Advanced Mechanics II. Phys 303

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Contents

LECTURE 1
Review of Phys 302. Newtonian formulation.

1.1. Introduction

- Syllabus. Exams. Homeworks. Grades. Office hours etc.
- Structure (tentative) of the course.
- Questions, interruptions etc.
- Homeworks, cheating, study groups.
- Homework sessions, Office hour.
- Past student evaluations:
  - Erasing from white board too quickly. Talk to me!
  - Not too friendly in office hours. I do not have peoples skills, I am a nerd. Factor this in. It's all cool.
  - Comments on the HW are not very descriptive. This is by design. After you have studied physics for two-three years it is time to learn how to find your own mistakes. Otherwise, ask!
  - How to prepare for the exams — banks of problems.
  - Study materials: my lecture notes, books, wikipedia, AI, ask questions.
- What is my role?
- Life is good and physics is great!

1.2. Newtonian formulation

- Inertial frame of references. Observer.
- The observer describes the position of a point-like particle by a time dependent position vector $\vec{r}(t)$.
- For this very observer the vectors of velocity and acceleration are the time derivatives of the position vector

$$\vec{v} = \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}}, \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} \equiv \ddot{\vec{r}}.$$

- Galilean invariance:
- By performing an experiment restricted to his/her own frame of reference, an observer cannot find out if he/she is moving.
- Galilean transformation:
  - Two observers observe the same object and each other.
  - The observer 1 observes the object O at time t at the position $\vec{r}_1(t)$.
  - He/she also observes the observer 2 at the position $\vec{R}(t)$.
  - The observer 2 observes the same object O at THE SAME time t at the position $\vec{r}_2(t)$.
  - At ANY time t we have

$$\vec{r}_1(t) = \vec{R}(t) + \vec{r}_2(t).$$

  - In Galilean/Newtonian world time is absolute, it goes the same for both observers. So we can differentiate the above expression over time (this implies,

that the increment of time for both observers is the same $dt_1 = dt_2$)

$$\vec{v}_1(t) = \vec{V}(t) + \vec{v}_2(t), \qquad \vec{v}_1 = \frac{d\vec{r}_1}{dt}, \quad \vec{v}_2 = \frac{d\vec{r}_2}{dt}, \quad \vec{V} = \frac{d\vec{R}}{dt}.$$

- The equation of motion of the particle for an observer is

$$\vec{F} = m\vec{a},$$

  where $\vec{F}$ is the (vector) sum of all the forces acting on the particle.
- A force is the result of INTERaction. If there is a force acting on a particle (1) then there MUST be another object (2) which applies this force. If the force from the object 2 acting on the particle 1 is $\vec{F}_{12}$, then the there is a force $\vec{F}_{21}$ which acts on the object 2 from the particle 1 and

$$\vec{F}_{12} = -\vec{F}_{21}.$$

- Write (draw) down all the forces that act on a particle. Remember, that the force is always a result of INTERaction.
- Chose a Cartesian system of coordinates in the inertial frame of reference.
- Write down the components of the forces in the chosen system of coordinates, and the components of the total force $\vec{F}$
- For each component write down the equation of motion

$$F^\alpha = ma^\alpha \equiv m\ddot{r}^\alpha, \qquad \dot{\vec{p}} = \vec{F},$$

  Remember, that generally the forces acting on a particle at position $\vec{r}$ may (and will) depend on the position $\vec{r}$. The also may depend on the velocity of the particle $\dot{\vec{r}}$ and on time t.
- Solve the resulting system of generally nonlinear second order differential equations.
- Use the initial conditions (initial position $\vec{r}(t=0)$ and initial velocity $\vec{v}(t=0) \equiv \dot{\vec{r}}(t=0)$) in order to find the motion $\vec{r}(t)$ of this particle..

Any extended body can be thought of as a collection of point-like particles. In this picture one write the equation of motion for EACH "particle". The total force $\vec{F}_i$ acting on a particle i includes the forces of interaction with external objects and all the forces of interaction with other "particles".

- Say, we have an extended body O.
- We split it into many particle-like (infinitesimal) pieces. A piece number i has position $\vec{r}_i(t)$ at time t and a mass dm_i.
- This particle experiences forces from outside the body O. We will call the sum of all such forces $\vec{f}_i^{ex}$.
- This particle also experiences the forces $\vec{f}_{i,j}$ — this is the force on the particle i due to its interaction with the particle j of the same body O. ($f_{i,i} = 0$.)
- This the sum of all the forces acting on the particle i is $\vec{f}_i = \vec{f}_i^{ex} + \sum_j \vec{f}_{i,j}$.
- So the equation of motion for the particle i is

$$dm_i \frac{d^2\vec{r}_i}{dt^2} = \vec{f}_i^{ex} + \sum_j \vec{f}_{i,j}.$$

- This equation is the same for any i. So we can sum up the equations for all i.

$$\frac{d^2}{dt^2}\sum_i \vec{r}_i dm_i = \sum_i \vec{f}_i^{ex}$$

where I used the fact that due to the third Newton's law $\vec{f}_{i,j} = -\vec{f}_{j,i}$, so $\sum_i \sum_j \vec{f}_{i,j} = 0$.

- Introducing the center of mass coordinate

$$\vec{R} = \frac{\sum_i \vec{r}_i dm_i}{M}, \qquad M = \sum_i dm_i \quad \text{— the total mass of the body } O,$$

we find

$$M\ddot{\vec{R}} = \vec{F} \equiv \sum_i \vec{f}_i^{ex}.$$

Notice,

  - You should simply sum up all the external forces acting on the body, you do not care which point/portion of the body the force is acting on.
  - This is the same equation as for a point-like object, so it is nonlinear second order differential equation.
  - It's solution will give $\vec{R}(t)$ — the motion of the center of mass, but it will not give the change of the orientation of the body with time.
  - Also typically this equation is not complete (except in the simple problems), as all $\vec{f}_i^{ex}$ depend on the orientation of the body. So one also needs to supply the equations for the orientation of the body as a function of time.

Pros:

- Very straight forward and intuitive.
- Very general – the nature of the forces does not matter, as long as you know them.

Cons:

- The symmetries and corresponding conservation laws are hidden.
- Difficult to use in anything but the inertial frame and Cartesian coordinates. (fictitious forces etc)
- Very quickly becomes cumbersome. Easy to make mistakes.

Examples. Wedge. Wedge with friction. Oscillator. Pendulum.

1.3. Conservation laws.

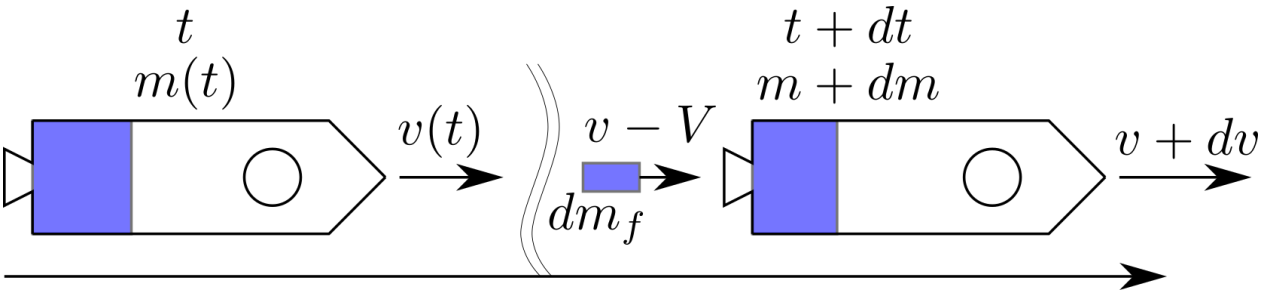
1.3.1. Momentum conservation law.

- Define your system!!
- Total momentum of the system $\vec{P}$ is the vector sum of the momenta of all the parts of the system.
- Find the total external force $\vec{F}$ acting on the system. The total external force is the vector sum of all external forces acting on all the parts of the system. External means external to the system you have defined!
- For the total momentum we can write

$$\dot{\vec{P}} = \vec{F}.$$

- If the total external force $\vec{F}$ is zero, then total momentum $\vec{P}$ is conserved, independent of all internal forces.

- Inelastic collision. Momentum conservation does not depend on if the internal forces are elastic or not.
- Rocket motion. The system is a rocket plus its fuel at time t.
- There is no external forces.
- Compare the momentum of the system at time t and time $t + dt$.



$$mv = (m+dm)(v+dv)+(v-V)dm_f, \qquad dm_f = -dm, \qquad \frac{dv}{V} = -\frac{dm}{m}, \qquad v_f - v_i = -V \log \frac{m_f}{m_i}.$$

LECTURE 2
Review of Phys 302. Angular momentum and Energy conservation.

2.1. Conservation laws.

2.1.1. Angular momentum conservation law.

- Angular momentum of a point-like particle:

$$\vec{J} = \vec{r} \times \vec{p}.$$

  The definition of the angular momentum contains the vector $\vec{r}$. This is a vector from the origin of your coordinate system to the object. So in order for this definition to work you MUST specify what is the origin of your coordinate system!
- For an extended body one has to split the body into infinitesimally small pieces, compute the vector of angular momentum for each piece, and then sum these vectors. If the body is rigid, there is a way to simplify the calculation — tensor of inertia.
- Torque of a force

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

  This same comment about the coordinate origin applies for the torque also. It makes no sense to talk about torques if you do not specify what the coordinate origin is. (except the case when the sum of all forces is zero. Why?)
- Differentiation $\vec{J}$ over time and taking into account that $\dot{\vec{r}} = \vec{v} \parallel \vec{p}$ and $\dot{\vec{p}} = \vec{F}$ we find that for each moment of time

$$\dot{\vec{J}} = \vec{\tau}.$$

- If we have extended body then we split it into infinitesimal pieces, find the angular momentum of each piece and then sum all of them up — this will be the total angular momentum of the body (at this moment of time).
- If all the internal forces between the infinitesimal pieces of the body are central then for the total angular momentum and total torque of all external (external to the body) forces for each moment of time

$$\dot{\vec{J}} = \vec{\tau}_{ex}.$$

Examples:

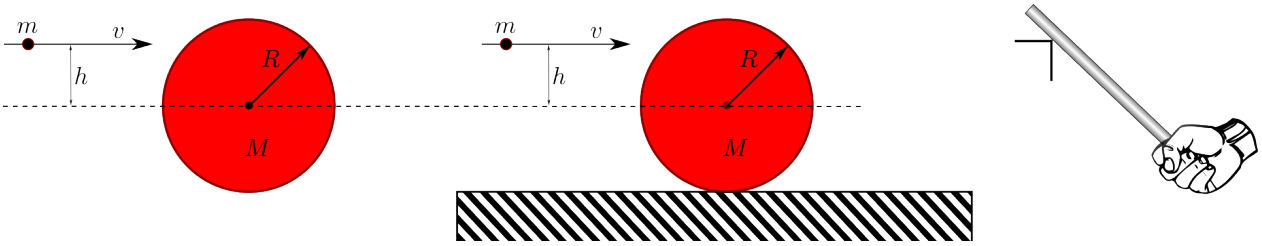


Figure 1. Examples of Angular momentum conservation..

- A bullet gets stuck in a uniform disk with the fixed axis. The mass of the bullet is negligible in comparison to the mass of the disk.
  - System: Bullet+disk.
  - Energy is NOT conserved: when the bullet gets stuck heat is produced.
  - Momentum is NOT conserved: there is an unknown force at the axis.
  - Angular momentum with respect to the disk's axis IS conserved: the torque of the force at the axis is zero.
- A bullet gets stuck in a uniform disc-like wheel with no friction. What height should the bullet strike for the wheel to roll without slipping? The mass of the bullet is negligible in comparison to the mass of the disk.
  - System: Bullet+disk.
  - Energy is NOT conserved: when the bullet gets stuck heat is produced.
  - Horizontal component of momentum IS conserved: there is no horizontal force acting on the system.
  - Angular momentum with respect to the disk's axis IS conserved: the torques of the forces of gravity and the table's reaction are zero.
  - No-slipping condition!
- At what point should the stick strike so that the striking hand feels good?

2.1.2. Energy conservation law.

- Work of a force $\vec{F}(\vec{r})$ which depends on coordinates (force field) is defined as
$$\mathcal{A} = \int_{A,\Gamma}^{B} \vec{F} \cdot d\vec{r}$$
The work depends on the path Γ from A to B.
- Consider a point-like particle which moves under the action of the force field $\vec{F}(\vec{r})$. There is no other forces acting on the particle.
$$\mathcal{A} = \int \vec{F} \cdot d\vec{r} = \int m\frac{d\vec{v}}{dt} \cdot \vec{v}dt = \int m\vec{v} \cdot d\vec{v} = \Delta\frac{m\vec{v}^2}{2}.$$
So the change of the kinetic energy equals to the work done by the force field $\vec{F}(\vec{r})$.

Conservative forces.

- A force $\vec{F}(\vec{r})$ is conservative if and only if there exists a function $U(\vec{r})$ such that
$$\vec{F} = -\frac{\partial U}{\partial \vec{r}}.$$
- The function $U(\vec{r})$ is called potential energy. If force depends on coordinates only it does not mean that the force is conservative and the function U exists — there

is a condition on the force $\vec{F}(\vec{r})$ which must be satisfied in order for $\vec{F}(\vec{r})$ to be conservative.

- On a closed contour the work of a conservative force equals zero.
- For a conservative force the work does not depend on the path.
- Total energy is defined as
$$E = \frac{m\vec{v}^2}{2} + U.$$
- If only conservative force given by the potential energy U acts on the particle, then the total energy is conserved
$$\frac{dE}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{\partial U}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} = \left(m\vec{a} - \vec{F}\right) \cdot \vec{v} = 0.$$
- Potential energy U is defined up to a constant. You have to define in advance at what point the potential energy is zero.

Examples:

- A wall with rope and a cart.
- Elastic 1D collision.
- Elastic collision in 2D (case of equal masses.)
- 1D motion under conservative force. (In 1D every force which depends only on the coordinate x is conservative.)

  The energy is given by
$$\frac{mv^2}{2} + U(x) = E.$$
  Using the initial conditions $x(t = t_0) = x_0$ and $v(t = t_0) = x_0$ we can find the value of the energy E.
$$E = \frac{mv_0^2}{2} + U(x_0)$$
  This energy is a conserved quantity, so it will be the same at any moment of time $t > t_0$.

  Writing energy conservation as
$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + U(x) = E$$
  and using the initial conditions, we find
$$\pm\sqrt{\frac{m}{2}}\int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}} = t - t_0$$
  Taking the integral we find the position of the particle as a function of time $x(t)$.
- $1D$, the graph $U(x)$. There is only two possibilities in $1D$:
  - Unbounded motion.
  - Periodic motion, with the period given by
$$T = \sqrt{2m}\int_{x_1}^{x_2} \frac{dx'}{\sqrt{E - U(x')}}, \qquad U(x_1) = U(x_2) = E.$$

LECTURE 3
Review of Phys 302. Lagrangian and Hamiltonian formulations.

