallel to the ones through the center of mass. The principal moment I_z does not change under such shift, the principal moment with respect to the axes x and y become by $I = I_x + ml^2$, where l is the distance from the point O to the center of mass.

$$\Omega_x = \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi$$
$$\Omega_y = -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi$$
$$\Omega_z = \dot\phi\cos\theta + \dot\psi$$

The kinetic energy of the symmetric top is

$$K = \frac{I_z}{2}\Omega_z^2 + \frac{I}{2}\left(\Omega_x^2 + \Omega_y^2\right) = \frac{I_z}{2}(\dot\psi + \dot\phi\cos\theta)^2 + \frac{I}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta)$$

The potential energy is simply $mgl\cos\theta$, so the Lagrangian is

$$L = \frac{I_z}{2}(\dot\psi + \dot\phi\cos\theta)^2 + \frac{I}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) - mgl\cos\theta$$

We see that the Lagrangian does not depend on ϕ and ψ – this is only correct for the symmetric top. The corresponding momenta $M_Z = \frac{\partial L}{\partial\dot\phi}$ and $M_3 = \frac{\partial L}{\partial\dot\psi}$ are conserved.

$$M_3 = I_z(\dot\psi + \dot\phi\cos\theta), \qquad M_Z = (I\sin^2\theta + I_z\cos^2\theta)\dot\phi + I_z\dot\psi\cos\theta.$$

The energy is also conserved

$$E = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta.$$

The values of M_Z, M_3, and E are given by the initial conditions.

So we have three unknown functions $\theta(t)$, $\phi(t)$, and $\psi(t)$ and three conserved quantities. The conservation lows then completely determine the whole motion.

From equations for M_Z and M_3 we have

$$\dot{\phi} = \frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$
$$\dot{\psi} = \frac{M_3}{I_3} - \cos\theta\frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$

We then substitute the values of the $\dot{\phi}$ and $\dot{\psi}$ into the expression for the energy and find

$$E' = \frac{1}{2}I\dot{\theta}^2 + U_{eff}(\theta),$$

where

$$E' = E - \frac{M_3^2}{2I_z} - mgl, \qquad U_{eff}(\theta) = \frac{(M_Z - M_3\cos\theta)^2}{2I\sin^2\theta} - mgl(1-\cos\theta).$$

This is an equation of motion for a $1D$ motion, so we get

$$t = \sqrt{\frac{I}{2}}\int\frac{d\theta}{\sqrt{E' - U_{eff}(\theta)}}.$$

This is an elliptic integral.

The effective potential energy goes to infinity when $\theta \to 0, \pi$. The function θ oscillates between θ_{min} and θ_{max} which are the solutions of the equation $E' = U_{eff}(\theta)$. These oscillations are called *nutations*. As $\dot{\phi} = \frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$ the motion depends on weather $M_Z - M_3\cos\theta$ changes sign in between θ_{min} and θ_{max}.

We can find a condition for the stable rotation about the Z axes. For such rotation $M_3 = M_Z$, so the effective potential energy is

$$U_{eff} = \frac{M_3^2}{2I}\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} - 2mgl\sin^2(\theta/2) \approx \left(\frac{M_3^2}{8I} - \frac{1}{2}mgl\right)\theta^2,$$

where the last is correct for small θ. We see, that the rotation is stable if $M_3^2 > 4Imgl$, or $\Omega_z^2 > \frac{4Imgl}{I_z^2}$.

Now assuming that $M_3^2 \approx 4Imgl$ we can find the effective energy close to the instability by going to the fourth order in θ. We get

$$U_{eff} \approx \frac{1}{2}mgl\left[\left(\frac{M_3^2}{4Imgl} - 1\right)\theta^2 + \frac{1}{12}\theta^4\right].$$

24.1. Euler equations.

Let's write the vector $\vec{M}$ in the following form

$$\vec{M} = I_x\Omega_x\hat{x} + I_y\Omega_y\hat{y} + I_z\Omega_z\hat{z}.$$

I want to use the fact that the angular momentum is conserved $\dot{\vec{M}} = 0$. In order to differentiate the above equation I need to use $\dot{\hat{x}} = \vec{\Omega} \times \hat{x}$ etc, then

$$0 = \dot{\vec{M}} = I_x\dot{\Omega}_x\hat{x} + I_y\dot{\Omega}_y\hat{y} + I_z\dot{\Omega}_z\hat{z} + I_x\Omega_x\vec{\Omega} \times \hat{x} + I_y\Omega_y\vec{\Omega} \times \hat{y} + I_z\Omega_z\vec{\Omega} \times \hat{z}.$$

Multiplying the above equation by $\hat{x}$, will find

$$0 = I_x\dot{\Omega}_x + I_y\Omega_y\vec{\Omega} \cdot [\hat{y} \times \hat{x}] + I_z\Omega_z\vec{\Omega} \cdot [\hat{z} \times \hat{x}],$$

or

$$I_x\dot{\Omega}_x = (I_y - I_z)\Omega_y\Omega_z.$$

Analogously for $\hat{y}$ and $\hat{z}$, and we get the Euler equations:

$$\begin{aligned} I_x\dot{\Omega}_x &= (I_y - I_z)\Omega_y\Omega_z \\ I_y\dot{\Omega}_y &= (I_z - I_x)\Omega_z\Omega_x \\ I_z\dot{\Omega}_z &= (I_x - I_y)\Omega_x\Omega_y \end{aligned}$$

One can immediately see, that the energy is conserved.

