Homework 1. Due August 31.

Problem 1.
How many moles of gases are in a gas tank if the volume of the tank is $10m^3$, and the pressure and the temperature of the medium are 0.9×10^5Pa and 17C, respectively?

Problem 2.
A balloon full with helium floats several kilometers above the ground. At this height the volume of the balloon is $800m^3$. Temperature and pressure of the helium is equal to the temperature and pressure of the surrounding air, which are -50C and 5.26×10^3Pa, respectively.

a. How many moles of helium are in the balloon?

b. What is the mass of the helium gas? (Molar mass of the helium is $M = 4$g/mol).

c. What was the volume of the balloon on the ground?

The temperature at the ground was 0C and the pressure of the helium was 10^5Pa.

Problem 3.
$8.31Wh$ (Watthours) heat is transmitted to 1 mole of single atomic ideal gas. The initial gas temperature is 27C and the pressure is constant during the process (slow process).

a. What is the gas' final temperature?

b. What is the change of the internal energy of the gas?

c. What is the work done on the gas?

Problem 4.
Find energy levels of a point particle of mass m in an infinitely deep one dimensional square potential well of length L.

Homework 2. Due September 07

Problem 1. *Work. I*

Consider a cyclic engine operating with n moles of an ideal monoatomic gas in the cycle $a \to b \to c \to d \to a$, where V_a and T_a are the volume and the temperature of the gas in point a.

$a \to b$ pressure doubles at constant volume.

$b \to c$ volume doubles at constant pressure.

$c \to d$ pressure halves constant volume.

$d \to a$ volume halves at constant pressure.

All processes are reversible.

a. What is the net entropy change of the gas in one cycle?

b. What is the net change of the energy of the gas in one cycle?

c. What work is done by the gas during one cycle?

d. How much heat the gas got during one cycle?

e. What is the change of entropy of the gas during each of the four processes $a \to b \to c \to d \to a$?

Problem 2. *Work II*

Consider a cyclic engine operating with one mole of an ideal monoatomic gas in the cycle $a \to b \to c \to d \to a$, where V_a and T_a are the volume and the temperature of the gas in point a.

$a \to b$ is isobaric increase of temperature from T_a to $3T_a$

$b \to c$ isothermal expansion to the volume $4V_a$

$c \to d$ decrease of temperature back to T_a at constant volume

$d \to a$ isothermal compression back to the volume V_a

All processes are reversible.

a. What is the net entropy change of the gas in one cycle?

b. What is the net change of the energy of the gas in one cycle?

c. What work is done by the gas during one cycle?

d. How much heat the gas got during one cycle?

e. What is the change of entropy of the gas during each of the four processes $a \to b \to c \to d \to a$?

Homework 3. Due September 14

Problem 1. *Jacobian*
Prove the Jacobian relations used in LL

a.

$$\frac{\partial(u,y)}{\partial(x,y)} = \left(\frac{\partial u}{\partial x}\right)_y$$

b.

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(u,v)}{\partial(y,x)}$$

c.

$$\frac{\partial(g,f)}{\partial(x,y)} = \frac{\partial(g,f)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}$$

Problem 2. *van der Waals*

a. Show that

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

b. For a van der Waals gas, show that the internal energy increases as the volume increases at constant temperature. What is the answer for the ideal gas? The van der Waals equation of state is

$$P = \frac{RT}{V-b} - \frac{a}{V^2}, \qquad a > 0, \quad b > 0, \quad V > b$$

Problem 3. *Ideal gas*
Using the ideal gas expression for entropy

$$S(E,V,N) = N\left\{\log\frac{V}{N} + c\log\frac{2E}{3N} + \log\frac{(2\pi m)^{3/2}e^{5/2}}{h^3}\right\}$$

a. find $E(S,V,N)$ and derive T, P from it;

b. find the free energy $F(T,V,N) = E - TS$ and derive S, and P from it;

c. What is the equation of state of an ideal gas? What is the condition for an adiabatic process for an ideal gas in terms of variables T and V?

d. find specific heats $C_V(T,V)$ and $C_P(T,P)$ for an ideal gas.

e. Find α, β_T, and β_S for an ideal gas.

Homework 4. Due September 21.