3.1. Lagrangian formulation.

- Coordinates: $\{q_i\}$ — complete and independent! The number N of coordinates is the number of degrees of freedom.
- Action

$$S[\{q_i(t)\}] = \int_{t_A}^{t_B} L(\{q_i\}, \{\dot{q}_i\}, t)dt, \qquad q_i(t_A) = q_i^A, \quad q_i(t_B) = q_i^B.$$

- The solution of the N Euler-Lagrange equations (one equation for each degree of freedom)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

  which satisfies the boundary conditions is the function (finctions) which gives the extremum of the functional S.
- Hamilton principle. For a conservative mechanical system there exists a functional which is at minimum on the classical trajectories. The functional is called an Action. The function L is called a Lagrangian.
- Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t)$ — is a function of generalized coordinates and generalized velocities!!!!!
- Classical mechanics states that if the Lagrangian is

$$L = K - U.$$

  Then the solution of the Euler-Lagrange equation gives a trajectory.
- The Euler-Lagrange equations are second order coupled (nonlinear) differential equations. They require two initial conditions for each degree of freedom.

3.1.1. Generalized momentum.

- Generalized momentum (canonical)

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

- Conservation of generalized momentum (ignorable coordinates). If the Lagrangian does not depend on a coordinate q_k, then the according to the E-L equation the corresponding generalized momentum $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is conserved

$$\frac{dp_k}{dt} = 0.$$

3.1.2. Energy.

- For a Lagrangian L we can define energy E

$$E = \sum_i p_i \dot{q}_i - L, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

- No explicit time dependence in the Lagrangian $\longrightarrow$ Energy is conserved.

3.1.3. Pros and Cons.

Cons:

- Only conservative forces.

Pros:

- General coordinates.
- Only one scalar function L needs to be constructed. Easier.
- Symmetries are more transparent.

Examples:

- Pendulum.
- Double pendulum.

The technique of minimizing a functional is not used in mechanics exclusively. There are a lot of problems where such techniques are useful. The conservation lows will also be applicable there, but will, in general, have different meaning.

3.2. Hamiltonian formulation

- Phase space. Typically it is $(\{q_i\}, \{p_i\})$ space, but it also can be arbitrary phase space coordinates ξ_i. The dimension of the phase space is twice the number of degrees of freedom.
- Poisson brackets $\{\xi_i, \xi_j\}$.
  - Antisymmetric.
  - Bilinear.
  - For a constant c, $\{f, c\} = 0$.
  - $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$.
  - Jacobi's identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.
- If we know all the Poisson brackets $\{\xi_i, \xi_j\}$ for the phase space coordinates $\{\xi_i\}$, then the Poisson brackets for any two functions on the phase space $g(\{\xi\})$ and $f(\{\xi\})$ can be computed as

$$\{g, f\} = \{\xi_i, \xi_j\} \frac{\partial g}{\partial \xi_i} \frac{\partial f}{\partial \xi_j}$$

(Einstein notations are used. Check the order of indexes.)

- Hamiltonian $H(\{q_i\},\{p_i\})$ is a function on the phase space. So the Hamiltonian is a function of the coordinates and momenta!!!!!
- Hamiltonian equation of motion: for any function f on phase space $\dot{f} = \{H, f\}$, in particular
$$\dot{p}_i = \{H, p_i\}, \qquad \dot{q}_i = \{H, q_i\}$$
- In canonical coordinates and momenta
$$\{p_i, q_j\} = \delta_{i,k}, \quad \{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0, \qquad \{g, f\} = \sum_i \left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right).$$
- In canonical coordinates and momenta the Hamiltonian equations of motion are:
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$
- If we know the Lagrangian, then the momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and coordinates q_i are canonical and the Hamiltonian is given by:
$$H(p_i, q_i) = \sum_i p_i \dot{q}_i - L, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$
The second equation must be used to express the generalized velocities through generalized coordinate and momenta $\dot{q}_j(\{q_i\},\{p_i\})$, then these function must be used in the right hand side of the first equation to eliminate all the generalized velocities.

Imoprtant:

- Lagrangian is a function of generalized coordinates and velocities.
- Hamiltonian is a function of generalized coordinates and momenta.

3.3. Motion in 2D in a central field

- For the central field the potential energy of a particle depends only on the distance to the center $U(r)$.
- For a motion in a central field the VECTOR of the angular momentum is conserved.
- As The direction of the vector is constant the motion is in one plane which contains the center of force. So the motion is $2D$ and we need 2 coordinates.
- Using polar coordinates (r, ϕ) with the center at the force's center, we find the Lagrangian
$$L = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} - U(r)$$
and the equations of motion:
$$\frac{d}{dt}m\dot{r} = mr\dot{\phi}^2 - \frac{\partial U}{\partial r}, \qquad \frac{d}{dt}mr^2\dot{\phi} = 0$$
So we the angular momentum
$$L_\phi = mr^2\dot{\phi}$$
is conserved and is GIVEN by initial conditions! We can use then $\dot{\phi} = \frac{L_\phi}{mr^2}$ and rewrite the equation for $r(t)$ as
$$m\ddot{r} = \frac{L_\phi^2}{mr^3} - \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{L_\phi^2}{2mr^2} + U(r) \right) = -\frac{\partial U_{eff}(r)}{\partial r}$$

This is a motion in $1D$ in the effective potential

$$U_{eff}(r) = U(r) + \frac{L_\phi^2}{2mr^2}.$$

We then know the solution. As $\frac{m\dot{r}^2}{2} + U_{eff}(r) = E$, where E is found from the initial conditions, is conserved, we have

$$\frac{dr}{\sqrt{E - U_{eff}(r)}} = \pm\sqrt{\frac{2}{m}}dt, \qquad dt = \frac{mr^2}{L_\phi}d\phi$$

or

$$\pm d\phi = \frac{L_\phi}{\sqrt{2m}}\frac{1}{r^2}\frac{dr}{\sqrt{E - U_{eff}(r)}}$$

- Kepler orbits. Let's use the gravitational potential energy

$$U(r) = -\frac{GMm}{r},$$

then we have

$$U_{eff} = -\frac{GMm}{r} + \frac{L_\phi^2}{2mr^2}$$

and

$$\phi - \phi_0 = \frac{L_\phi}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr'}{r'^2}\frac{1}{\sqrt{E + \frac{GMm}{r'} - \frac{L_\phi^2}{2mr'^2}}}$$

- For $E < 0$ $(0 < \epsilon < 1)$ this expression will give $r(\phi)$ for Kepler orbits $\frac{1}{r} = \frac{1}{c}(1 + \epsilon\cos\phi)$.
- For $E > 0$ $(\epsilon \geq 1)$ it will give the unbounded orbits. In particular we can use it to compute the angle between the incoming and outgoing velocities at infinity — the scattering angle.

LECTURE 4
Math preliminaries. Probability density. Solid angle.

We are going to discuss what is called scattering problem. We start from some math preliminaries.

4.1. Probability density.

Classical mechanics is fully deterministic. The randomness and hence probability appears because we do not have full control over the initial conditions and over the measurement of the final state.

4.1.1. Probability for a random process with discrete outcomes.

Consider a random process which randomly outputs some numbers k from a given discrete subset A. This subset is a set of all possible outcomes of the process. We run this process N times and count now many times n_k the outcome k has appeared. Notice, that $\sum_{k\in A} n_k = N$.

For this process we define the probability of an outcome k as

$$p_k = \lim_{N\to\infty} \frac{n_k}{N}.$$

Notice, that we always have

$$p_k \geq 0, \qquad \sum_{k\in A} p_k = 1.$$

We can also compute the average outcome $\langle k\rangle$.

$$\langle k\rangle = \lim_{N\to\infty} \sum_{k\in A} \frac{kn_k}{N} = \sum_{k\in A} kp_k.$$

In fact, for any function $f(k)$ of k we can define the average value of this function: the measurement $f(k)$ occurs with the probability p_k, so

$$\langle f(k)\rangle = \sum_{k\in A} f(k)p_k.$$

Examples:

- One tosses a dice and gets number k from 1 to 6. Assuming the game is fair the probability of each outcome must be equal, so $p_1 = p_2 = \cdots = p_6 \equiv p$. As the sum of all probabilities must be 1, we find $p = 1/6$. One can compute the average outcome

$$\langle k\rangle = \sum_{k\in A} kp_k = p\sum_{k=1}^{6} k = 3.5.$$

Or one can compute $\langle k^2 \rangle$

$$\langle k^2 \rangle = \sum_{k \in A} k^2 p_k = p \sum_{k=1}^{6} k^2 = \frac{91}{6}.$$

- A random process has M possible outcomes. An outcome k has a probability p_k. We run the process N times. What is the probability P_n that the outcome k happens exactly n times? The answer is

$$P_n = \frac{N!}{N!(N-n)!} p_k^n (1-p_k)^{N-n}.$$

This is called binomial distribution. You should check, that $\sum_{n=0}^{N} P_n = 1$.

- On average "Crust&Physics" pizzeria has λ customers per hour. Assuming, that the customers come randomly and independently from each other, what is the probability P_n that the pizzeria has n customers in a given hour? The answer is

$$P_n = \frac{\lambda^n}{n_!} e^{-\lambda}.$$

This is called Poisson distribution. You should check, that $\sum_{n=0}^{N} P_n = 1$ and $\langle n \rangle = \lambda$.

4.1.2. Probability density

What if the outcome of a random process is not discrete? An outcome is ANY number x from the interval $[a, b]$. In this case we split the interval into (infinitesimally) small intervals, each of length dx. We run the process N times and count how many times $dn(x)$ the process gives an outcome in the interval dx at value x. The probability then is $dp(x) = \lim_{N \to \infty} \frac{dn(x)}{N}$. If the process is smooth, then $dp(x)$ will be proportional to the length dx of the interval. So we can write $dp(x) = \rho(x)dx$. The function $\rho(x)$ is called the probability density. Thus the probability density $dp(x) = \rho(x)dx$ is the probability to find an outcome in the small interval $[x, x + dx]$. The probability density $\rho(x)$ is the characteristic of the random process.

- Positive definite. The probability MUST be positive, so $\rho(x) > 0$ for any x.
- Normalization. The probability to find the outcome somewhere in the full interval $[a, b]$ is the sum of the probabilities to find the outcome in all small intervals dx. So this probability is $\int_a^b \rho(x)dx$. But this probability MUST be 1 — the process does give SOME number. So the probability density MUST be normalized

$$\int_a^b \rho(x)dx = 1.$$

- Averaging. Let's compute the average value of x in a random process which gives a random x from the interval $[a, b]$ with probability density $\rho(x)$. The probability to find x in the infinitesimal interval $[x, x + dx]$ is $dp(x) = \rho(x)dx$. So the average x is given by the sum of $xdp(x)$ over all infinitesimal intervals, so $\langle x \rangle = \int_a^b x\rho(x)dx$, similarly $\langle x^2 \rangle = \int_a^b x^2\rho(x)dx$, or more generally

$$\langle f(x) \rangle = \int_a^b f(x)\rho(x)dx.$$

Examples:

- Uniform distribution.
  - The probability for x is uniformly distributed in the interval $[a, b]$.

- The probability density is a constant $\rho(x) = A$.
- The constant must be found from the normalization condition:

$$\int_a^b \rho(x)dx = 1, \quad \text{so} \quad A = \frac{1}{b-a}.$$

- Average x

$$\langle x \rangle = \int_a^b x\rho(x)dx = \frac{1}{b-a}\int_a^b xdx = \frac{a+b}{2}.$$

as expected.

- We also can compute average of x^2:

$$\langle x^2 \rangle = \int_a^b x^2\rho(x)dx = \frac{1}{b-a}\int_a^b x^2dx = \frac{b^2+ab+a^2}{3}.$$

Notice, $\langle x^2 \rangle \neq \langle x \rangle^2$.

- $1D$ quantum harmonic oscillator:
  - Potential energy $U(x) = \frac{m\omega^2 x^2}{2}$: the wave function of the ground state is $\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$, and $-\infty < x < \infty$.
  - The probability density for the coordinate is

$$\rho(x) = |\psi(x)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{\hbar}x^2}, \qquad \int_{-\infty}^{\infty} \rho(x)dx = 1.$$

These are the results of Quantum mechanics. We here take them for granted and only use the resulting probability density.

  - Average position in the ground state is given by

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx = 0.$$

  - However,

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\rho(x)dx = \frac{1}{2}\frac{\hbar}{m\omega}.$$

  - In particular average potential energy

$$\langle U(x) \rangle = \int_{-\infty}^{\infty} U(x)\rho(x)dx = \frac{m\omega^2}{2}\langle x^2 \rangle = \frac{1}{4}\hbar\omega.$$

  - We also can ask what is probability density for the potential energy U? We see, that $x = \sqrt{2U/m\omega^2}$, so $dx = \frac{dU}{\sqrt{2Um\omega^2}}$.

$$\rho(x)dx = \rho\left(\sqrt{2U/m\omega^2}\right)\frac{dU}{\sqrt{2Um\omega^2}} = \frac{1}{\sqrt{2\pi U\hbar\omega}}e^{-\frac{U}{\hbar\omega}}dU.$$

so

$$\rho_U(U) = \frac{1}{\sqrt{2\pi U\hbar\omega}}e^{-\frac{U}{\hbar\omega}}, \qquad \int_0^{\infty} \rho_U(U)dU = 1.$$

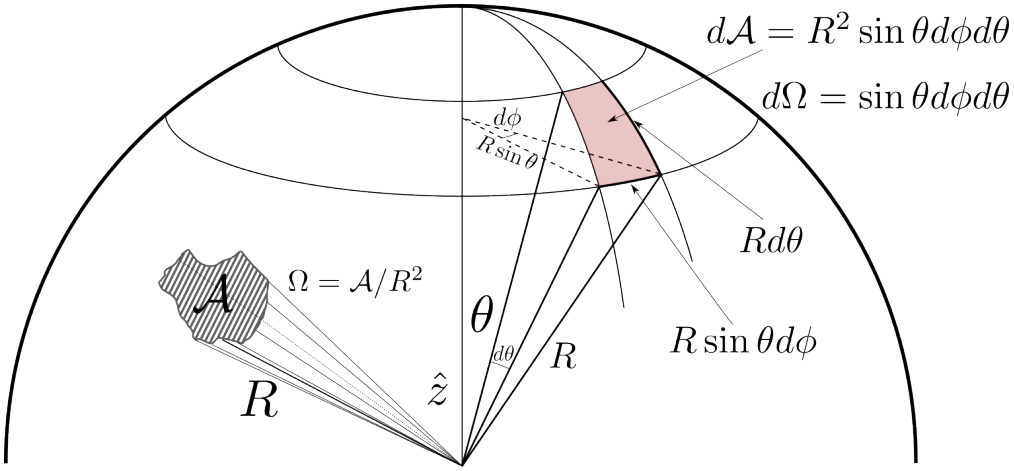


Figure 1. Solid Angle.

4.1.2.1. *Change of variables.* Let's say we have a probability density $\rho(v)$ to find a speed between v and $v + dv$. We want to find the probability density to find a kinetic energy between K and $K + dK$. The probability is

$$dp = \rho(v)dv$$

The kinetic energy is $K = \frac{mv^2}{2}$, or $v = \sqrt{2K/m}$, so $dv = \frac{dK}{\sqrt{2mK}}$, and

$$dp = \rho\left(\sqrt{2K/m}\right)\frac{1}{\sqrt{2mK}}dK, \qquad \rho_K(K) = \rho\left(\sqrt{2K/m}\right)\frac{1}{\sqrt{2mK}}.$$

- Notice the change of differential!
- Notice, that if the distribution is uniform in one variable it may not be uniform in the other!

4.1.3. Solid angle.

- An angle in radians is the ratio of the arc's length to the radius.
- Analogously, the solid angle is the ration of the area of the patch on the surface of a sphere to the square of the radius of the sphere.
- A small solid angle $d\Omega$ is given by $d\Omega = \sin\theta d\theta d\phi$, see figure 1.