For a symmetric top $I_y = I_x$ we find that $\Omega_z = \text{const.}$, then denoting $\omega = \Omega_z\frac{I_z - I_x}{I_x}$ we get

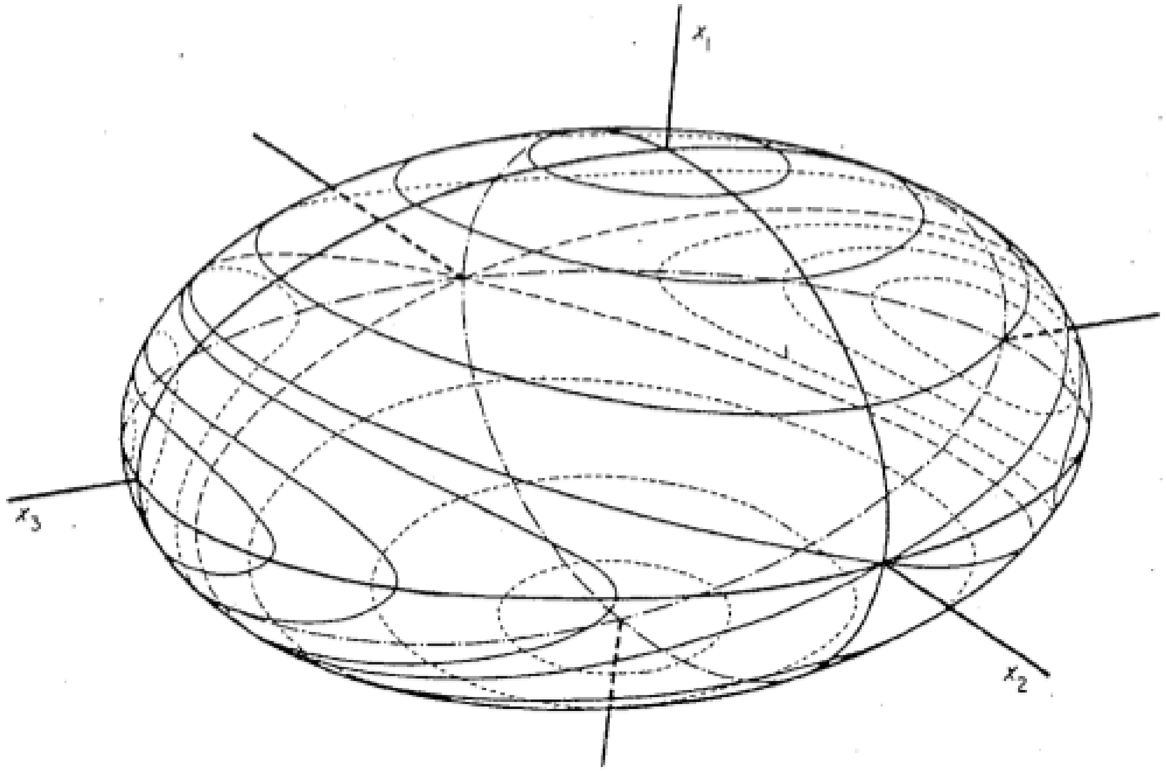
$$\begin{aligned} \dot{\Omega}_x &= -\omega\Omega_y \\ \dot{\Omega}_y &= \omega\Omega_x \end{aligned}$$

The solution is

$$\Omega_x = A\cos\omega t, \qquad \Omega_y = A\sin\omega t.$$

So the vector $\vec{\Omega}$ rotates around the z axis with the frequency ω. So does the vector $\vec{M}$ – this is the picture in the moving frame of reference. It is the same as the one before.