Read: LL 19,15,24

Problem 1. *A Brick*

For a solid body (a brick) C_P and C_V are almost the same, as the thermal expansion is small. In a large range of temperatures the heat capacity can be considered to be independent of temperature.

If temperature of a brick with heat capacity C has changed from T_1 to T_2 what is the change of energy ΔE and entropy ΔS?

Problem 2. *A Brick and an Iceberg.*

The thermodynamic system consists of a hot brick of temperature T_1 and specific heat C and of an iceberg at $T_2 = 0°\mathrm{C}$. What is the maximal work W that can be performed by bringing this system to the state of thermal equilibrium? Consider the effects due to the change of the total volume of the system as negligible.

Problem 3. *Two bricks*

There are two identical bricks with heat capacitance C each. The temperature of the first brick is T_1, the temperature of the second is T_2. What will be the finite temperature of the bricks

a. if we bring the bricks to a contact and leave them touching for a long time?

b. if we extract as much work as possible from the equilibration process?

c. In which case the final temperature is smaller?

d. What is the maximal amount of work we were able to extract in the part **b**?

Problem 4. *Three bricks*

There are three identical bricks with temperatures T_1, T_2, and T_3. You are allowed to use any engines, but you cannot use external work or heat. What is the largest temperature you can give to one of the bricks?

Homework 5. Due date September 28.

Problem 1. *Magnetization*
In the presence of magnetic field H one defines magnetization of the body as $M = -(\partial E/\partial H)_{S,V}$. Then one thinks about the energy as of function of three independent thermodynamic variables $E = E(S, V, H)$ with $dE = TdS - PdV - MdH$.

Assume that we know the complete equation of state $P(T, V, H)$, the temperature dependence of (constant volume, constant magnetic field) specific heat $C_V(T, V_0, H_0)$ at some given V_0 and H_0 and temperature and magnetic field dependence of magnetization $M(T, V_0, H)$ at the same given V_0 .

Find the full dependence of $M(T, V, H)$ from this data.

Problem 2. *Membrane*
A circular membrane has a temperature dependent surface tension $\sigma(T)$. How does the heat capacity of the membrane depend on the small displacement h of the center of the membrane perpendicular to the membrane?

Problem 3. *Equilibrium*
Let's consider a macroscopic system which has an internal parameter ϕ. This parameter is not a conserved quantity (for example, magnetization). Assume that we know the free energy $F(\phi, T, V)$ of the system as a function of all variables, including the variable ϕ.

a. Show that

$$\left(\frac{\partial S}{\partial \phi}\right)_{E,V} = -\frac{1}{T}\left(\frac{\partial F}{\partial \phi}\right)_{T,V} .$$

b. There is no **b.** point for this problem.

Problem 4. *Second order*
Let's consider a macroscopic system, which has the free energy of the form

$$F(\phi, T) = F_0(T) + a(T - T_c)\phi^2 + b\phi^4,$$

where $a > 0$, $b > 0$, $T_c > 0$ are temperature independent constants and ϕ is a parameter. The value of this parameter is defined by the equilibrium condition.

a. Calculate the value of ϕ in equilibrium, for both $T > T_c$ and $T < T_c$.

b. Calculate the value of F in equilibrium, for both $T > T_c$ and $T < T_c$.

c. Calculate the value of C_V for both $T > T_c$ and $T < T_c$.

Hint: ϕ is a physical parameter, and as such cannot be a complex number.

Homework 6. Due October 5.

Problem 1. *Gas and vacuum*
A cylinder of volume $V_0 + V_1$ is divided by a partition into two parts of volume V_0 and V_1, where $V_1 \ll V_0$. The volume V_0 contains 1 mole of a gas at temperature T_0 and pressure P_0. The volume V_1 has vacuum. You know the gas's coefficient of thermal expansion $\alpha(P_0, T_0)$, its $\beta_T(P_0, T_0)$, and its heat capacity $C_V(T, V)$. At some moment the partition disappears.

1. What will be the temperature of the gas after it equilibrates?
2. A piston now adiabatically returns the gas back to the volume V_0, what will be it's temperature?
3. What are these results for the ideal gas?

Problem 2. *Oscillator*
N particles of mass m of ideal gas at temperature T are in a $3D$ harmonic potential $u(r) = \frac{m\omega^2 r^2}{2}$. Find the particle density at distance r from the center.