LECTURE 5
Disintegration of many particles.

- HW sessions, Fridays, 1:50-2:40 p.m., MPHY 213.
- Volunteer for Physics Festival (April 13).

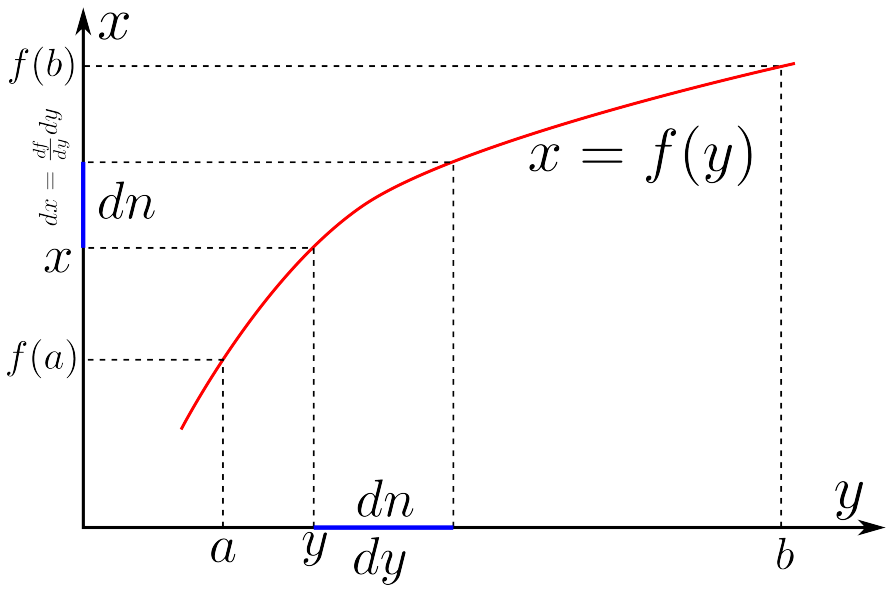
5.1. Change of variables in probability density.

Let's assume that we have a probability density $\rho_x(x)$ for some variable x. Let's assume that we want to use a variable y instead of x and that we know how to convert x to y, namely we know $x = f(y)$. The question is what will be the probability density for the variable y?

In order to answer this question we need to go back to the definition of the probability density. If we have a random process which gives us the numbers x, we run this process N times and we count how many times dn the outcome of the process is within a small interval dx at a point x. The probability density $\rho_x(x)$ is defined as

$$\lim_{N\to\infty} \frac{dn}{N} = \rho_x(x)dx.$$

The index x in ρ_x is here to remind us, that this function must be multiplied by dx in order to get the probability.



Case of monotonic $f(y)$.
Now we use the variable y instead. The point x is expressed through y as $x = f(y)$, the interval dx in variable y is expressed as $dx = \frac{\partial f}{\partial y} dy$. Now all outcomes withing our dx will be outcomes within dy, so

$$\lim_{N\to\infty} \frac{dn}{N} = \rho_x(f(y)) \left|\frac{\partial f}{\partial y}\right| dy.$$

- The absolute value is needed in the case when the derivative is negative in the case of monotonically decreasing function.
- In this case smaller x correspond to larger y, so when we change x from small to large y goes in the "wrong" direction, from large to small.
- The absolute value fixes these problems.

But by definition $\lim_{N\to\infty} \frac{dn}{N} = \rho_y(y)dy$, so

$$\rho_y(y) = \rho_x(f(y)) \left|\frac{\partial f}{\partial y}\right|.$$

Case of non-monotonic $f(y)$.
In this case there are at least two intervals dx which correspond to one dy, so one has to sum the probabilities in these intervals.

- Notice, that in any case the change of variables is not just a simple substitution of f in ρ_x.
- The probability density makes sense in calculations only inside the integrals. In fact this is integration measure. So when you change the variables you must change them as change of variables in integration measure.

5.2. Disintegration of many particles in a beam.

We want to consider the following problem:

- There is a beam of identical particles. We know the velocity of these particles and the density of the particles in the beam, or we know particle flux — how many particles crosses the unit area of a beam per unit of time.
- Initial particle has internal energy ϵ.
- These initial particles spontaneously disintegrate releasing their internal energy into two different particles: 1 and 2, which will have some velocities after the disintegration. The direction at which these particle fly is random.
- The disintegration happens inside of a detector. The detector can measure the kinetic energy of a particle 1. The detector cannot measure the direction of the velocity.

We want to know what is the kinetic energy probability distribution the detector will measure.

5.3. Disintegration of a particle.

We want to consider the following problem: A particle has an internal energy ϵ. At some point in time this energy is released and the particle disintegrates into two particles of masses m_1 and m_2. In the laboratory we can measure the direction and magnitude of the velocity of the particle 1. The initial particle has velocity V in the laboratory frame of references.

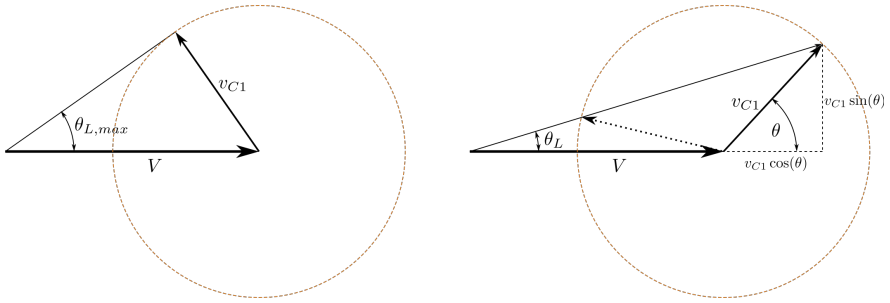


Figure 1. Illustrations to the equations (5.1) and (5.2).

- In the center of mass frame of reference the initial momentum is zero and the initial energy is ϵ, so the conservation laws give

$$m_1 v_{C1} + m_2 v_{C2} = 0, \qquad \frac{m_1 v_{C1}^2}{2} + \frac{m_2 v_{C2}^2}{2} = \epsilon,$$

$$\frac{m_1 v_{C1}^2}{2} + \frac{m_1^2 v_{C1}^2}{2m_2} = \frac{m_1 v_{C1}^2}{2}\left(1 + \frac{m_1}{m_2}\right) = \epsilon$$

  The direction of the velocity v_{C1} is arbitrary with the uniform distribution – this means that there is no control over the orientation of the initial particle.
- We are observing the process in the laboratory frame of references. In this frame the center of mass has a velocity V. In the laboratory frame of reference the velocity $\vec{v}_{L1}$ of the particle 1 is given by

$$\vec{v}_{L1} = \vec{V} + \vec{v}_{C1}.$$

- Kinematics show (see figure)
  - In the case $V > v_{C1}$ there is a maximum of the angle measured in the laboratory.

$$\sin(\theta_{L,max}) = \frac{v_{C1}}{V}, \quad \text{if} \quad V > v_{C1} \tag{5.1}$$

  - There is a relation between the angle θ in the center of mass frame with the angle θ_L in the laboratory frame.

$$\tan\theta_L = \frac{v_{C1}\sin(\theta)}{v_{C1}\cos(\theta) + V}, \tag{5.2}$$

  or if we measure θ_L — this is what we measure — then we know θ in the center of mass frame.

$$\cos\theta = -\frac{V}{v_{C1}}\sin^2(\theta_L) \pm \cos(\theta_L)\sqrt{1 - \frac{V^2}{v_{C1}^2}\sin^2(\theta_L)}.$$

  For $v_{C1} > V$ the result is one-to-one, we must take the + sign, so that $\theta(\theta_L = 0) = 0$. For $v_{C1} < V$ the result is not one-to-one: for a single θ_L in laboratory frame, there are two θs in the center of mass frame.

5.3.1. Disintegration of many particles.

- All detectors are on a sphere of a large radius R_d.
- We are watching only particle number 1.
- Our detector is very simple. It only measures the kinetic energy of particles 1.

- In each disintegration we know the speed v_{C1} if we know the internal energy of the initial particle ϵ, or vice verse
$$\frac{m_1 v_{C1}^2}{2}\left(1+\frac{m_1}{m_2}\right)=\epsilon.$$
The speed in the center of mass reference frame is the same for all particles number 1 which are the products of disintegrating initial particles.
- The direction of the vector $\vec{v}_{C1}$ is arbitrary. We assume that there is no preferential direction (the way the initial particle was set up) and any direction of $\vec{v}_{C1}$ is equally probable.
- In the center of mass reference frame the probability to find the particle 1 in the solid angle $d\Omega$ is uniform and is given by
$$dp=\frac{R^2 d\Omega}{4\pi R^2}=\frac{\sin(\theta)d\theta d\phi}{4\pi}$$
The angle θ is the angle between the vectors of the velocity of the center of mass $\vec{V}$ and velocity $\vec{v}_{C1}$ of the particle 1 *in the center of mass reference frame!!!!!*
- The probability density to find the velocity $\vec{v}_{C1}$ direction between the angle θ and $\theta+d\theta$ is (we do not care about the angle ϕ.)
$$dp=d\theta\int_0^{2\pi}\frac{\sin(\theta)d\phi}{4\pi}=\frac{1}{2}\sin(\theta)d\theta.$$
(One should check that the total is 1.)
- Now we have everything (see previous section) to find the distribution of particles over θ_L – the angle measured in laboratory frame of references, as we know the function $\theta(\theta_L)$.
- Instead of doing that we will find the distribution of the kinetic energy of the particles 1 in the laboratory frame of ref. (**Important note:** in the center of mass frame of reference the distribution of the kinetic energies is trivial, as all particles 1 have the same speed v_{C1}, so they have the same kinetic energy. However, in the laboratory frame of reference it is not so.)
- In the laboratory reference frame $\vec{v}_{L1}=\vec{V}+\vec{v}_{C1}$, so
$$v_{L1}^2=V^2+v_{C1}^2+2Vv_{C1}\cos(\theta).$$
where θ is the angle in the center of mass ref. frame.
- The kinetic energy of the particle 1 in the laboratory reference frame is
$$K=\frac{m_1 v_{L1}^2}{2}=\frac{m_1 V^2}{2}+\frac{m_1 v_{C1}^2}{2}+m_1 V v_{C1}\cos(\theta),$$
$$dK=-m_1 V v_{C1}\sin(\theta)d\theta.$$
or
$$\sin(\theta)d\theta=-\frac{1}{m_1 V v_{C1}}dK$$
- We can ignore the minus sign in the above (it only tells us that the particles with larger kinetic energy have smaller angle)
$$dp=\frac{1}{2}\sin\theta d\theta=\frac{1}{2m_1 V v_{C1}}dK.$$
The uniform distribution: $\rho_K=\frac{1}{2m_1 V v_{C1}}$

- The maximum kinetic energy is $K_{max} = \frac{m_1(V+v_{C1})^2}{2}$. The minimum is $K_{min} = \frac{m_1(V-v_{C1})^2}{2}$.
- One can (should) check that

$$\int_{K_{min}}^{K_{max}} \rho_K dK = 1$$

- One can measure $\rho(K)$ by simply measuring how many initial particles went to the detector and how many particles with kinetic energies between K and $K + dK$ the detector measures (dK is just a small enough interval of energies) during the same time, and taking the ratio of the two numbers. If $\rho_K(K)$ is uniform, then one knows that what happens is a simple particle disintegration. Measuring the width of the distribution, one finds v_{C1}. Then one can find ϵ — learn something about the initial particles!
- Also $\rho_K(K)$ measured by the procedure described above will not be normalized by 1, the total integral will be smaller, as not all the particles that went into the detector will necessarily disintegrate. Thus by measuring this discrepancy we will learn the probability of disintegration!

LECTURE 6
Scattering. Scattering cross-section.

6.1. General.

6.1.1. Experimental setup and measurements.

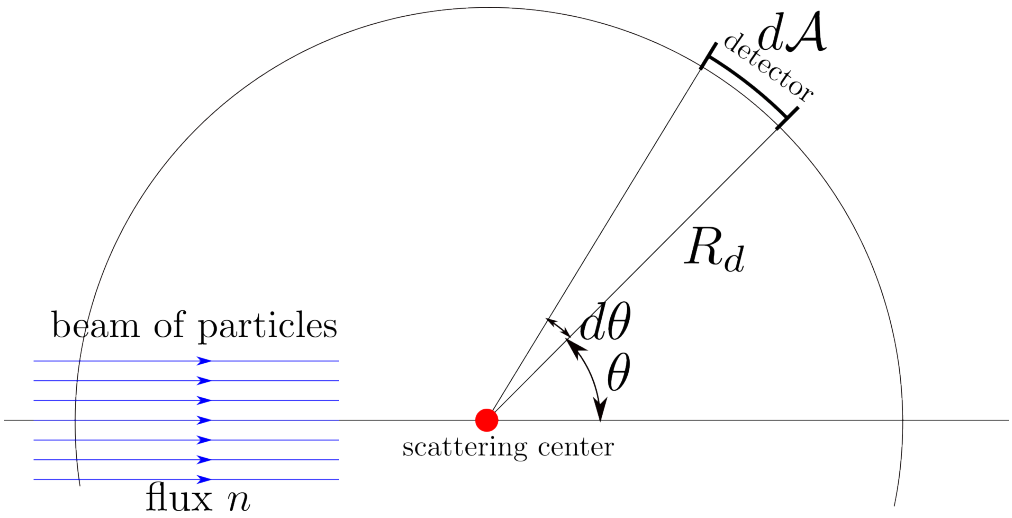


Figure 1. Left: Individual scattering event. Right: experimental setup.

- Set up of a scattering problem. Experiment, detectors, etc.
- Detectors are placed around the large sphere of radius R_d. Each detector has some (known) θ and ϕ and some (known) area $d\mathcal{A}$.
- The scattering center is at the center of the sphere.
- Flux of particle. Same energy, different impact parameters, different scattering angles.
- The scattering problem, n — the beam flux, number of particles per unit area per unit time in the beam. dN the number of particles scattered into a detector at the position θ, ϕ per unit time, or the number of particles scattered into the detector of solid angle $d\omega = d\mathcal{A}/R_d^2$ at angles θ and ϕ per unit time. A suitable quantity to describe the scattering

$$d\sigma = \frac{dN}{n}.$$

- Let's check the units

$$[dN] = \frac{\#}{\text{time}}, \quad [n] = \frac{\#}{\text{Area} \cdot \text{time}}, \quad [d\sigma] = \text{Area}.$$

It has the units of area and is called differential cross-section.

- This quantity $d\sigma$ is what we measure, by simply counting the number of particles captured by a detector at the position θ, ϕ. This quantity will be proportional to the "size" of the detector $d\omega$.

6.1.2. What does $d\sigma$ tell us?

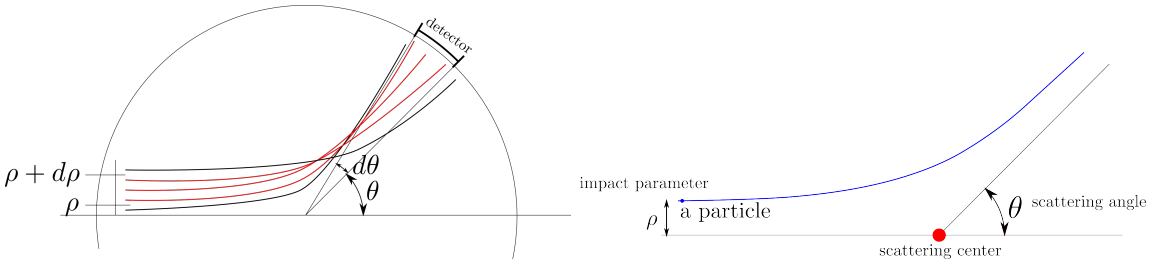


Figure 2. Illustration for the differential scattering cross-section and scattering angle.