24.2. Stability of the free rotation of a asymmetric top.



Conservation of energy and the magnitude of the total angular momentum read

$$\frac{I_x\Omega_x^2}{2} + \frac{I_y\Omega_y^2}{2} + \frac{I_z\Omega_z^2}{2} = E$$
$$I_x^2\Omega_x^2 + I_x^2\Omega_x^2 + I_x^2\Omega_x^2 = M^2$$

In terms of the components of the angular momentum these equations read

$$\frac{M_x^2}{2I_x} + \frac{M_y^2}{2I_y} + \frac{M_z^2}{2I_z} = E$$
$$M_x^2 + M_y^2 + M_z^2 = M^2$$

The first equation describes an ellipsoid with the semiaxes $\sqrt{2I_xE}$, $\sqrt{2I_yE}$, and $\sqrt{2I_zE}$. The second equation describes a sphere of a radius M. The initial conditions give us E and M, the true solution must satisfy the conservation lows at all times. So the vector $\vec{M}$ will lie on the lines of intersection of the ellipsoid, and sphere. Notice, how different these lines

The instability of rotation around the intermediate axis is sometimes called "tennis rocket theorem" or "Dzhanibekov effect" (see `https://en.wikipedia.org/wiki/Tennis_racket_theorem`)

24.3. Asymmetric top.

I will consider the asymmetric top with $I_z > I_y > I_x$. We rewrite the Euler equations in the form

$$\dot{M}_x = \frac{I_y - I_z}{I_y I_z} M_y M_z$$
$$\dot{M}_y = \frac{I_z - I_x}{I_z I_x} M_z M_x$$
$$\dot{M}_z = \frac{I_x - I_y}{I_x I_y} M_x M_y$$

These equations are valid at all times, so differentiating the second one over time and substituting the first and the third equation in the result we get

$$\ddot{M}_y = \frac{I_z - I_x}{I_z I_x}\left(\dot{M}_z M_x + M_z \dot{M}_x\right) = \frac{I_z - I_x}{I_z I_x} M_y \left(\frac{I_x - I_y}{I_x I_y} M_x^2 + \frac{I_y - I_z}{I_y I_z} M_z^2\right).$$

Using the total momentum $M^2 = M_x^2 + M_y^2 + M_z^2$ and energy $E = \frac{M_x^2}{2I_x} + \frac{M_y^2}{2I_y} + \frac{M_z^2}{2I_z}$ we can express M_x^2 and M_z^2 through M_y^2 and get

$$\ddot{M}_y = 2\left(\frac{1}{I_y} - \frac{1}{I_z}\right)\left(\frac{1}{I_x} - \frac{1}{I_y}\right)\left(M_y^2 - \tilde{M}_y^2\right) M_y,$$

where I introduced

$$\tilde{M}_y^2 = M^2 - I_y\left(E - \frac{M^2}{2I_y}\right)\left(\frac{I_x}{I_y - I_x} - \frac{I_z}{I_z - I_y}\right).$$

The equation of motion for M_y can now be written in the form

$$\ddot{M}_y = -\frac{\partial U(M_y)}{\partial M_y}, \qquad U(M_y) = -\frac{1}{2}\left(\frac{1}{I_y} - \frac{1}{I_z}\right)\left(\frac{1}{I_x} - \frac{1}{I_y}\right)\left(M_y^4 - 2\tilde{M}_y^2 M_y^2.\right)$$

We see, that now this motion can be analyzed as a motion of $1D$ particle in the potential U. This potential has to maximums at $M_y = \pm\tilde{M}_y$ and a minimum at $M_y = 0$.

As we know such motion can be fully integrated using “energy” conservation law. The total energy of our $1D$ motion is given by

$$\tilde{E} = \frac{1}{2}\dot{M}_y^2 + U(M_y).$$

Using the second of the Euler equations we find that $\dot{M}_y^2 = \left(\frac{I_z - I_x}{I_z I_x}\right)^2 M_z^2 M_x^2$. Again using energy E and angular momentum M^2 we can re-express $\tilde{E}$ through M^2 and E (the dependence on M_y drops out, as it should)

$$\tilde{E} = \frac{1}{2}\left(\frac{1}{I_y} - \frac{1}{I_z}\right)\left(\frac{1}{I_x} - \frac{1}{I_y}\right)\left[M^2 + 2I_y \frac{I_x}{I_x - I_y}\left(E - \frac{M^2}{2I_y}\right)\right]\left[M^2 + 2I_y \frac{I_z}{I_z - I_y}\left(E - \frac{M^2}{2I_y}\right)\right].$$

LECTURE 25
Statics. Strain and Stress.

Static conditions:

- Sum of all forces is zero. $\sum \vec{F}_i = 0$.
- Sum of all torques is zero: $\sum \vec{r}_i \times \vec{F}_i = 0$.

If the sum of all forces is zero, then the torque condition is independent of where the coordinate origin is.

$$\sum (\vec{r}_i + \vec{a}) \times \vec{F}_i = \sum \vec{r}_i \times \vec{F}_i + \vec{a} \times \sum \vec{F}_i$$

Examples

- A bar on two supports.
- A block with two legs moving on the floor with μ_1 and μ_2 coefficients of friction.
- A ladder in a corner.