Problem 3. *1983-Spring-SM-U-3*
The heat capacity of a normal metal C_n at low temperatures is given by $C_n = \gamma T$, where γ is a constant. If the metal is superconducting below T_c, then the heat capacity C_S in the temperature range $0 < T < T_c$ is given by the relation $C_S = \alpha T^3$, where α is a constant. The entropy S_n, S_S of the normal metal and superconducting metal are equal at T_c, also $S_n = S_S$ as $T \to 0$.

1. Find the relation between C_S and C_n at T_c.
2. Is the transition first or second order?

Problem 4. *EXTRA PROBLEM, just for fun, will not be included in grading.*
Consider a gas of electrons at temperature T. The concentration of electrons is small. There is no gravity.

a. The gas is on top of infinitely large the positively charged plate with uniform charge σ per area. The number of electrons is N per area of the plate. Find the concentration as a function of distance to the plate.

b. The total number of electrons is N. Find the concentration of the electrons as a function of a distance from a uniformly and positively charged sphere of radius R and total charge Q

Homework 7. Due October 12.

No homework. Exam week.

Homework 8. Due October 19.

Problem 1. *Distribution function. 1983-Fall-SM-G-6*
A classical one dimensional particle, confined to the region $y \geq y_0$ is in a potential

$$V(y) = V_0 \log(y/y_0)$$

The statistical distribution is given by $\varrho(p, q) = Ae^{-E(p,q)/T}$, where A is a normalization parameter, T is a parameter which is called temperature, and $E(p, q)$ is the particle's energy.

a. Find the normalization constant A. Determine the critical temperature T_c above which the particle escapes to infinity (You need to figure out what it means.).

b. Write down the normalized positional distribution function $f(y)$ (i.e. the probability per unit distance to find the particle between y and $y + dy$) for this particle for $T < T_c$.

c. Find the average distance $\langle y \rangle$ for the particle. What happens if $0 < T_c/2 - T \ll T_c/2$

Problem 2. *Distribution, average, and fluctuations*
You through a dice N times count the number of times you have a "3". You repeat this activity many many times.

a. With what probability the number of 3s is n?

b. What is the average number of 3s?

c. What is the standard deviation (r.m.s. fluctuation)?

Problem 3. *Distribution function of an oscillator.*
A classical one dimensional oscillator ($V(x) = \frac{m\omega^2 x^2}{2}$) has a statistical distribution function $\varrho(p, q) = Ae^{-E(p,q)/T}$, where A is a normalization parameter, T is a parameter which is called temperature, and $E(p, q)$ is the oscillator's energy.

a. Find the normalization constant A.

b. Find the average coordinate of the particle.

c. Find the average momentum of the particle.

d. Find the r.m.s. fluctuations of the particle' coordinate and momentum.

e. Find the average energy of the particle. Find the heat capacity.

f. Find the distribution function for a quantity $f = f(p, x)$, where $f(p, x) = p - \omega m x$.

Problem 4. *EXTRA PROBLEM, will not be included in grading.*
In a huge cavity of temperature T there are two neutral atoms at large distance R from each other. Each atom has a magnetic susceptibility χ. Find the force between the two atoms. Neglect the retardation due to the finite speed of light and the electric dipole susceptibility.

Homework 9. Due October 26.

Problem 1. *Traveling frog.*
Consider a one dimensional frog. After every τ seconds it hops with probability 1/2 one meter to the left and with probability 1/2 one meter to the right. At $t = 0$ the frog is at $x = 0$.

1. Consider a function $p(x, t)$ – the probability for the frog to be at point x at time t. Find $p(x, t + \tau)$.
2. Consider a limit of large distances and long times. Find a differential equation for $p(x, t)$.
3. What is the initial condition for this equation?
4. Solve the equation.
5. What is the average coordinate of the frog? How does it change with time?
6. What is the average deviation of the frog's coordinate? How does it change with time?
7. Repeat all the steps for the situation when the frog hops with probability q to the left and probability $1 - q$ to the right.

Problem 2. *Electrons in wire*
A current I flows in the wire. Treat electrons as point like classical particles.

1. What is the probability that exactly n electrons cross through a wire cross-section in time T.
2. What is the average number of electrons which crossed a wire cross-section in time T?
3. What is the standard deviation of that number?

Homework 10. Due November 2.