We assume the axial symmetry of the problem. It means that nothing depends on the angle ϕ. It also means that it makes more sense to integrate the data from all the detectors with the same θ, but different ϕ (this way we collect more data). For such a case we can think that a single detector is a belt on the sphere between the angles θ and $\theta + d\theta$.

- All particles in the beam have the same velocity (kinetic energy E), but they have different impact parameters.
- A single particle in the beam has an impact parameter ρ. After interaction this particle is scattered at angle θ. This is a simple one particle problem and solving it gives (for the given energy E) the function $\theta(\rho)$, or inverting this function we get the function $\rho(\theta)$.
- If we know the function $\rho(\theta)$, then only the particles which are in between $\rho(\theta)$ and $\rho(\theta + d\theta)$ are scattered at the angle between θ and $\theta + d\theta$, see the red trajectories on the figure.
- The number of particles in the beam crossing the ring between ρ and $\rho + d\rho$ per unit time is $n2\pi\rho d\rho$. So $dN = n2\pi\rho d\rho$, or

$$d\sigma = 2\pi\rho d\rho = 2\pi\rho(\theta) \left| \frac{d\rho(\theta)}{d\theta} \right| d\theta$$

(The absolute value is needed because the derivative is usually negative.)

- Often $d\sigma$ refers not to the scattering between θ and $\theta + d\theta$. If one is interested in the scattering to the solid angle $d\omega = 2\pi \sin\theta d\theta$. Then

$$d\sigma = \frac{\rho}{\sin\theta} \left| \frac{d\rho}{d\theta} \right| d\omega.$$

- The strategy thus is the following: if we assume the interaction potential $U(r)$. We need to compute $\rho(\theta)$ for individual scattering event. Then we compute $d\sigma$. We then

can compare the computed $d\sigma$ with the measured one and draw a conclusion if our $U(r)$ is correct.

- In a certain subset of cases it is possible to take the measured $d\sigma$ and derive $U(r)$. This is called the inverse scattering problem.

6.2. Example: Scattering from a perfectly rigid sphere

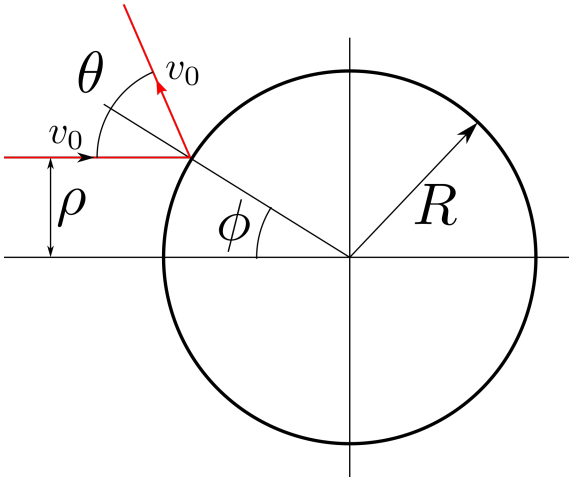


Figure 3. Scattering from a rigid sphere.

- Cross-section for scattering of particles from a perfectly rigid sphere of radius R.
  - The scattering angle $\theta = 2\phi$.
  - $R\sin\phi = \rho$, so $\rho = R\sin(\theta/2)$.
  - $$d\sigma = \frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| d\omega = \frac{1}{4}R^2 d\omega$$
  - Independent of the incoming energy. The scattering does not probe what is inside.
  - The total cross-section area is
$$\sigma = \int d\sigma = \frac{1}{4}R^2 2\pi \int_0^\pi \sin\theta d\theta = \pi R^2$$

6.3. Example: Scattering from a spherical potential well.

Here we compute cross-section for scattering of particles from a spherical potential well of depth U_0 and radius R.

$$U(r) = \begin{cases} 0, & \text{if } r > R \\ -U_0, & \text{if } r < R \end{cases}$$

At first we consider the situation $U_0 > 0$.

- Energy conservation

$$\frac{mv_0^2}{2} = \frac{mv^2}{2} - U_0, \qquad v = v_0\sqrt{1 + \frac{2U_0}{mv_0^2}} = v_0\sqrt{1 + U_0/E}$$

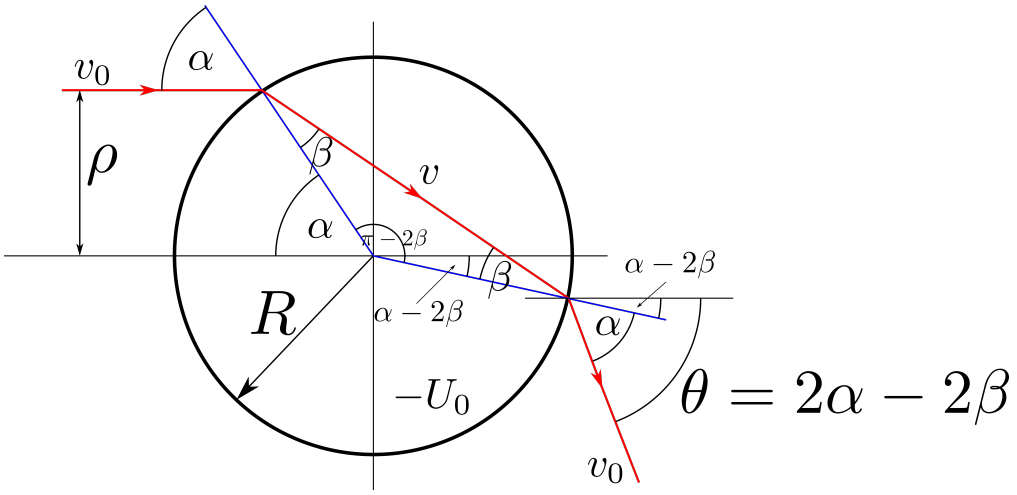


Figure 4. The scattering processes from a spherical square potential $-U_0$.

- Angular momentum conservation

$$v_0 \sin\alpha = v \sin\beta, \qquad \sin\alpha = n(E)\sin\beta, \qquad n(E) = \sqrt{1 + U_0/E} > 1$$

- Scattering angle (see figure)

$$\theta = 2(\alpha - \beta)$$

- Impact parameter

$$\rho = R\sin\alpha$$

- So we have

$$\frac{\rho}{R} = \sin(\alpha) = n\sin(\beta) = n\sin(\alpha - \theta/2) = n\sin\alpha\cos(\theta/2) - n\cos\alpha\sin(\theta/2)$$
$$= n\frac{\rho}{R}\cos(\theta/2) - n\sqrt{1 - \rho^2/R^2}\sin(\theta/2)$$

- And we have an equation connecting ρ and θ

$$\frac{\rho}{R} = n\frac{\rho}{R}\cos(\theta/2) - n\sqrt{1 - \rho^2/R^2}\sin(\theta/2).$$

- Solving this equation for ρ we get

$$\rho^2 = R^2\frac{n^2\sin^2(\theta/2)}{1 + n^2 - 2n\cos(\theta/2)}.$$

- The differential cross-section is

$$d\sigma = \frac{R^2n^2}{4\cos(\theta/2)}\frac{(n\cos(\theta/2) - 1)(n - \cos(\theta/2))}{(1 + n^2 - 2n\cos(\theta/2))^2}d\omega$$

- Differential cross-section depends on E/U_0, where E is the energy of incoming particles. By measuring this dependence we can find U_0 from the scattering.
- The scattering angle changes from 0 ($\rho = 0$) to θ_{max}, where $\cos(\theta_{max}/2) = 1/n$ (for $\rho = R$). The total cross-section is the integral

$$\sigma = \int_0^{\theta_{max}} d\sigma = \pi R^2.$$

It does not depend on energy or U_0. (If you want to check this result, and you should, then remember that $d\omega = 2\pi \sin\theta d\theta$.)

6.3.1. Negative U_0.

Here we consider a negative U_0 – this is not a well, but a bump.

- In this case
$$n = \sqrt{1 - |U_0|/E}$$
- We see, that n is imaginary for $E < U_0$. This is a rigid sphere — the particle cannot get into the potential.
- For $E > U_0$, n is real and $n < 1$, so there is no solution of equation $\sin(\alpha) = n\sin(\beta)$ for $\alpha > \alpha_{cr} = \sin^{-1}(n)$, or for $\rho > \rho_{cr} = R\sin(\alpha_{cr}) = nR$.
- So for ρ in the range $nR < \rho < R$ the particle does not penetrate inside the potential (total internal reflection).
- In this range of impact parameters we will have a reflection from a "rigid" sphere.

6.4. Arbitrary central potential

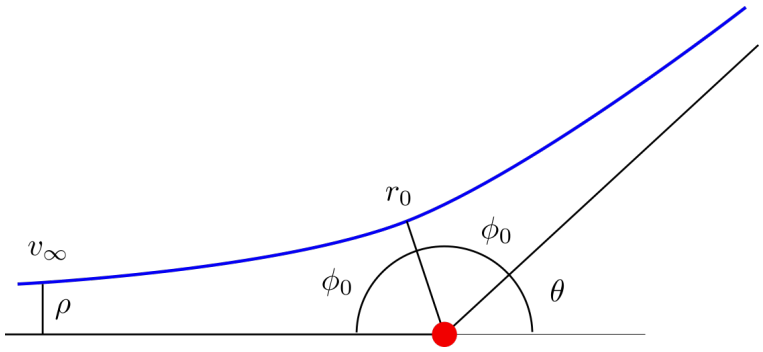


Figure 5. Scattering in central potential.

Consider the scattering of a particle of initial velocity v_∞ from the central force given by the potential energy $U(r)$.

- From the initial state — a particle in the beam
  - The energy is
$$E = \frac{mv_\infty^2}{2}.$$
  - The angular momentum is given by
$$L_\phi = mv_\infty\rho = \rho\sqrt{2mE},$$
where ρ is the impact parameter.
- The trajectory is given by
$$\pm(\phi - \phi_0) = \frac{L_\phi}{\sqrt{2m}}\int_{r_0}^{r}\frac{1}{r^2}\frac{dr}{\sqrt{E - U_{eff}(r)}}, \qquad U_{eff}(r) = U(r) + \frac{L_\phi^2}{2mr^2}$$

where r_0 and ϕ_0 are some distance and angle on the trajectory.

At some point the particle is at the closest distance r_0 to the center. The angle at this point is ϕ_0 (the angle at the initial infinity is zero.) Let's find the distance r_0. As the energy and the angular momentum are conserved and at the closest point the velocity is perpendicular to the radius we have

$$E = \frac{mv_0^2}{2} + U(r_0), \qquad L_\phi = mr_0v_0.$$

so we find that the equation for r_0 is

$$U_{eff}(r_0) = E.$$

This is, of course, obvious from the picture of motion in the central field as a one dimensional motion in the effective potential $U_{eff}(r)$.

The angle ϕ_0 is then given by

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}. \tag{6.1}$$

From geometry the scattering angle θ is given by the relation

$$\theta + 2\phi_0 = \pi. \tag{6.2}$$

(For attractive potential it might be more convenient to define $\theta = 2\phi_0 - \pi$.)

So we see, that for a fixed v_0 the energy E is given, but the angular momentum L_ϕ depends on the impact parameter ρ. The equation (6.1) then gives the dependence of ϕ_0 on ρ. Then the equation (6.2) gives the dependence of the scattering angle θ on the impact parameter ρ. If we know that dependence, we can calculate the scattering cross-section.

$$d\sigma = \frac{\rho}{\sin\theta} \left| \frac{d\rho}{d\theta} \right| d\omega$$

LECTURE 7
Rutherford's formula.

Historical facts:

- 1897, J. J. Thomson's measurement of e/m. Cathode ray deflected by magnetic field experiment established that electricity is carried by particles.
- 1899, Rutherford (in McGill University, Canada) and Paul Villard (working in Paris) separated radiation into three types: eventually named alpha, beta, and gamma by Rutherford.
- 1904, Thomson model of the atom.
- 1905, Einstein's photo-electric explanation: Light is a particle.
- 1909, Robert Millikan measured e, oil drop experiment.
- 1911, Bohr–Van Leeuwen theorem. Magnetism is impossible in classical mechanics.
- 1911, Rutherford's experiment, α particles scattered of a gold film, experiment established the structure of an atom (experiment was done in 1909).
- 1913, Bohr's model of the atom.
- 1924, Louis de Broglie, electron is a wave.
- 1926, Schrödinger equation.

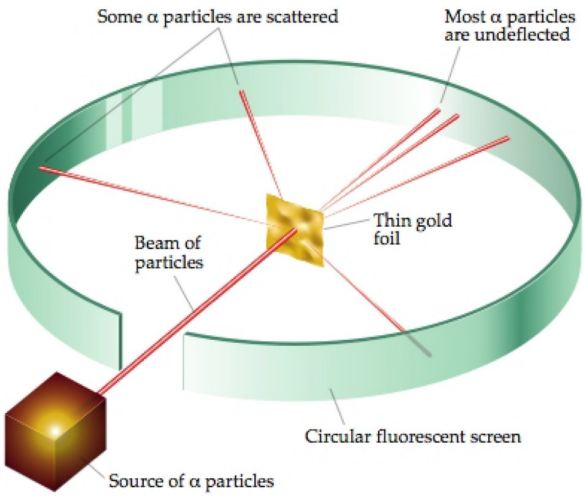


Figure 1. Left: Rutherford experiment set up. Right: Scattering in Coulomb potential.

7.1. Rutherford experiment.

- What is the question?
- Experiment set up.
- Expected result from Thomson model.
  - If the pudding is soft, then there should not be much of the back scattering.
  - If the pudding is hard, then according to our result for the hard sphere $d\sigma = \frac{1}{4}R^2 d\omega$ there should be not dependence on energy E.
- Obtained result.
  - There is a considerable back scattering.
  - The scattering strongly depends on the energy of the incoming particles.

7.2. Coulomb potential.

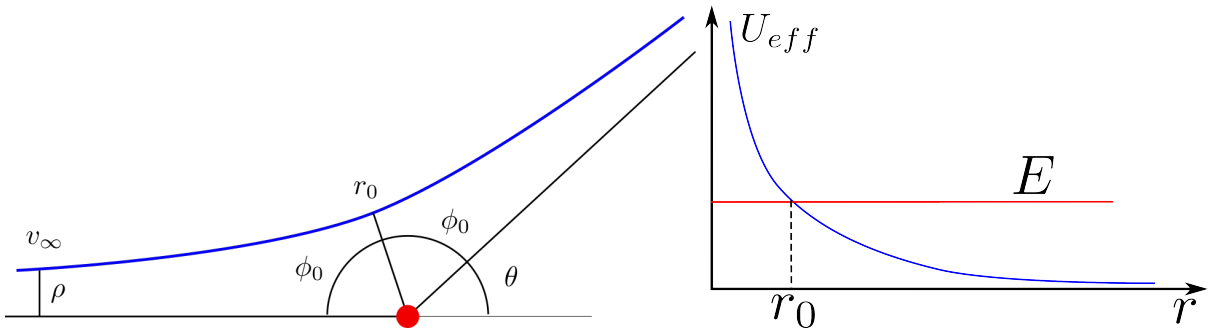


Figure 2. Left: Scattering in Coulomb potential. Right: sketch of $U_{eff}(r)$.