A problem for students in class:

- A bar on three supports.

Elastic deformations:

- Continuous media. Scales.
- Small, only linear terms.
- No nonelastic effects.
- Static.
- Isothermal.

25.1. Strain

Let the unstrained lattice be given positions x_i and the strained lattice be given positions $x_i' = x_i + u_i$. The distance dl between two points in the unstrained lattice is given by $dl^2 = dx_i^2$. The distance dl'^2 between two points in the strained lattice is given by

$$\begin{aligned} dl'^2 = dx_i'^2 &= (dx_i + du_i)^2 = dx_i^2 + 2dx_i du_i + du_i^2 \\ &= dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k} dx_j dx_k \\ &= dl^2 + 2u_{ik} dx_i dx_k, \end{aligned} \tag{25.1}$$

where

$$u_{ik} = \frac{1}{2}(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}) \tag{25.2}$$

Normally we will take only the case of small strains, for which

$$u_{ik} \approx \frac{1}{2}(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}). \tag{25.3}$$

Can diagonalize the real symmetric u_{ik}, and get orthogonal basis set. In that local frame $(1, 2, 3)$ have $dx'_1 = dx_1(1 + u_{11})$, etc. Hence the new volume is given by

$$\begin{aligned} dV' = dx'_1 dx'_2 dx'_3 &\approx dx_1 dx_2 dx_3 (1 + u_{11} + u_{22} + u_{33}) \\ &= dV(1 + u_{ii}), \end{aligned} \tag{25.4}$$

where the trace u_{ii} is invariant to the coordinate system used. Hence the fractional change in the volume is given by

$$\frac{\delta(dV)}{dV} = u_{ii}. \tag{25.5}$$

25.2. Stress

The forces are considered to be short range.

Consider a volume V that is acted on by internal stresses. The force on it due to the internal stresses is given by

$$\mathcal{F}_i = \int \frac{d\mathcal{F}_i}{dV} dV = \int F_i dV. \tag{25.6}$$

However, because the forces are short-range it should also be possible to write them as an integral over the surface element $dS_i = n_i dS$, where $\hat{n}$ is the outward normal (L&L use df_i for the surface element). Thus we expect that

$$\mathcal{F}_i = \int \sigma_{ij} dS_j \tag{25.7}$$

for some σ_{ij}. Thinking of it as a set of three vectors (labeled by i) with vector index j, we can apply Gauss's Theorem to rewrite this as

$$\mathcal{F}_i = \int \frac{\partial \sigma_{ij}}{\partial x_j} dV, \tag{25.8}$$

so comparison of the two volume integrals gives

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{25.9}$$

Because there are no self-forces (by Newton's Third Law), these forces must come from material that is outside V.

In equilibrium when only the internal stresses act we have $F_i = \frac{\partial \sigma_{ij}}{\partial x_j} = 0$. If there is a long-range force, such as gravity acting, with force $F_i^g = \rho g_i$, where ρ is the mass density and g_i is the gravitational field, then in equilibrium $F_i + F_i^g = 0$. This latter case is important for objects with relatively small elastic constant per unit mass, because then they must distort significantly in order to support their weight.

When no surface force is applied, the stress at the surface is zero. When there is a surface force P_i per unit area, this determines the stress force $\sigma_{ij}\hat{n}_j$, so

$$P_i = \sigma_{ij}\hat{n}_j \tag{25.10}$$

If the surface force is a pressure, then $P_i = -P\hat{n}_i = \sigma_{ij}\hat{n}_j$. The only way this can be true for any $\hat{n}$ is if

$$\sigma_{ij} = -P\delta_{ij}. \tag{25.11}$$

Just as the force due to the internal stresses should be written as a surface integral, so should the torque. Each of the three torques is an antisymmetric tensor, so we consider

$$\begin{aligned} M_{ik} &= \int (F_i x_k - F_k x_i) dV = \int (\frac{\partial \sigma_{ij}}{\partial x_j} x_k - \frac{\partial \sigma_{kj}}{\partial x_j} x_i) dV \\ &= \int \Big(\frac{\partial(\sigma_{ij} x_k)}{\partial x_j} - \frac{\partial(\sigma_{kj} x_i)}{\partial x_j} - (\sigma_{ik} - \sigma_{ki})\Big) dV \\ &= \int (\sigma_{ij} x_k - \sigma_{kj} x_i) dS_j - \int (\sigma_{ik} - \sigma_{ki}) dV. \end{aligned} \tag{25.12}$$

To eliminate the volume term we require that

$$\sigma_{ik} = \sigma_{ki}. \tag{25.13}$$

LECTURE 26
Work, Stress, and Strain.