Problem 1. *Statistical matrix*
A quantum oscillator of frequency ω and mass m is in a mixed state which is characterized by the following statistical matrix

$$w_{m,n} = \begin{cases} 0, & \text{for } n \neq m \\ Ae^{-\beta n}, & \text{for } n = m \end{cases}$$

1. Find A.
2. Find $\bar{x}$, and $\bar{p}$ where $\hat{x}$ is the coordinate and $\hat{p}$ is the momentum.
3. Find $\bar{x^2}$, and $\bar{p^2}$.
4. Find $\bar{E}$, the average energy.

Problem 2. *N independent particles*
There is a system consisting of N independent particles. Each particle can have only one of the two energy levels 0 and $+\epsilon_0$.

a. Find the stat. weight $\Delta\Gamma$ of a state with the total energy $E = M\epsilon_0$, $(M = 0, \ldots, N)$.

b. Calculate the entropy of the system as a function of its energy.

c. Find the relation between the temperature and the energy of the system.

d. How does the energy of the system depend on the temperature for $T \ll \epsilon_0$ and $T \gg \epsilon_0$?

Problem 3. *An oscillator.*
N classical particles with mass m are in the $3D$ harmonic oscillator trap potential $V = \frac{m\omega^2 r^2}{2}$.

a. Calculate the number of states (volume of phase space) with the total energy $E < E_0$.

b. Then using $w(E) = Ae^{-E/T}$, calculate the average energy of the system. (Do not forget normalization)

c. Calculate the heat capacity of the system.

d. Calculate the entropy of the system.

Homework 11. Due November 9.

Problem 1. *Quantum oscillator*

A quantum particle of charge q is in the potential $V(x) = \frac{m\omega^2}{2}x^2$ in $1D$ at temperature T.

a. Find the heat capacity of this system.

b. Find the electric dipole susceptibility of the system. ($d = -(\partial F/\partial \mathcal{E})_{V,N,T}$, $\chi = (\partial d/\partial \mathcal{E})_{V,N,T}$, where $\mathcal{E}$ is the electric field.)

c. For the oscillator state ψ_n in the presence of electric field $\mathcal{E}$ calculate $x_n = \langle \psi_n | \hat{x} | \psi_n \rangle$, and then average displacement $\bar{x} = \sum_n x_n w_n$.

Problem 2. *Averages.*

For a gas in $3D$

1. Find $\left\langle \frac{1}{v} \right\rangle$, where v is the magnitude of the particle's velocity. Express the result through $\frac{1}{\langle v \rangle}$.
2. Find $\left\langle \frac{1}{\vec{v}^2} \right\rangle$, where $\vec{v}$ is the particle's velocity. Express the result through $\frac{1}{\langle \vec{v}^2 \rangle}$.
3. Find $\left\langle e^{\vec{v}\cdot\vec{l}} \right\rangle$, where $\vec{v}$ is the particle's velocity, and $\vec{l}$ is some arbitrary vector.
4. Find $\left\langle \log\left(\frac{m\vec{v}^2}{2T}\right) \right\rangle$, where v is the particle's velocity.

Problem 3. *Small hole.*

A vessel with an ideal gas is held at constant temperature T. There is a small whole in the wall of the vessel. Calculate how the density of the gas is changing with time if outside of the vessel is vacuum.

Homework 12. Ising and friends. Due November 16.

Problem 1. *A spin* 1/2
A spin 1/2 is in a magnetic field $\mathcal{H}$ pointing in z direction. The spin is at equilibrium with heat bath at temperature T.

a. Calculate the average components of the spin.

b. Calculate the average square of the components.

c. Calculate the r.m.s. fluctuation of the components.

Problem 2. *N spins* 1/2
N non-interacting spins 1/2 are in a magnetic field $\mathcal{H}$ pointing in z direction. They are in equilibrium with heat bath at temperature T.

a. Calculate the free energy of the system?

b. What is the heat capacity of the system?

c. What is the magnetization $\mathcal{M}$ (total spin) of the system?

d. What is the magnetic susceptibility of the system? (magnetic susceptibility is defined as $\chi = (\partial \mathcal{M}/\partial \mathcal{H})_T$)

Plot the answers for the last three questions.