Let's say that we have a repulsive Coulomb interaction

$$U = \frac{\alpha}{r}, \qquad \alpha > 0$$

In this case the geometry gives

$$\theta = \pi - 2\phi_0.$$

In order to compute the differential cross-section we need the function $\rho(\theta)$, so we start by computing ϕ_0 as a function of ρ for a given energy E.

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}}, \qquad E = \frac{mv_\infty^2}{2}, \qquad L_\phi = mv_\infty \rho = \rho\sqrt{2mE},$$

where r_0 is the value of r, where the expression under the square root is zero.

Let's take the integral

$$\int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E - \alpha x - x^2 \frac{L_\phi^2}{2m}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E + \frac{\alpha^2 m}{2L_\phi^2} - \frac{L_\phi^2}{2m}(x + \frac{\alpha m}{L_\phi^2})^2}}$$

$$= \sqrt{\frac{2m}{L_\phi^2}} \int_0^{1/r_0} \frac{dx}{\sqrt{\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4} - (x + \frac{\alpha m}{L_\phi^2})^2}}$$

changing $\sqrt{\frac{2mE}{L_\phi^2}+\frac{\alpha^2m^2}{L_\phi^4}}\sin\psi = x+\frac{\alpha m}{L_\phi^2}$ we find that the integral is

$$\sqrt{\frac{2m}{L_\phi^2}}\int_{\psi_1}^{\pi/2}\frac{\cos\psi}{\cos\psi}d\psi = \sqrt{\frac{2m}{L_\phi^2}}\int_{\psi_1}^{\pi/2}d\psi,$$

where $\sin(\psi_1)=\frac{\alpha m}{L_\phi^2}\left(\frac{2mE}{L_\phi^2}+\frac{\alpha^2m^2}{L_\phi^4}\right)^{-1/2}$ So we find that

$$\phi_0 = \pi/2-\psi_1,$$

or remembering, that $\phi_0=\pi/2-\theta/2$ and hence $\theta/2=\psi_1$ we get

$$\sin\frac{\theta}{2}=\sin\psi_1=\frac{\alpha m}{L_\phi^2}\left(\frac{2mE}{L_\phi^2}+\frac{\alpha^2m^2}{L_\phi^4}\right)^{-1/2}.$$

Using $L_\phi=\rho\sqrt{2mE}$ this gives

$$\sin\frac{\theta}{2}=\frac{\alpha}{2E}\left(\rho^2+\frac{\alpha^2}{4E^2}\right)^{-1/2},$$

or

$$\rho^2=\frac{\alpha^2}{4E^2}\cot^2\frac{\theta}{2}.$$

This we have the function $\rho(\theta)$.

The differential cross-section then is

$$d\sigma=\frac{d\rho^2}{d\theta}\frac{1}{2\sin\theta}d\omega=\left(\frac{\alpha}{4E}\right)^2\frac{1}{\sin^4(\theta/2)}d\omega.$$

or using $d\omega=2\pi\sin\theta d\theta$

$$d\sigma=4\pi\left(\frac{\alpha}{4E}\right)^2\frac{\cos(\theta/2)}{\sin^3(\theta/2)}d\theta.$$

7.2.1. Rutherford formula analysis.

- Notice, that the differential cross-section diverges at small scattering angles as $1/\theta^4$.
- Not only it diverges, but the integral $\sim\int\frac{d\theta}{\theta^3}$ – the total scattering cross-section — also diverges at small angles.
- This means that the total number of particles our detectors collect in a unit of time is infinite.
- This is obviously un-physical, as the total number of particles entering our device per unit time is finite.
- So the divergence must be cut off. In the ideal situation it is cutoff simply by the finite width of the beam. The finite width means, that the impact parameter ρ cannot be larger than the radius of the beam $\rho(\theta)<R_{\text{beam}}$. So the scattering angle cannot be smaller than a certain value.
- Such divergence can be traced to the fact, that the Coulomb interaction is a long-range interaction.

7.3. Rutherford experiment analysis.

Rutherford's formula.

$$d\sigma = \frac{d\rho^2}{d\theta}\frac{1}{2\sin\theta}d\omega = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}d\omega$$

- The beam. How do you characterize it?
- What is measured?
- The statistics. How much data we need to collect to get certainty of our results?
- The beam again. Interactions.
- Final state interactions.
- The forward scattering diverges.
- The cut off of the divergence is given by the size of the atom.
- Back scattering. Almost no dependence on θ.
- Energy dependence $1/E^2$.
- Contrast this to the Thomson model:
  - If the pudding is soft, then there should not be much of the back scattering.
  - If the pudding is hard, then according to our result for the hard sphere there should be not dependence on energy E.
- Plot $d\sigma$ at any angle as a function of $1/(4E)^2$, expect a straight line at large $1/(4E)^2$.
- The slope of the line gives α^2.
- What is the behavior at very large E? What is the crossing point?
- The crossing point tells us the size of the nucleus $d\sigma = \frac{R^2}{4}d\omega$. (Rutherford's experiment was not accurate enough to determine the radius of the nucleus, but it was accurate enough to show that this radius is by far less than the size of the atom.)

LECTURE 8
Oscillations. Many degrees of freedom.

8.1. Small oscillations. One degree of freedom.

Problem with one degree of freedom: $U(x)$. The Lagrangian is

$$L = \frac{m\dot{x}^2}{2} - U(x).$$

The equation of motion is

$$m\ddot{x} = -\frac{\partial U}{\partial x}$$

If the function $U(x)$ has an extremum at $x = x_0$, then $\frac{\partial U}{\partial x}\Big|_{x=x_0} = 0$.

- Then $x(t) = x_0$ is a (time independent) solution of the equation of motion. But it will not satisfy arbitrary initial conditions.

Consider a small deviation from the solution $x(t) = x_0 + \delta x(t)$. We assume that $\delta x(t)$ stays small at all times during the motion. So the particle is always in the neighborhood of x_0. It means that we only need to know the behavior of the function $U(x)$ in the small neighborhood of x_0. Using the Taylor expansion we write.

$$U(x) = U(x_0 + \delta x) \approx U(x_0) + U'(x_0)\delta x + \frac{1}{2}U''(x_0)\delta x^2 = U(x_0) + \frac{1}{2}U''(x_0)\delta x^2.$$

- Notice, that we kept only the first non-trivial term is the Taylor expansion.
- There are higher order terms — terms with higher powers of δx — but we ignore them, as for small enough δx these terms are small in comparison to the term we kept.

The equation of motion becomes

$$m\ddot{\delta x} = -U''(x_0)\delta x$$

- If $U''(x_0) > 0$, then we have small oscillations with the frequency

$$\omega^2 = \frac{U''(x_0)}{m}$$

  This is a stable equilibrium.
- If $U''(x_0) < 0$, then the solution grows exponentially, and at some point our approximation becomes invalid. The equilibrium is unstable.

Look at what it means graphically.

8.1.1. Noise and dissipation.

Generality: consider a system with infinitesimally small dissipation and external perturbations. The perturbations will kick it out of any unstable equilibrium. The dissipation will bring it down to a stable equilibrium. It may take a very long time.

After that the response of the system to small enough perturbations will be defined by the small oscillations around the equilibrium

8.2. Examples in $1D$ for one degree of freedom.

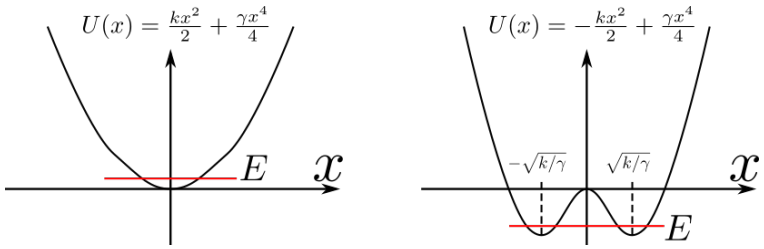


Figure 1. Examples of $1D$ potentials.

8.2.1. $U(x) = \frac{kx^2}{2} + \frac{\gamma x^4}{4}$, where $k > 0$ and $\gamma > 0$.

- First, we find the equilibrium positions. The equation is
$$0 = \frac{\partial U}{\partial x} = x(k + \gamma x^2).$$
for $\gamma > 0$, $k > 0$ there is only one real solution: $x = 0$.
- We need to compute
$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=0} = k > 0.$$
- The frequency of the small oscillations is
$$\omega^2 = \frac{1}{m}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=0} = k/m.$$
- Notice, that the frequency does not depend on γ. It depends only on quadratic part of the potential.

8.2.2. $U(x) = -\frac{kx^2}{2} + \frac{\gamma x^4}{4}$, where $k > 0$ and $\gamma > 0$.

- First, we find the equilibrium positions. The equation is
$$0 = \frac{\partial U}{\partial x} = x(-k + \gamma x^2)$$
for $\gamma > 0$, $k > 0$ there are three real solution: $x = 0$, $x_0 = \pm\sqrt{k/\gamma}$.
- We need to compute
$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=0} = -k < 0, \qquad \left.\frac{\partial^2 U}{\partial x^2}\right|_{x=\pm\sqrt{k/\gamma}} = 2k > 0.$$

So the equilibrium point $x = 0$ is unstable. The two equilibrium points $x = \pm\sqrt{k/\gamma}$ are stable. The small oscillations are only possible around the stable points. So we have

$$\omega^2 = \frac{1}{m}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=\pm\sqrt{k/\gamma}} = 2k/m.$$

Look at what it means graphically.

8.3. Full solution in $1D$ for one degree of freedom.

The equation

$$\ddot{x} = -\omega^2 x, \qquad \omega^2 = U''(x_0)/m > 0, \qquad U'(x_0) = 0$$

General solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t + \phi) = \Re\tilde{C}e^{i\omega t}, \qquad \tilde{C} = Ce^{i\phi}.$$

The two arbitrary constants A and B, or C and ϕ, or one complex $\tilde{C}$ must be found from the initial conditions, say, at $t = t_0$ the position was x_0 and the velocity was v_0, then

$$x(t = t_0) = x_0, \qquad \dot{x}(t = t_0) = v_0$$

are two linear equations for the two unknown (arbitrary) constants in the general solution.

8.4. Special $\omega = 0$ case.

- In the case $\omega = 0$, the equation becomes

  $$\ddot{x} = 0, \qquad x(0) = x_0, \quad \dot{x}(0) = v_0$$

  with an obvious solution

  $$x(t) = x_0 + v_0 t.$$

- Let's obtain the same result from the general solution

  $$x(t) = A\cos(\omega t) + B\sin(\omega t).$$

- If one naively plugs $\omega = 0$ in it, one gets $x(t) = A$, which is obviously incorrect. It is physically incorrect, but also mathematically, as a solution of a second order differential equation must depend on 2 arbitrary constant, while here we have only one.
- To do it correctly and to see where the problem is, one has to first satisfy the initial conditions and write

  $$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t).$$

- Now we see, that we cannot simply plug $\omega = 0$, as we cannot divide by zero. Instead we must take a limit $\omega \to 0$ at fixed time t.
- Then the argument of the sin function is small. Using $\sin(\omega t) \sim \omega t$, we restore the right answer.

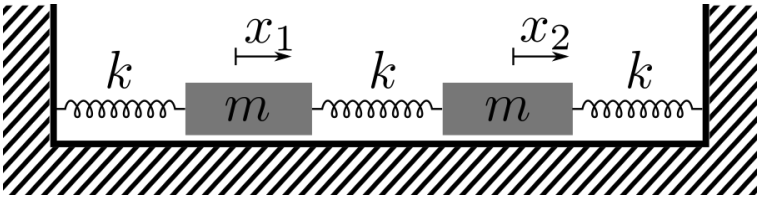


Figure 2. Two blocks.

8.5. Many degrees of freedom.

Consider two blocks of equal masses in $1D$ connected by springs of constant k to each other and to the walls.

There are two coordinates: x_1 and x_2.

What do we expect? There are two modes $x_1 - x_2$ and $x_1 + x_2$.

The potential energy of the system is

$$U(x_1, x_2) = \frac{kx_1^2}{2} + \frac{k(x_1 - x_2)^2}{2} + \frac{kx_2^2}{2}$$

The Lagrangian

$$L = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{kx_1^2}{2} - \frac{k(x_1 - x_2)^2}{2} - \frac{kx_2^2}{2}$$

The equations of motion are

$$m\ddot{x}_1 = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -2kx_2 + kx_1$$

- These are two linear second order differential equations.
- They must have four linearly independent solutions.
- The general solution must depend on four arbitrary constants.
- These constants must be found from the initial conditions.

Let's look for the solutions in the form

$$x_1 = A_1 e^{i\omega t}, \qquad x_2 = A_2 e^{i\omega t}$$

(These functions are complex. We need to take the real parts of these function, but as our equations are **linear with real coefficients**, we can first solve them with complex functions and only after that take the real parts.) then

$$-\omega^2 m A_1 = -2kA_1 + kA_2$$
$$-\omega^2 m A_2 = -2kA_2 + kA_1$$

Notice, that $e^{i\omega t}$ drops out (because ω is the same for x_1 and x_2) so instead of two differential equations we now have to linear algebraic equations. Let's write them in the form:

$$(2k - m\omega^2)A_1 - kA_2 = 0$$
$$-kA_1 + (2k - m\omega^2)A_2 = 0$$

or

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

There is always a trivial solution $A_1 = A_2 = 0$. In order for this set of equations to have a non trivial solution we must have

$$\det \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} = 0, \qquad (2k - m\omega^2)^2 - k^2 = 0, \qquad (k - m\omega^2)(3k - m\omega^2) = 0$$

There are two modes with the frequencies

$$\omega_a^2 = k/m, \qquad \omega_b^2 = 3k/m$$

and corresponding eigen vectors

$$\begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix} = A^a e^{i\phi_a} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix} = A^b e^{i\phi_b} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where A^a, A^b, ϕ_a, ϕ_b are arbitrary **real** numbers.

The general solution is the linear combination of the two solutions and is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Re \left[A^a e^{i(\omega_a t + \phi_a)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A^b e^{i(\omega_b t + \phi_b)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

or

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_a t + \phi_a) + A^b \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_b t + \phi_b)$$

- The equations of the form as above are to be read line-by-line. Here, this is simply the shorthand for two equations

$$x_1(t) = A^a \cos(\omega_a t + \phi_a) + A^b \cos(\omega_b t + \phi_b)$$
$$x_2(t) = A^a \cos(\omega_a t + \phi_a) - A^b \cos(\omega_b t + \phi_b)$$

- There are four arbitrary constants A^a, A^b, ϕ_a, ϕ_b which must be obtained from the four initial conditions — initial coordinate and initial velocity of each block.
- Picture of the eigen modes $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- Symmetry.

What will happen if the masses and springs constants are different?

Repeat the previous calculation for arbitrary m_1, m_2, k_1, k_2, k_3.