26.1. Work by Internal Stresses

If there is a displacement δu_i, the work per unit volume done on V by the internal stress force F_i (due to material outside V) is given by $\delta R = F_i \delta u_i$. Hence the total work done by the internal stresses is given by

$$
\begin{aligned}
\delta W &= \int \delta R dV = \int F_i \delta u_i dV = \int \frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i dV \\
&= \int \frac{\partial(\sigma_{ij}\delta u_i)}{\partial x_j} dV - \int \sigma_{ij} \frac{\partial(\delta u_i)}{\partial x_j} dV. \qquad (26.1)
\end{aligned}
$$

If we transform the first integral to a surface integral, by Gauss's Theorem, and take $\delta u_i = 0$ on the surface — we fix the boundary, then we eliminate the first term. If we use the symmetry of σ_{ik} and the small-amplitude form of the strain, then the last term can be rewritten so that we deduce that

$$\delta R = -\sigma_{ik} \delta u_{ki}. \qquad (26.2)$$

26.1.1. Thermodynamics

We now assume the system to be in thermodynamic equilibrium. Using the energy density $d\epsilon$ and the entropy density s, the first law of thermodynamics gives

$$d\epsilon = T ds - dR = T ds + \sigma_{ik} du_{ki}. \qquad (26.3)$$

Defining the free energy density $F = \epsilon - Ts$ we have

$$dF = -s dT + \sigma_{ik} du_{ki}. \qquad (26.4)$$

In the next section we consider the form of the free energy density as a function of T and u_{ik}.

26.2. Elastic Energy

The elastic equations must be linear, as this is the accuracy which we work with. The energy density then must be quadratic in the strain tensor. We thus need to construct a scalar out of the strain tensor in the second order. If we assume that the body is isotropic, then the only way to do that is:

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ik}^2. \tag{26.5}$$

Here λ and μ are the only parameters (in the isotropic case). They are called *Lamé coefficients,* and in particular μ is called the *shear modulus* or *modulus of rigidity.* Note that u_{ii} is associated with a volume change, by (25.5). The quantity

$$\tilde{u}_{ik} = u_{ik} - \frac{1}{3}\delta_{ik}u_{jj} \tag{26.6}$$

satisfies $\tilde{u}_{ii} = 0$, and is said to describe a pure shear.

With this definition we have

$$u_{ik} = \tilde{u}_{ik} + \frac{1}{3}\delta_{ik}u_{jj} \tag{26.7}$$

$$u_{ik}^2 = \tilde{u}_{ik}^2 + \frac{2}{3}\tilde{u}_{ii}u_{kk} + \frac{1}{3}u_{jj}^2 = \tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2. \tag{26.8}$$

Hence (26.5) becomes

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu(\tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2) = F_0 + \frac{1}{2}K u_{ii}^2 + \mu\tilde{u}_{ik}^2. \qquad (K \equiv \lambda + \frac{2}{3}\mu) \tag{26.9}$$

In this form the two elastic terms are independent of one another. For the elastic energy to correspond to a stable system, each of them must be positive, so $K > 0$ and $\mu > 0$.

26.2.1. Stress

On varying u_{ik} at fixed T the free energy of (26.9) changes by

$$\begin{aligned} dF =& K u_{ii} du_{kk} + 2\mu\tilde{u}_{ik}d\tilde{u}_{ik} = K u_{ii} du_{kk} + 2\mu\tilde{u}_{ik}(du_{ik} - \frac{1}{3}\delta_{ik}du_{jj}) \\ =& K u_{ii} du_{kk} + 2\mu\tilde{u}_{ik}du_{ik} = K u_{jj}\delta_{ik}du_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}\right)du_{ik}, \end{aligned} \tag{26.10}$$

so comparison with (26.4) gives

$$\sigma_{ik} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}). \tag{26.11}$$

Note that $\sigma_{jj} = 3K u_{jj}$, so that

$$u_{jj} = \frac{\sigma_{jj}}{3K}. \tag{26.12}$$

We now solve (26.11) for u_{ik}:

$$\begin{aligned} u_{ik} =& \frac{1}{3}\delta_{ik}u_{jj} + \frac{\sigma_{ik} - K u_{jj}\delta_{ik}}{2\mu} \\ =& \frac{\sigma_{ik}}{2\mu} + \delta_{ik}(\frac{1}{3} - \frac{K}{2\mu})\frac{\sigma_{jj}}{3K} \\ =& \delta_{ik}\frac{\sigma_{jj}}{9K} + \frac{\sigma_{ik} - \frac{1}{3}\sigma_{jj}\delta_{ik}}{2\mu}. \end{aligned} \tag{26.13}$$

In the above the first term has a finite trace and the second term has zero trace.

LECTURE 27
Elastic Modulus'.