Problem 3. *Ising chain.*
Ising chain, or one dimensional Ising model is the following. There is a $1D$ chain with sites enumerated by $i = 1 \ldots N$. in each site there is Ising variable $\sigma_i = \pm 1$. The Hamiltonian of the chain is given by

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}.$$

Calculate specific heat of this chain. Is there a phase transition?

Homework 13. Due November 23

Problem 1. *Occupation numbers.*
Calculate the fluctuations of the occupation numbers for

a. Fermi gas.

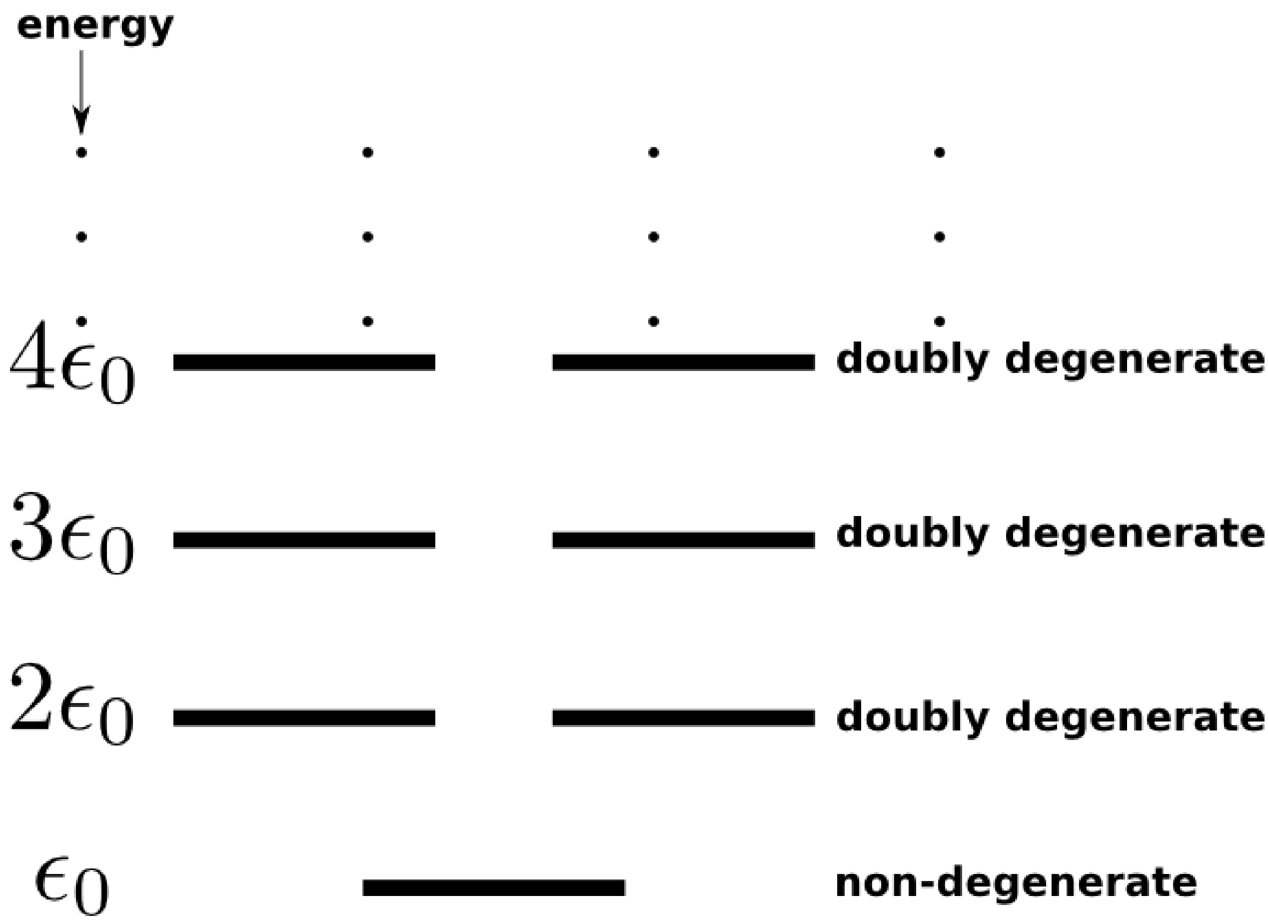
b. Bose gas.

c. Classical gas.