The Lagrangian

$$L = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} - \frac{k_1 x_1^2}{2} - \frac{k_2 (x_1 - x_2)^2}{2} - \frac{k_3 x_2^2}{2}.$$

The equations of motion

$$m_1 \ddot{x}_1 = -(k_1 + k_2) x_1 + k_2 x_2$$
$$m_2 \ddot{x}_2 = k_2 x_1 - (k_3 + k_2) x_2.$$

We look for the solution in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{i\omega t} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

The equations become

$$\begin{pmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

There is a non-trivial solution iff

$$\det \begin{pmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 \end{pmatrix} = 0.$$

- As it is the determinant of a 2×2 matrix with ω^2 in the diagonal elements, the equation is a quadratic equation for ω^2.
- It will have two solutions for ω^2 (it may be degenerate).
- Correspondingly it will have two eigen vectors.

$$\omega^2 = \omega_a^2 \longrightarrow \begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix}, \qquad \omega^2 = \omega_b^2 \longrightarrow \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix}.$$

The general solution then is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix} \cos(\omega_a t + \phi_a) + b \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix} \cos(\omega_b t + \phi_b).$$

The four constants a, ϕ_a, b, and ϕ_b must be found from the initial conditions.

Notice the following:

- We had 2×2 matrix, because we had 2 degrees of freedom. If we had N degrees of freedom the matrix would have been $N \times N$.
- We had quadratic equation for ω^2 because we had 2×2 matrix for 2 degrees of freedom. If we have N degrees of freedom the matrix is $N \times N$ and the equation for ω^2 is N-degree equation for ω^2.
- Although in general such an equation has N complex roots, it will be proven later that if you are in a minimum of the potential energy all roots for ω^2 will be real and positive.
- So for N degrees of freedom we will have N solutions for ω^2 (some may coincide).
- For each root there will be a mode! even if the roots coincide the corresponding modes will be different.
- N degrees of freedom — N modes!

IMPORTANT!!!!

- Each eigen frequency has a corresponding eigen mode! The frequency and the mode come together.
- The number of eigen modes EQUALS to the number of degrees of freedom! Some of the eigen frequencies can coincide (degenerate case) but the eigen modes will be different!

General scheme.

LECTURE 9
Oscillations. Many degrees of freedom.

9.1. Examples.

9.1.1. Two masses, splitting of symmetric and anitsymmetric modes.

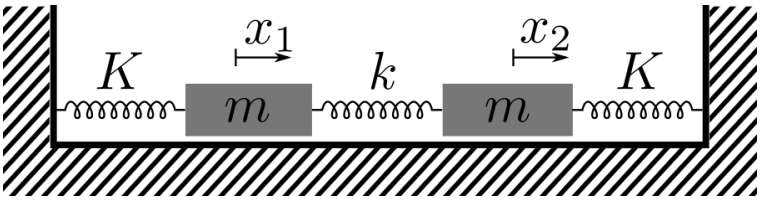


Figure 1. Two blocks.

This is the problem of two identical masses m connected with each other with the spring k and to the walls with the springs K.

Notice, that the problem is symmetric with respect left↔right. As it has only two degrees of freedom with experience one can guess right away, that the normal modes will be

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

After that one can compute the frequencies of these modes.

$$\omega_-^2 = \frac{K+2k}{m}, \qquad \text{and} \qquad \omega_+^2 = \frac{K}{m}.$$

However we want to solve this problem without any guesses.

In order to describe the system we need to coordinates x_1 and x_2.

- We have two degrees of freedom. We will have two normal modes.

The Lagrangian is almost the same as we have derived before

$$L = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{Kx_1^2}{2} - \frac{k(x_1-x_2)^2}{2} - \frac{Kx_2^2}{2}$$

The equations of motion are

$$m\ddot{x}_1 = -(K+k)x_1 + kx_2$$
$$m\ddot{x}_2 = -(K+k)x_2 + kx_1$$

Looking for a solution in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t},$$

we get

$$\begin{pmatrix} -\omega^2 m + K + k & -k \\ -k & -\omega^2 m + K + k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

The nontrivial solution exists if and only if the determinant of the matrix is zero.

$$\det \begin{pmatrix} \omega^2 m - K - k & k \\ k & \omega^2 m - K - k \end{pmatrix} = 0.$$

To find the eigen/normal modes we need to find such ω^2 at which the determinant of this matrix is zero.

Notice that

- If the masses were not interacting $k = 0$, then the matrix would be already diagonal and we would have two degenerate eigen frequencies $\omega^2 = K/m$.
- The interaction between the masses (degrees of freedom) appears as the off-diagonal matrix element k.
- This off-diagonal matrix element will split the degeneracy of the two modes.

The equation is

$$\left(\omega^2 m - K - k\right)^2 - k^2 = 0,$$

with the solutions

$$\omega_-^2 = \frac{K + 2k}{m}, \quad A_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \omega_+^2 = \frac{K}{m}, \quad A_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These are symmetric and antisymmetric modes. It is a very general situation for a symmetric system, that the interaction splits the degeneracy and the new modes are symmetric/antisymmetric pair.

The full general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\omega_- t + \phi_-\right) + a_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\omega_+ t + \phi_+\right)$$

Four constants $a_\pm$ and $\phi_\pm$ must be calculated from initial conditions.

9.1.2. Zero mode, or $\omega = 0$.

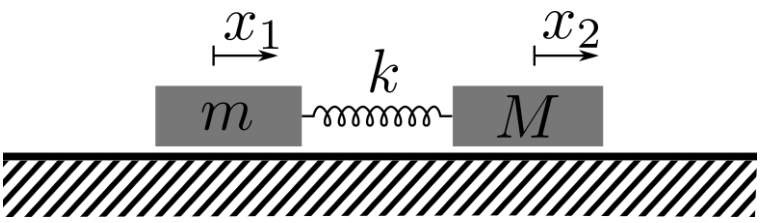


Figure 2. Two blocks, zero mode.

A rail and two objects with masses m and M connected by a spring k.

In order to describe the system we need to coordinates x_1 and x_2.

- We have two degrees of freedom. We will have two normal modes.

The Lagrangian is

$$L = \frac{m\dot{x}_1^2}{2} + \frac{M\dot{x}_2^2}{2} - \frac{k(x_1 - x_2)^2}{2}.$$

Notice, that the Lagrangian depends only on $x_1 - x_2$ and velocities.

The equations of motion are

$$m\ddot{x}_1 = -k(x_1 - x_2)$$
$$M\ddot{x}_2 = -k(x_2 - x_1)$$

Looking for a solution in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t},$$

then

$$\begin{pmatrix} -\omega^2 m + k & -k \\ -k & -\omega^2 M + k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

So the determinant equal zero requirement gives

$$\omega^4 mM - k(m + M)\omega^2 = 0.$$

Two solutions

$$\omega^2 = 0, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \omega^2 = k\frac{m + M}{mM}, \quad \begin{pmatrix} M \\ -m \end{pmatrix}.$$

The full general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (v_0 t + x_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} M \\ -m \end{pmatrix} \cos(\omega t + \phi)$$

- Role of symmetry for zero mode.
  - As the Lagrangian depends only on $x_1 - x_2$ (and the velocities), the Lagrangian will not change if we add the same constant ϵ to both x_1 and x_2.
  - This is the symmetry of the problem.
  - It means that if we find a solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ which minimizes the Action, then the solution $\begin{pmatrix} x_1(t) + \epsilon \\ x_2(t) + \epsilon \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will also minimize the Action (but it may not satisfy the initial conditions)
  - Notice, that ϵ multiplies the mode that corresponds to $\omega = 0$.
  - This is a very general situation.

9.1.3. Mode disappearing.

A rail a wall and to objects with masses m and M connected by a spring k with each other and by the spring k through mass m with the wall. We want to consider two cases $m = 0$ and $m \to 0$.

The Lagrangian is

$$L = \frac{m\dot{x}_1^2}{2} + \frac{M\dot{x}_2^2}{2} - \frac{kx_1^2}{2} - \frac{k(x_1 - x_2)^2}{2}$$

Notice, that now the Lagrangian depends not only on the combination $x_1 - x_2$, but also on x_1. This breaks the symmetry discussed before. It means that we do not expect $\omega = 0$ mode.

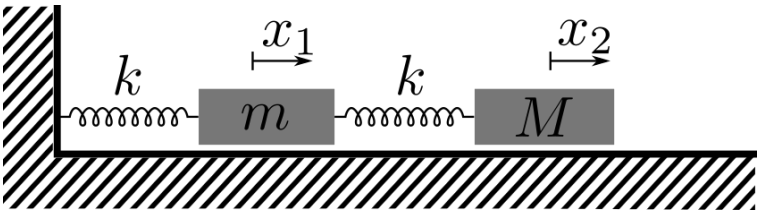


Figure 3. Two blocks, mode disappearing.

The equations of motion are

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$
$$M\ddot{x}_2 = -k(x_2 - x_1)$$

Looking for a solution in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t},$$

then

$$\begin{pmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 M + k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

So the determinant equal zero requirement gives

$$\omega^4 mM - k(m + 2M)\omega^2 + k^2 = 0.$$

- Notice, that if we put $m = 0$ directly in the equation we will get a single solution $\omega^2 = k/2M$.
- This happens because m multiplies ω^4, so when $m = 0$ we get a linear in ω^2 equation instead a quadratic.
- However, for any, no matter how small $m \neq 0$ the equation is quadratic in ω^2, so we must have two solutions — two modes! What happens to the second mode as $m \to 0$?

The two solutions are

$$\omega^2 = k\frac{2M + m \pm \sqrt{4M^2 + m^2}}{2mM}$$

For $m \ll M$, we need to keep only linear in m terms in the numerator

$$\omega^2 = k\frac{m + 2M \pm 2M}{2mM} = \begin{cases} \frac{2k}{m}, & \text{for the "+" sign} \\ \frac{k}{2M}, & \text{for the "−" sign} \end{cases},$$

or two normal frequencies

$$\omega_-^2 = \frac{k}{2M}, \qquad \omega_+^2 = \frac{2k}{m} \to \infty.$$

Physical picture:

- Mode ω_-: when m is very small, we have two springs in series (spring constant $k/2$) acting on mass M.
- Mode ω_+: when m is very small, M almost does not move we have two springs in parallel (spring constant $2k$) acting on mass m.
- When $m \to 0$ acceleration goes to infinity, so does the corresponding frequency.

LECTURE 10
Small oscillations. General solution.

Notice, in all examples we considered so far we were doing the same thing again and again:

(a) Wrote the Lagrangian.
(b) Derived the equations of motion.
(c) Looked for a solution in a particular form.
(d) Derived a system of homogeneous linear equations.
(e) Demanded, that the determinant were zero and found the frequencies of the normal modes.
(f) Found the modes.
(g) Wrote the general solution.

The procedure is not difficult, but has several drawbacks.

- Notice the absence of a very important step: finding the stable equilibrium!
- Also in the general situation the equations of motion are NOT linear. The step of linearizing the equations is missing.
- The procedure is simple enough for a few degrees of freedom, but when there are many is becomes to cumbersome with many unneeded steps.

In this lecture we want to fix these problems.

10.1. General situation.

Let's consider a general situation in detail. We have N degrees of freedom which are described be a set of N generalized coordinates $\{q_i\}$.

- Please, keep in mind this number N — the number of degrees of freedom, or the number of generalized coordinates we need to describe the configuration of our system.

We start from an arbitrary Lagrangian

$$L = K(\{\dot{q}_i\}, \{q_i\}) - U(\{q_i\})$$

Very generally the kinetic energy is zero if all velocities are zero. It will also increase if any of the velocities increase. We assume, that the function $U(\{q_i\})$ has a minimum. Then this minimum is a stable equilibrium

- We expect harmonic oscillations around the stable equilibrium.
- The harmonic oscillator equations are linear.

- In order to have linear equations we need the Lagrangian only up to the second order in both velocities and coordinate **shifts** from the equilibrium position.

First we need to find the coordinates $q_i = q_{i0}$ of the minimum of the potential energy.

$$\left.\frac{\partial U}{\partial q_i}\right|_{\{q_j\}=\{q_{j0}\}} = 0$$

These are N (generally nonlinear) **coupled** equations for the equilibrium position $\{q_{i0}\}$.

Let's first change the definition of the coordinates $x_i = q_i - q_{0i}$. We rewrite the Lagrangian in these new coordinates.

$$L = K(\{\dot{x}_i\}, \{x_i\}) - U(\{x_i\})$$

We can take the potential energy to be zero at $\{x_i\} = 0$, also as $\{x_i\} = 0$ is a minimum we must have $\partial U/\partial x_i|_{\{x_j\}=0} = 0$.

Let's now assume, that the motion has very small amplitude. We then can use Taylor expansion in both $\{\dot{x}_i\}$ and $\{x_i\}$ up to the second order.

The time reversal invariance demands that only even powers of velocities can be present in the expansion. Also as the kinetic energy is zero if all velocities are zero, we have $K(0, \{x_i\}) = 0$, so we have

$$K(\{\dot{x}_i\}, \{x_i\}) \approx \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 K}{\partial \dot{x}_i \partial \dot{x}_j}\right|_{\dot{x}=0, x=0} \dot{x}_i \dot{x}_j = \frac{1}{2} k_{ij} \dot{x}_i \dot{x}_j,$$

(in the last expression the Einstein notations are assumed) where the constant $N \times N$ matrix k_{ij} is symmetric and positive definite. (I use the small letter k_{ij} to denote it to remind that it originates from the kinetic energy. Physically, it is more as a mass matrix.)

For the potential energy we have

$$U(\{x_i\}) \approx \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 U}{\partial x_i \partial x_j}\right|_{x=0} x_i x_j = \frac{1}{2} u_{ij} x_i x_j,$$

(in the last expression the Einstein notations are assumed) where the constant $N \times N$ matrix u_{ij} is symmetric. If $x = 0$ is indeed a minimum, then the matrix u_{ij} is also symmetric and positive definite. (I use the small letter u_{ij} to denote it to remind that it originates from the potential energy. Physically, it is more as a spring constant matrix.)

The Lagrangian is then

$$L = \frac{1}{2} k_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} u_{ij} x_i x_j$$

where k_{ij} and u_{ij} are just constant matrices and summation over the repeated indexes i and j is assumed. The Lagrange equations are

$$k_{ij} \ddot{x}_j = -u_{ij} x_j,$$

where the summation over the repeated index j on both sides is assumed.

- This is a system of N second order linear differential equations.

We are looking for the (non-trivial) solutions in the form

$$x_j^a = A_j^a e^{i\omega_a t}.$$

On notations

- We expect N normal modes, where N is the number of degrees of freedom.
- The letter a denotes/labels the eigen/normal modes: $a = 1, \ldots, N$.
- Each mode has its own frequency ω_a and its own vector A_j^a.

Substituting the solution in this form in the equations of motion we get:

$$\left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a = 0, \tag{10.1}$$

where the summation over the index j is assumed.

In order for this linear equation to have a nontrivial solution we must have

$$\det\left(\omega_a^2 k_{ij} - u_{ij}\right) = 0$$

After solving this equation we can find all N of eigen/normal frequencies ω_a and the eigen/normal modes of the small oscillations A_i^a.

- The number of normal (eigen) modes must be equal to the number of degrees of freedom.

The general solution of the equations of motion is then

$$x_j(t) = \sum_{a=1}^{N} A_j^a a_a \sin(\omega_a t + \phi_a) \equiv \sum_{a=1}^{N} A_j^a \left(\tilde{a}_a \sin(\omega_a t) + \tilde{b}_a \cos(\omega_a t)\right).$$

- If a particular $\omega_b = 0$, then one must substitute $a_b \sin(\omega_b t + \phi_b) \to v_b t + d_b$.
- The general solution depends on $2N$ arbitrary numbers $\{a_a\}$ and $\{\phi_a\}$ (or $\{\tilde{a}_a\}$ and $\{\tilde{b}_a\}$). These numbers must be found from the initial conditions.