27.1. Bulk Modulus and Young's Modulus

For hydrostatic compression $\sigma_{ik} = -P\delta_{ik}$, so (26.12) gives

$$u_{jj} = -\frac{P}{K}. \qquad \text{(hydrostatic compression)} \tag{27.1}$$

We can think of this as being a δu_{jj} that gives a $\delta V/V$, by (25.5), due to $P = \delta P$, so

$$\frac{1}{K} = -\frac{\delta u_{jj}}{\delta P} = -\left.\frac{1}{V}\frac{\partial V}{\partial P}\right|_T. \tag{27.2}$$

Now let there be a compressive for per unit area P along z for a system with normal along z, so that $\sigma_{zz} = -P$, but $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{ii} = -P$. By (26.13) we have $u_{ik} = 0$ for $i \neq k$, and

$$u_{xx} = u_{yy} = \frac{P}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right), \tag{27.3}$$

$$u_{zz} = -\frac{P}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right) = -\frac{P}{E}, \qquad E \equiv \frac{9K\mu}{3K+\mu}. \tag{27.4}$$

Notice, that for positive pressure (compression) u_{zz} is always negative, as both $K > 0$ and $\mu > 0$, and hence $E > 0$.

The coefficient of P is called the *coefficient of extension.* Its inverse E is called *Young's modulus,* or the *modulus of extension.*

In particular a spring constant can be found by

$$\Delta z = u_{zz}L = -\frac{PL}{E} = -\frac{L}{AE}F, \qquad k = \frac{AE}{L}$$

We now define *Poisson's ratio* σ via

$$u_{xx} = -\sigma u_{zz}. \tag{27.5}$$

Then we find that

$$\sigma = -\frac{u_{xx}}{u_{zz}} = \frac{(\frac{1}{2\mu} - \frac{1}{3K})}{(\frac{1}{3K} + \frac{1}{\mu})} = \frac{1}{2}\frac{3K - 2\mu}{3K+\mu}. \tag{27.6}$$

Since K and μ are positive, the maximum value for σ is $\frac{1}{2}$ and the minimum value is -1. All materials in Nature (except some) have $\sigma > 0$.

It is instructive to see, how the volume changes in this experiment

$$\delta V/V = u_{ii} = u_{zz} + u_{xx} + u_{yy} = (1-2\sigma)u_{zz}.$$

In particular if $\sigma = 1/2$, then $\delta V = 0$. This is a liquid. One can also see, that $\sigma = 1/2$ means $\mu = 0$.

Often one uses E and σ instead of K and μ. We leave it to the reader to show that

$$\lambda = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}, \tag{27.7}$$

$$\mu = \frac{E}{2(1+\sigma)}, \tag{27.8}$$

$$K = \frac{E}{3(1-2\sigma)}. \tag{27.9}$$

27.2. Twisted rod.

Let's take a circular rod of radius a and length L and twist its end by a small angle θ. We want to calculate the torque required for that.

- We first guess the right solution.

Two cross-section a distance dz from each other are twisted by the angle $\frac{\theta}{L}dz$ with respect to each other. So a point at distance r from the center on the cross-section at $z + dz$ is shifted by the vector $d\vec{u} = r\frac{\theta}{L}dz\vec{e}_\phi$ in comparison to that point in the cross-section at z. We thus see that the strain tensor is

$$u_{z\phi} = u_{\phi z} = \frac{1}{2}\frac{du_\phi}{dz} = \frac{1}{2}r\frac{\theta}{L}$$

and all other elements are zero.

The relation between u_{ij} and σ_{ij} is local, so we can write them in any local system of coordinates. So as the strain tensor is trace-less

$$\sigma_{z\phi} = \sigma_{\phi z} = \mu r\frac{\theta}{L}.$$

and all other elements are zeros.

- Notice, that for that stress tensor $\frac{\partial \sigma_{z\phi}}{\partial z} = \frac{\partial \sigma_{z\phi}}{\partial \phi} = 0$, so the condition of equilibrium is satisfied and our guess was right.

Now we calculate the torque on we need to apply to the end. To a small area ds at a point at distance r from the end we need to apply a force $dF_\phi \vec{e}_\phi = \sigma_{\phi z} dS \vec{e}_\phi$. The torque of this force with respect to the center is along z direction and is given by $d\tau = rF_\phi = r\sigma_{\phi z}dS$. So the total torque is

$$\tau = \int r\sigma_{\phi z}dS = \int r\mu r\frac{\theta}{L}rdrd\phi = \mu\frac{\theta}{L}\int r^3 drd\phi = \frac{\pi}{2}\frac{\mu}{L}a^4\theta.$$

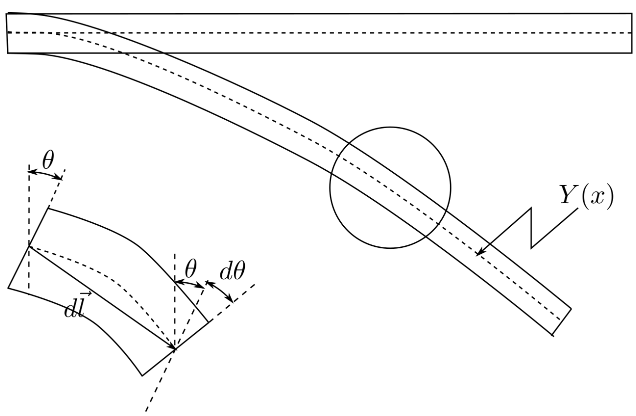
So we can measure μ in this experiment by the following way

(a) Prepare rods of different radii and lengths.
(b) For each rod measure torque τ as a function of angle θ.