Problem 2. *Dielectric constant of ideal gas.*
Consider an ideal classical gas of rigid dipolar molecules in an electric field E. The dipole moment of each molecule is μ. Calculate the linear dielectric constant ϵ of the gas as a function of temperature T and density $\rho = N/V$.

Problem 3. *1993-Fall-SM-U-3*

1. Shown in the figure below is a set of available single-particle states and their energies. By properly filling these single-particle states, find the energies and degeneracies for a system of four identical non-interacting particles in the system's lowest energy level and in its first excited energy level, assuming that these particles are:
   (a) identical spinless bosons,
   (b) identical spinless fermions.
2. At a finite temperature T, calculate the ratio $r = P_1/P_0$, again for the two cases defined above. P_1 is the probability for finding the four-particle system in the first excited energy level, and P_0 is the probability for finding the same system in the lowest energy level.


energy
$4\epsilon_0$
doubly degenerate
$3\epsilon_0$
doubly degenerate
$2\epsilon_0$
doubly degenerate
ϵ_0
non-degenerate

Homework 14. Due November 30

Problem 1. *1984-Spring-SM-G-4.*

There are three quantum states of energies 0, ϵ, and 2ϵ. Consider a system of two indistinguishable non-interacting particles which can occupy these states. The system is coupled with the heat bath at temperature T.

a. Calculate the free energy of the system in case the particles are Bosons.

b. Calculate the free energy of the system in case the particles are Fermions.

c. What is the ration of occupation probability of the highest energy state of the system to the lowest energy state in each of these cases?

Problem 2. *??*

a. Find the mean speed of the fermions in an ideal D-dimensional Fermi gas at $T = 0$ (the dispersion is $\epsilon_p = p^2/2m$) in terms of v_F – Fermi velocity (velocity at ϵ_F).

b. Find the average kinetic energy of a fermion in terms of v_F.

c. Estimate Fermi energy ϵ_F and Fermi velocity for a typical metal.

Problem 3. *??*

Find how the heat capacity of the D dimensional Fermi gas with $\epsilon(p) = \vec{p}^2/2m$ depends on the density at small temperatures.

Problem 4. *??*

a. Find the temperature dependence of the chemical potential of the D dimensional gas of fermions at small temperatures.

b. Are you sure your answer is correct for $D = 2$?

c. What is the condition for the temperature to be small enough?

d. Estimate this temperature for an electron gas in a typical metal.

Homework 15. The last one! Due December 7

Problem 1. *??*
Consider a gas of non-interacting Bose particles with spin $S = 0$ and mass m. In the ultrarelativistic limit, one can approximate the dispersion relation by $E(p) = cp$.

1. Write down a general integral expression for the statistical average of the total number of particles not in the zero-energy ground state.
2. Determine the Bose-Einstein condensation temperature T_0 of the gas as a function of the gas density $\rho = N/V$.
3. Determine the fraction N_0/N of the particles in the zero-energy ground state as a function of temperature T and density ρ.

You may gnd the following formula useful:

$$\int_0^\infty \frac{z^{x-1}}{e^z - 1} dz = \Gamma(x)\xi(x)$$

x	$3/2$	$5/2$	3	5
Γ	$\sqrt{\pi}/2$	$3\sqrt{\pi}/4$	2	24
ξ	2.612	1.341	1.202	1.037

Problem 2. *??*
According to the principles of quantum statistical mechanics, the pressure of blackbody radiation inside a volume V may be calculated by treating the radiation as a photon gas, and using the relation

$$\bar{p} = \frac{1}{\beta}\frac{\partial \log Z}{\partial V},$$

where $\bar{p}$ is the mean pressure, Z the partition function, and $\beta = 1/k_B T$ is kept constant. You may assume that the volume V is a cubic box of edge length $L = V^{1/3}$, with walls maintained at temperature T.

1. Express the partition function Z in terms of the energies ϵ_s, of a set of independent photon states (*i.e.*, normal modes) in the volume, and use it to show that

$$\bar{p} = -\sum_s \frac{\partial \epsilon_s}{\partial V} \bar{n}_s,$$

where n_s is the mean population of the state s with energy ϵ_s.

2. Using the above result, obtain an explicit relation between the mean pressure $\bar{p}$ and the mean energy density $\bar{u}$ $(= \bar{E}/V)$ of the photon gas.