We can prove, that all ω_a^2 are positive (if U is at minimum.) Let's substitute the solutions ω_a and A_j^a into equation (10.1), multiply it by $(A_i^a)^*$ and sum over the index i.

$$\sum_{ij} \left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a \left(A_i^a\right)^* = 0.$$

From here we see

$$\omega_a^2 = \frac{\sum_{ij} u_{ij} A_j^a \left(A_i^a\right)^*}{\sum_{ij} k_{ij} A_j^a \left(A_i^a\right)^*}$$

As both matrices k_{ij} and u_{ij} are symmetric and positive definite, we have the ratio of two positive real numbers in the RHS. So ω_a^2 must be positive and real.

A note on linear algebra: The equation for ω_a^2 is the same as the equation for the eigen values of the matrix $\hat{u}$ with the scalar/dot product given by the matrix $\hat{k}$.

10.2. Examples.

10.2.1. Two masses.

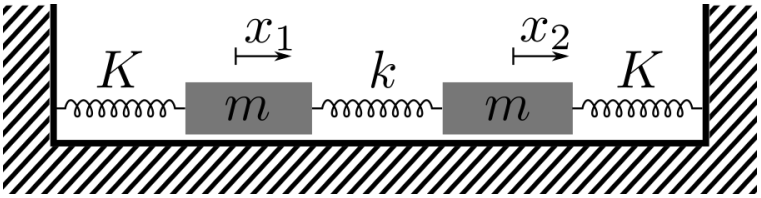


Figure 1. Two blocks.

This is the problem of two identical masses m connected with each other with the spring k and to the walls with the springs K.

The kinetic and potential energies are

$$K = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2}$$
$$U = \frac{Kx_1^2}{2} + \frac{k(x_1 - x_2)^2}{2} + \frac{Kx_2^2}{2} = \frac{1}{2}\left[(K+k)x_1^2 - 2kx_1x_2 + (K+k)x_2^2\right]$$

Using "a vector"

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

they can be written as

$$K = \frac{1}{2}(\dot{x}_1, \dot{x}_2)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \qquad U = \frac{1}{2}(x_1, x_2)\begin{pmatrix} K+k & -k \\ -k & K+k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

so the matrices $\hat{k}$ and $\hat{u}$ are

$$\hat{k} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad \hat{u} = \begin{pmatrix} K+k & -k \\ -k & K+k \end{pmatrix},$$

and the matrix $\omega^2\hat{k} - \hat{u}$ is

$$\begin{pmatrix} \omega^2 m - K - k & k \\ k & \omega^2 m - K - k \end{pmatrix}.$$

To find the eigen/normal modes we need to find such ω^2 at which the determinant of this matrix is zero. This is exactly the same matrix, as we have obtained for this problem in the previous lecture.

10.2.2. Three masses on a ring. Symmetries. Zero mode.

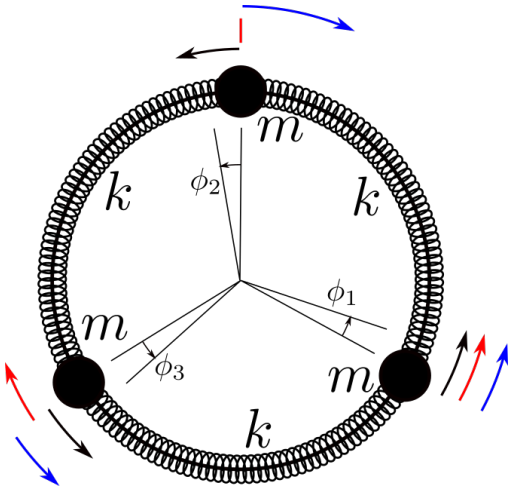


Figure 2. Three masses on a ring.

The Kinetic and potential energies are

$$K = \frac{mR^2\dot{\phi}_1^2}{2} + \frac{mR^2\dot{\phi}_2^2}{2} + \frac{mR^2\dot{\phi}_3^2}{2}$$
$$U = \frac{kR^2(\phi_1-\phi_2)^2}{2} + \frac{kR^2(\phi_2-\phi_3)^2}{2} + \frac{kR^2(\phi_3-\phi_1)^2}{2}$$
$$= \frac{1}{2}\left[2kR^2\phi_1^2 + 2kR^2\phi_2^2 + 2kR^2\phi_3^2 - 2kR^2\phi_1\phi_2 - 2kR^2\phi_2\phi_3 - 2kR^2\phi_3\phi_1\right]$$

Using the vector

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

The corresponding matrices are

$$\hat{k} = \begin{pmatrix} mR^2 & 0 & 0 \\ 0 & mR^2 & 0 \\ 0 & 0 & mR^2 \end{pmatrix}, \qquad \hat{u} = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix}.$$

So we need to solve the equation

$$\det \begin{pmatrix} \omega^2 mR^2 - 2kR^2 & kR^2 & kR^2 \\ kR^2 & \omega^2 mR^2 - 2kR^2 & kR^2 \\ kR^2 & kR^2 & \omega^2 mR^2 - 2kR^2 \end{pmatrix} = 0,$$

which gives

$$\omega^2 m\left(\omega^2 m - 3k\right)^2 = 0.$$

We see, that there are three modes: one is the zero mode $\omega = 0$, the other two are **double degenerate** $\omega^2 = 3k/m$. The corresponding eigen modes are

$$\omega = 0 \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \omega^2 = 3k/m \longrightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

(Any linear combination of the degenerate modes is also a normal mode.)

- The zero mode exists because of the continuous symmetry. It would be present even if all the masses and all the springs were different.
- The degeneracy is the consequence of the discrete symmetry, the fact that all the springs are the same and all masses are the same. This degeneracy would be lifted if we take, say, one of the masses to be different form the other two.

The modes are shown on the figures with different colors.

10.3. General procedure

(a) Choose the N coordinates $\{q_i\}$.
(b) Write the Potential energy $U(\{q_i\})$.
(c) Find the position of extremum $\{q_{j0}\}$ by solving $\frac{\partial U}{\partial q_i}\Big|_{\{q_j\}=\{q_{j0}\}} = 0$. If there is more than one solution the rest of the procedure must be done for all of them one-by-one.
(d) Change the coordinates to $x_i = q_i - q_{i0}$.
(e) Write the Taylor expansion for the function $U(\{x_i\})$ for small $\{x_i\}$ (around $\{x_i\} = 0$) up to the second order in $\{x_i\}$.

(f) Read the matrix u_{ij} from the expansion.
(g) Write the kinetic energy $K(\{q_i\},\{\dot{q}_i\})$ in terms of $\{q_i\}$ and $\{\dot{q}_i\}$.
(h) Change the variables to $\{x_i\}$ and $\{\dot{x}_i\}$ to find $K(\{x_i\},\{\dot{x}_i\})$.
(i) Taylor expand the function $K(\{x_i\}=0,\{\dot{x}_i\})$ for small $\{\dot{x}_i\}$ up to the second order in $\{\dot{x}_i\}$.
(j) Read the matrix k_{ij} from the expansion.
(k) Construct the matrix $\omega^2 k_{ij} - u_{ij}$.
(l) Find all values for ω^2 for which the determinant of this matrix is zero. You MUST have N real positive(non-negative) solutions for ω^2, we label them ω^a, $a = 1 \dots N$ (some might be degenerate).
(m) If any of the obtained ω^2 is negative, then $\{q_{i0}\}$ is NOT a stable equilibrium. Full stop here. Repeat the procedure for another extremum.
(n) For each ω^a you found, find the corresponding mode A^a_j (you do not need to normalize these vectors, if you do choose to normalize them, then use the matrix k_{ij} as the metric).
(o) The general solution is

$$x_j(t) = \sum_{a=1}^{N} a_a A^a_j \sin(\omega_a t + \phi_a).$$

It depends on $2N$ arbitrary constants a_a and ϕ_a.

LECTURE 11
Oscillations with time-dependent parameters. Kapitza pendulum $\Omega \gg \omega$.

11.1. Oscillations with time-dependent parameters.

- Oscillations with parameters depending on time.

$$L = \frac{1}{2}m(t)\dot{x}^2 - \frac{1}{2}k(t)x^2.$$

The Lagrange equation

$$\frac{d}{dt}m(t)\frac{d}{dt}x = -k(t)x.$$

We change the definition of time

$$m(t)\frac{d}{dt} = \frac{d}{d\tau}, \qquad \frac{d\tau}{dt} = \frac{1}{m(t)}$$

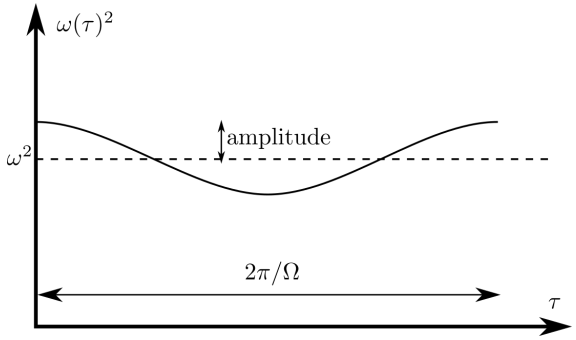
then the equation of motion is

$$\frac{d^2x}{d\tau^2} = -mkx.$$

So without loss of generality we can consider an equation

$$\ddot{x} = -\omega^2(\tau)x$$

- The most interesting is the situation when $\omega(\tau)$ is by itself a periodic function with frequency Ω. In this case the system returns back to where it was after time $2\pi/\Omega$.
- We call Ω the frequency of *change* of ω.
- Different time scales. Three different cases: $\Omega \gg \omega$, $\Omega \ll \omega$, and $\Omega \sim \omega$.



- $\Omega \gg \omega$ — Kapitza pendulum. (demo) Criteria: $\overline{\left(\dot{\xi}\right)^2} > gl$.
  - Importance of the time scale separation.
  - Averaging out fast processes – a natural thing to do.
  - Importance of non-linearity.
  - Universal mechanism – averaging over fast degrees of freedom leads to the change of the dynamics of the slow degree of freedom through non-linearity.
- $\Omega \ll \omega$ — Foucault pendulum as an example of slow change of the parameter $\Delta\phi$ =solid angle of the path. (quantum: Berry phase 1984; classical: Hannay angle 1985.)
  - Topological in nature.
  - Universal.
- $\Omega \sim \omega$ — parametric resonance ($\omega_p = \frac{2}{n}\omega_0$)

$$\ddot{x} = -\omega^2(t)x, \qquad \omega^2(t) = \omega_0^2(1 + h\cos(\omega_p t)), \quad h \ll 1$$

  Different from the usual resonance:
  - If the initial conditions $x(t=0) = 0$, $\dot{x}(t=0) = 0$, then $x(t) = 0$.
  - Frequency ω_p is a fraction of ω_0.
  - At finite dissipation one must have a finite amplitude h in order to get to the resonance regime.

11.2. The Kapitza pendulum. Set up of the problem.

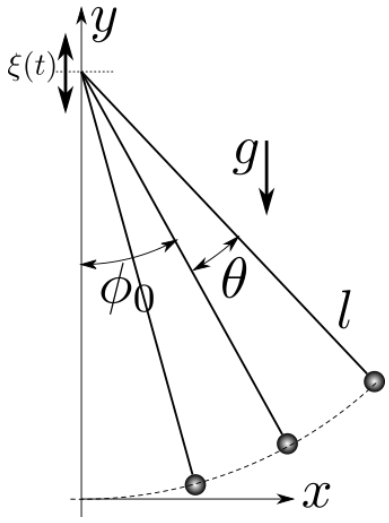
- We have a standard rigid pendulum of length l in the gravitation field g. Somebody/something is shaking the pivot point of the pendulum vertically or horizontally very fast, but with a small amplitude. So the position of the pivot point with respect to its average is given by $\xi(t)$, where $|\xi(t)| \ll l$.
- Time scales difference. The shaking is very fast. So the frequency of the function $\xi(t)$, which we call Ω is much larger than the natural frequency of the pendulum $\omega = \sqrt{g/l}$, or $\Omega \gg \omega$.
- Expected results. We assume that the function $\xi(t)$ is so fast, that we do not actually see the motion of the pivot point. The pendulum then will perform some very complicated motion, but this motion can be split into a very fast one, which we do not see, and a slow "average" one. We are interested in that slow motion only. We will be able to clearly define these two motion because there is large separation of the time scales. We then expect, that the fast motion will change — "renormalize" — the slow motion in comparison with the motion of a normal pendulum of length l.

We will consider two cases — vertical and horizontal shaking — separately. One can also consider a general case, when both are present.

11.3. Vertical displacement.

11.3.1. Exact Lagrangian and exact equation of motion.

The position of the pendulum bob is described by a single coordinate ϕ. The pivot point is shifting vertically very fast. The position of the pivot point is given by the known function



$\xi(t)$. On average the pivot point is not moving

$$\overline{\xi(t)} = 0.$$

The Cartesian coordinates of the pendulum bob are (see figure)

$$\begin{array}{ll} x = l\sin\phi & \dot{x} = l\dot{\phi}\cos\phi \\ y = l(1-\cos\phi) + \xi \ , & \dot{y} = l\dot{\phi}\sin\phi + \dot{\xi} \end{array}$$

The Lagrangian is

$$L = \frac{ml^2}{2}\dot{\phi}^2 + ml\dot{\phi}\dot{\xi}\sin\phi + mgl\cos\phi,$$

where I dropped the terms that are independent of ϕ and $\dot{\phi}$. The equation of motion is

$$\ddot{\phi} + \frac{\ddot{\xi}}{l}\sin\phi = -\omega^2\sin\phi, \qquad \omega^2 = g/l.$$

(Check the units.)
This is a non-linear second order differential equation with time dependent coefficients. We cannot find the exact solution of it, but we do not want to. The exact solution will have both slow and fast motions. What we want is to "average" over the fast motion and to find an effective equation for the slow motion only.

11.3.2. Averaging over fast motion.

We think that the motion will have two "parts" fast and slow. It means that there is a fast changing of ϕ on the background of slow motion of the "average" ϕ. Accordingly, we look for the solution in the following form:

$$\phi(t) = \phi_0(t) + \theta(t), \qquad \overline{\theta(t)} = 0,$$

where $\phi_0(t)$ is slow and $\theta(t)$ is fast. We are interested only in the function $\phi_0(t)$. The function $\theta(t)$ is so fast, that we cannot observe it. What we would like to do is to "average" the exact equation of motion so that $\theta(t)$ is "averaged out" and only $\phi_0(t)$ is left. In order to do that we need to define what do we mean by "average". **What does "averaging" mean?**

- Separation of the time scales $2\pi/\Omega \ll 2\pi/\omega$. Consider the time T such that $2\pi/\Omega \ll T \ll 2\pi/\omega$. During this time the fast motion goes over many cycles, while the slow motion almost does not happen.
- The averaging of any function, say $f(t)$, then means
$$\overline{f}(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} f(t')dt'.$$
Notice, that this is not full averaging. The full averaging would require taking a limit $T \to \infty$. We do not do that.
- Instead we average over time T. During this time T the fast functions $\theta(t)$ and $\xi(t)$ go over many many cycles, while the slow function $\phi_0(t)$ almost does not change.
- Notice, that with this averaging procedure we write
$$\overline{\theta(t)} = \frac{1}{T}\int_{t-T/2}^{t+T/2} \theta(t')dt' = 0, \quad \text{while} \quad \overline{\phi_0(t)} = \frac{1}{T}\int_{t-T/2}^{t+T/2} \phi_0(t')dt' = \phi_0(t),$$
as $\phi_0(t)$ can be considered to be a constant during time T. This also means that
$$\overline{\phi(t)} = \overline{\phi_0(t) + \theta(t)} = \overline{\phi_0(t)} + \overline{\theta(t)} = \phi_0(t).$$
- Such a procedure only makes sense if the result does not depend on T as long as $2\pi/\Omega \ll T \ll 2\pi/\omega$. The validity of separating the scales must also be checked afterwards.