(c) For each rod plot τ as a function of θ. Verify, that for small enough angle τ/θ does not depend on θ and is just a constant. This constant is a slope of each graph at small θ.

(d) Plot this constant as a function of $\frac{\pi a^4}{2L}$. Verify, that the points are on a straight line for small $\frac{\pi a^4}{2L}$. The slope of this line at small $\frac{\pi a^4}{2L}$ is the sheer modulus μ.

LECTURE 28
Small deformation of a beam.



Let's consider a small deformation of a (narrow) beam with rectangular cross-section under gravity.

- x coordinate is along undeformed beam, y is perpendicular to it.
- Nothing depends on z.
- Part of the beam is compressed, part is stretched.
- *Neutral surface.* The coordinates of the neutral surface is $Y(x)$.
- Deformation is small, $|Y'(x)| \ll 1$.

The vector $d\vec{l} = \begin{pmatrix} 1 \\ Y'(x) \end{pmatrix} dx \approx \vec{e}_x dx$. Under these conditions the angle $\theta(x) \approx Y'(x)$. So the change of the angle $\theta(x)$ between two near points is $d\theta = Y''(x)dx$.

The neutral surface is neither stretched, nor compressed. The line which is a distance y from this surface is stretched (compressed) in x direction by $du_x = yd\theta = yY''dx$, so we have

$$u_{xx} = \frac{\partial u_x}{\partial x} = y\frac{\partial^2 Y(x)}{\partial x^2}.$$

- The stretching (compression) proportional to the second derivative, as the first derivative describes the uniform rotation of the beam.

There is no confining in the y or z directions, so we find that

$$\sigma_{xx} = -Eu_{xx} = -Ey\frac{\partial^2 Y(x)}{\partial x^2}.$$

Consider a cross-section of the beam at point x. The force in the x direction of the $dydz$ element of the beam is $\sigma_{xx}dzdy$. The torque which acts from the left part on the right is

$$\tau(x) = \int y\sigma_{xx}dydz = -E\frac{\partial^2 Y(x)}{\partial x^2}\int y^2 dzdy = -IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad I = \frac{\int y^2 dydz}{\int dydz}.$$

The beam is at equilibrium. So if we take a small portion of it, between x and $x + dx$, the total force and torque on it must be zero. If the total y component of the force in a cross-section is F, then we have

$$F(x+dx)-F(x)=\rho gA dx, \qquad \frac{\partial F}{\partial x}=\rho gA.$$

The total torque acting on this portion is

$$\tau(x+dx)-\tau(x)-F(x)dx+\frac{1}{2}m\rho gA(dx)^2=0, \qquad \frac{\partial \tau}{\partial x}=F(x).$$

From these equations we find

$$\frac{\partial^2\tau}{\partial x^2}=\frac{\partial F}{\partial x}=\rho gA, \qquad IAE\frac{\partial^4 Y(x)}{\partial x^4}=-\rho gA.$$

The general solution of this equation is simply

$$Y(x)=-\frac{\rho g}{24IE}x^4+\frac{C_3}{6}x^3+\frac{C_2}{2}x^2+C_1x+C_0.$$

$$\tau(x)=-IAE\frac{\partial^2 Y(x)}{\partial x^2}$$

$$F(x)=-IAE\frac{\partial^3 Y(x)}{\partial x^3} \tag{28.1}$$

The constants must be found from the boundary conditions.

28.1. A beam with free end. Diving board.

We need to determine four unknown constants. C_0, C_1, C_2, and C_3.

We take $y=0$ at $x=0$ — fixing the position of one end — which gives $C_0=0$. Another condition is that at $x=0$ the board is horizontal – **the end is clamped** ,

$$Y'(x=0)=0$$

This determines $C_1=0$.

At the other end (distance L) both the force and the momentum are zero — it is a free end, so we get the conditions

$$F(x=L)=\left.\frac{\partial^3 Y(x)}{\partial x^3}\right|_{x=L}=0, \qquad \tau(x=L)=\left.\frac{\partial^2 Y(x)}{\partial x^2}\right|_{x=L}=0.$$

These two conditions will define $C_3=\frac{\rho g}{IE}L$ and $C_2=-\frac{\rho g}{2IE}L^2$.

$$Y(x)=-\frac{\rho g}{24IE}x^2\left(x^2-4xL+6L^2\right).$$

In particular,

$$Y(x=L)=-\frac{\rho g}{8IE}L^4.$$

Notice the proportionality to the fourth power.

Different modes for the boundary conditions.

- Clamped.
- Supported.
- Free.

28.2. A rigid beam on three supports.

Consider an absolutely rigid $E = \infty$ horizontal beam with its ends fixed. Let's see how the force on the central support changes as a function of height h of this support. For $h < 0$ the force is zero. For $h > 0$ the force is infinite and $h \to 0_-$ and $h \to 0_+$ are very different. So the situation is unphysical. It means that the order of limits first $E \to \infty$ and then $h \to 0$ is wrong. We need to take the limits in the opposite order: first take $h = 0$ and then $E \to \infty$. In this order the limits are well defined. So we need to solve the static horizontal beam on three supports for large, but finite E and then take the limit $E \to \infty$ at the very end, when we already know the solution. Luckily we know how to solve this problem for large E!

The beam is of length L. The central support has a coordinate $x = 0$ and is at the distance l_2 from the left end and at the distance l_1 from the right end ($l_1 + l_2 = L$).

The central support exerts a force F_2 on the beam. It means that there is a jump in the internal elastic forces at $x = 0$. We then need to consider the shape of the beam to be given by two functions: $Y_L(x)$ and $Y_R(x)$. As all supports are at the same height we must have $Y_L(x = 0) = Y_L(x = -l_2) = Y_R(x = 0) = Y_R(x = l_1) = 0$, so

$$Y_L = -\tfrac{\rho g}{24IE} x(x + l_2)\left(x^2 + C_1^L x + C_0^L\right) \qquad \text{for } -l_2 < x < 0$$
$$Y_R = -\tfrac{\rho g}{24IE} x(x - l_1)\left(x^2 + C_1^R x + C_0^R\right) \qquad \text{for } 0 < x < l_1$$

First let's calculate the force F_2. It is given by

$$F_2 = -IAE\left(\frac{d^3 Y_R}{dx^3}\bigg|_{x=0} - \frac{d^3 Y_L}{dx^3}\bigg|_{x=0}\right) = -\frac{\rho g A}{4}\left(C_1^R - C_1^L - l_1 - l_2\right).$$

Check the units.

The boundary conditions are

- The beam is smooth at $x = 0$: $\frac{\partial Y_L}{\partial x}\Big|_{x=0} = \frac{\partial Y_R}{\partial x}\Big|_{x=0}$.
- The torques on both ends are zero, $\frac{\partial^2 Y_L}{\partial x^2}\Big|_{x=-l_2} = \frac{\partial^2 Y_R}{\partial x^2}\Big|_{x=l_1} = 0$.
- The torque at $x = 0$ is continuous: $\frac{\partial^2 Y_L}{\partial x^2}\Big|_{x=0} = \frac{\partial^2 Y_R}{\partial x^2}\Big|_{x=0}$.

We thus have four conditions and four unknowns.

We now see what the boundary conditions give one by one:

- $$l_2 C_0^L = -l_1 C_0^R.$$
- $$3l_1^2 + 2C_1^R l_1 + C_0^R = 0, \qquad 3l_2^2 - 2C_1^L l_2 + C_0^L = 0.$$
- $$C_0^L + l_2 C_1^L = C_0^R - l_1 C_1^R.$$

These are four linear equation for four unknowns. We only need a combination $C_1^R - C_1^L$ from them. Solving the equations we find

$$C_1^R - C_1^L = -\frac{1}{2}(l_1 + l_2)\frac{l_1^2 + l_1 l_2 + l_2^2}{l_1 l_2}.$$

and hence the force is

$$F_2 = \frac{\rho g A}{8}(l_1 + l_2)\left(1 + \frac{(l_1 + l_2)^2}{l_1 l_2}\right) = \frac{Mg}{8}\left(1 + \frac{L^2}{l(L - l)}\right).$$

where l is the distance between the left end and the central support.

After this we find that

$$F_L = \frac{Mg}{8}\left(3 + \frac{l}{L} - \frac{L}{l}\right), \qquad F_L = \frac{Mg}{8}\left(3 + \frac{L-l}{L} - \frac{L}{L-l}\right).$$

In particular

- The answer does not depend on E! So the limit $E \to \infty$ is well defined!
- If $l = L/2$, we have $F_2 = \frac{5}{8}Mg$, $F_L = F_R = \frac{3}{16}Mg$. The guy at the center carries more than half of the total weight!
- If $l \to 0$ ($l \to L$), then F_2 and F_L (F_R) diverges. Why?