We expect θ to be small, but $\dot\theta$ and $\ddot\theta$ are **NOT** small. θ is a very fast function, so its time derivatives are large!

Using the smallness of θ and the Taylor expansion we have
$$\ddot\phi_0 + \ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 = -\omega^2\sin\phi_0 - \omega^2\theta\cos\phi_0$$

- Notice, that the last terms on both sides are due to non-linearity of the system. Especially important will be the last term in the left hand side.

The frequency of the function ϕ_0 is small, so the functions ϕ_0 and $\ddot\phi_0$ are slow, while the functions θ, $\ddot\theta$, and $\ddot\xi$ are fast. Let's group the fast terms to the left hand side of the equation and the slow term to the right hand side

$$\ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 + \omega^2\theta\cos\phi_0 = -\omega^2\sin\phi_0 - \ddot\phi_0 \tag{11.1}$$

The slow functions cannot cancel out the fast ones, so the fast oscillating functions must (almost) cancel each other:
$$\ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 + \omega^2\theta\cos\phi_0 = 0.$$
In this equation we notice, that θ is small, while $\ddot\theta$ and $\ddot\xi$ are not small. So the last two terms are much smaller than the first two terms. Small terms cannot cancel the large ones, so the large terms mast (almost) cancel each other. So we get $\ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 = 0$, or
$$\theta = -\frac{\xi}{l}\sin\phi_0.$$
As $\bar\xi = 0$, the requirement $\bar\theta = 0$ fixes the other terms coming from the integration of the second order differential equation.

- Notice, that we again used the fact that θ and ξ are fast, while ϕ_0 is slow, so in the above equation ϕ_0 can be treated as a constant.

Now we take the equation (11.1) and average it over the time T. In this procedure we can treat ϕ_0 as a constant. In addition for any bounded small function $f(t)$ the average of the time derivative is very small

$$\overline{\dot{f}(t)} = \frac{1}{T}\int_0^T \dot{f}(t)dt = \frac{f(T)-f(0)}{T}$$

The result is small, as it has large T in the denominator. So the averaging of the equation (11.1) gives.

$$\overline{\theta\ddot{\xi}}\frac{1}{l}\cos\phi_0 = -\omega^2\sin\phi_0 - \ddot{\phi}_0 \tag{11.2}$$

We now we use our result $\theta = -\frac{\xi}{l}\sin\phi_0$ in the left hand side

$$\overline{\theta\ddot{\xi}} = -\overline{\xi\ddot{\xi}}\frac{1}{l}\sin\phi_0.$$

To simplify it a bit we use

$$\overline{\xi\ddot{\xi}} = \frac{1}{T}\int_0^T \xi\ddot{\xi}dt = -\frac{1}{T}\int_0^T(\dot{\xi})^2dt = -\overline{(\dot{\xi})^2}.$$

(One important technical note. The expression $\overline{(\dot{\xi})^2}$ means that you first take the derivative, then square, then average. The order of operations is important. Any other order of the same operations will give you zero.)

Our averaged equation (11.2) then becomes

$$\ddot{\phi}_0 = -\left(\omega^2\sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0\right) = -\frac{\partial}{\partial\phi_0}\left(-\omega^2\cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0\right)$$

11.3.3. Effective potential.

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2\cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0.$$

Notice:

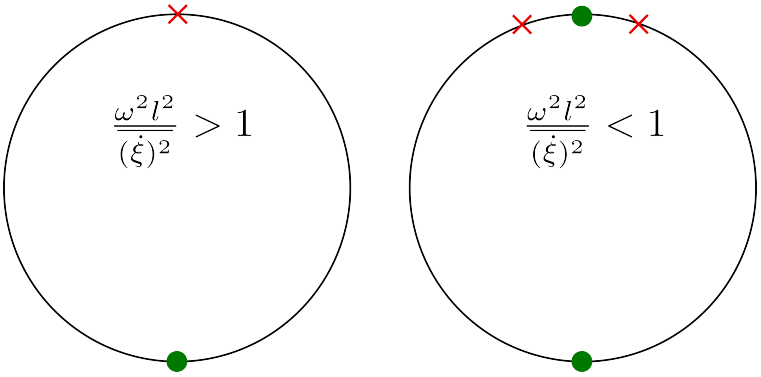
- the first term in the effective potential energy is simply the standard gravitational term;
- the second term, however, comes from the "averaging" procedure;
- the second term is **NOT** small in comparison to the first, as $\dot{\xi}$ is not small.

Now we analyze this effective potential energy the usual way.

11.3.3.1. *The equilibrium.* The equilibrium positions are given by

$$\frac{\partial U}{\partial\phi_0} = \omega^2\sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0 = 0, \qquad \left(\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_0\right)\sin\phi_0 = 0$$

- We see, that a pair of solutions $\phi_0 = 0$ and $\phi_0 = \pi$ (these are the solutions of $\sin\phi_0 = 0$) always exists.



- We see, that if $\frac{\omega^2 l^2}{\overline{(\dot\xi)^2}} < 1$, a pair of new solutions appears. These are the solutions of the equation

$$\cos\phi_0 = -\frac{l^2\omega^2}{\overline{(\dot\xi)^2}}$$

11.3.3.2. *The stability of equilibrium points.* The stability is defined by the sign of

$$\frac{\partial^2 U}{\partial\phi_0^2} = \omega^2\cos\phi_0 + \frac{\overline{(\dot\xi)^2}}{l^2}\cos 2\phi_0$$

One sees, that

- $\phi_0 = 0$ is always a stable solution $\frac{\partial^2 U}{\partial\phi_0^2} = \omega^2 + \frac{\overline{(\dot\xi)^2}}{l^2} > 0$.
- $\phi_0 = \pi$ is unstable for $\frac{\omega^2 l^2}{\overline{(\dot\xi)^2}} > 1$; $\frac{\partial^2 U}{\partial\phi_0^2} = -\omega^2 + \frac{\overline{(\dot\xi)^2}}{l^2} < 0$, but becomes stable if $\frac{\omega^2 l^2}{\overline{(\dot\xi)^2}} < 1$; $\frac{\partial^2 U}{\partial\phi_0^2} = -\omega^2 + \frac{\overline{(\dot\xi)^2}}{l^2} > 0$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot\xi)^2}} < 1$; $\frac{\partial^2 U}{\partial\phi_0^2} = -\frac{\overline{(\dot\xi)^2}}{l^2}\left[1 - \left(\frac{\omega^2 l^2}{\overline{(\dot\xi)^2}}\right)^2\right] < 0$ are unstable.

11.3.3.3. *The oscillations around the equilibrium.* For ϕ_0 close to π we can introduce $\phi_0 = \pi + \tilde\phi$, where $\tilde\phi$ is small.

$$\ddot{\tilde\phi} = -\omega^2\left(\frac{\overline{(\dot\xi)^2}}{l^2\omega^2} - 1\right)\tilde\phi$$

We see, that for $\frac{\overline{(\dot\xi)^2}}{l^2\omega^2} > 1$, the oscillations in the upper point have the frequency

$$\tilde\omega^2 = \omega^2\left(\frac{\overline{(\dot\xi)^2}}{l^2\omega^2} - 1\right)$$

Remember, that above calculation is correct if Ω of the ξ is much larger than ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot\xi)^2} \approx \Omega^2\xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime ($\overline{(\dot\xi)^2}/l^2\omega^2 \sim 1$) is at

$$\Omega^2 \sim \omega^2\frac{l^2}{\xi_0^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

LECTURE 12
Kapitza pendulum. What is going on? Horizontal case.

12.1. What is going on?

In the previous lecture we considered the Kapitza pendulum. We obtained a surprising result, that at fast enough shaking of the pivot point the position straight up becomes a stable equilibrium. However, the physics of why it happens is obscured by the mathematics. Here we try to understand what really happens.

Let's consider a shaken pendulum without the gravitation force acting on it. The fast shaking is given by a fast time dependent vector $\vec{\xi}(t)$. This vector defines a direction in space. I will call this direction $\hat{z}$, so $\vec{\xi}(t) = \hat{z}\xi(t)$.

The amplitude ξ is small $\xi \ll l$, where l is the length of the pendulum, but the shaking is very fast $\Omega \gg \omega$, the frequency of the pendulum motion (without gravity it is not well defined, but we will keep in mind that we are going to include gravity later.)

Let's now use a non inertial frame of reference connected to the point of attachment of the pendulum. In this frame of reference there is an artificial force which acts on the pendulum bob. This force is

$$\vec{f} = -\ddot{\xi} m \hat{z}.$$

If the pendulum makes an angle ϕ with respect to the axis $\hat{z}$, then the position vector of the bob $\vec{r}$ is given by (see figure for the axes. The positive direction of the axis y is defined by the angle, the naming of the x and y axes is defined by the right hand rule.)

$$\vec{r} = \hat{x} l \sin\phi + \hat{z} l \cos\phi$$

and the torque of the force $\vec{f}$ is

$$\vec{\tau} = \vec{r} \times \vec{f} = -\ddot{\xi} m \left[\hat{x} l \sin\phi + \hat{z} l \cos\phi\right] \times \hat{z} = \hat{y} m \ddot{\xi} l \sin\phi$$

The vector of angular velocity of the bob is $\vec{\omega} = \hat{y}\dot{\phi}$. So the equation of motion

$$ml^2 \dot{\vec{\omega}} = \vec{\tau}$$

becomes

$$\hat{y} m l^2 \ddot{\phi} = \vec{\tau} = \hat{y} m \ddot{\xi} l \sin\phi, \qquad \ddot{\phi} = \frac{\ddot{\xi}}{l} \sin\phi$$

Now we split the angle onto slow motion described by ϕ_0 – a slow function of time, and fast motion $\theta(t)$ a fast oscillating function of time such that $\bar{\theta} = 0$.

We then have

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0 + \theta)$$

Notice the non linearity of the RHS.

As $\theta \ll \phi_0$, we can use the Taylor expansion

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0) + \frac{\ddot{\xi}\theta}{l}\cos(\phi_0) \tag{12.1}$$

(Red term is the term which appears due to non-linearity. It plays the major role in the effect.)

Double derivatives of θ and ξ are very fast, so all the fast terms must cancel each other. In addition the term proportional to $\ddot{\xi}\theta$, although it is fast, it is quadratic in small amplitude, so it can be neglected in comparison to the other fast terms. As the result we get

$$\ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0), \qquad \theta = \frac{\xi}{l}\sin(\phi_0).$$

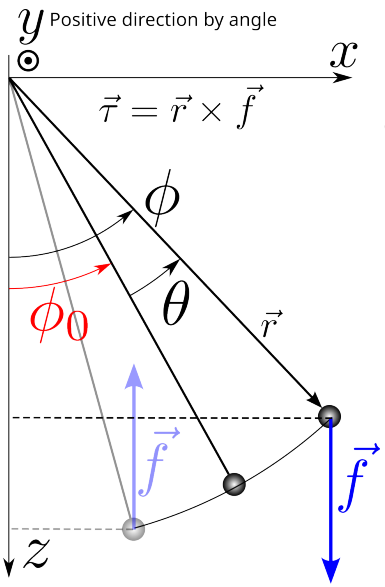


Figure 1. The Kapitza pendulum.

Now averaging the equation (12.1) in the way described in the previous lecture we get

$$\ddot{\phi}_0 = \frac{\overline{\ddot{\xi}\theta}}{l}\cos(\phi_0) = \frac{\overline{\ddot{\xi}\xi}}{l^2}\sin(\phi_0)\cos(\phi_0)$$

or, after using integration by parts in averaging, the final equation of motion for ϕ_0 is

$$\ddot{\phi}_0 = -\frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0)$$

What is happening is illustrated on the figure.

- For $0 < \phi_0 < \pi/2$, the force $f \sim -\ddot{\xi} \sim -\ddot{\theta}$.
- As θ is oscillating function, when θ is positive $\ddot{\theta}$ is negative, and when θ is negative $\ddot{\theta}$ is positive. Or $\ddot{\theta} \sim -\theta$.
- This means $f \sim \theta$. This is what is shown in the figure.
- The magnitude of the force is the same for θ and $-\theta$, but the directions are opposite. If we were to average the force we would get zero.
- However, the torque of the force f is larger for $\theta > 0$, as the "shoulder" of the force is larger. (This is, in fact, the consequence of the non-linearity.)
- So the net torque after averaging is not zero and pushes the bob towards $\phi_0 = 0$!

So the reason that there is a net torque which keeps the pendulum around $\phi_0 = 0$ (or $\phi_0 = \pi$) even without gravity is the correlation between the direction of the force and the length of the "shoulder" of the force.

12.2. Vertical.

Now we can get the result from the previous lecture. We just need to add the gravitational term $-\omega^2 \sin\phi_0$.

$$\ddot{\phi}_0 = -\omega^2 \sin\phi_0 - \frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0).$$

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2\cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0$$

The equilibrium positions are given by

$$\frac{\partial U}{\partial\phi_0} = \omega^2\sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0 = 0, \qquad \sin\phi_0\left(\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_0\right) = 0$$

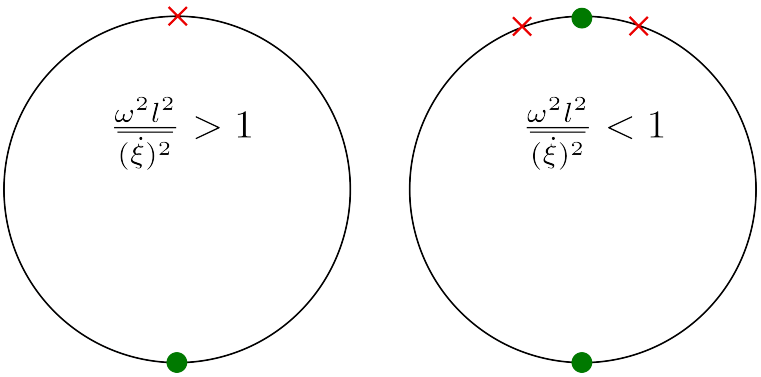
We see, that if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$, a pair of new solutions appears.

The stability is defined by the sign of

$$\frac{\partial^2 U}{\partial\phi_0^2} = \omega^2\cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos 2\phi_0$$

One sees, that

- $\phi_0 = 0$ is always a stable solution $\frac{\partial^2 U}{\partial\phi_0^2} = \omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2} > 0$.
- $\phi_0 = \pi$ is always an equilibrium. As $\frac{\partial^2 U}{\partial\phi_0^2} = -\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}$ it is unstable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} > 1$, but becomes stable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$; $\frac{\partial^2 U}{\partial\phi_0^2} = -\frac{\overline{(\dot{\xi})^2}}{l^2}\left[1 - \left(\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}}\right)^2\right] < 0$ are unstable.



12.2.1. Oscillations around the equilibrium at $\phi_0 = \pi$.

For ϕ_0 close to π we can introduce $\phi_0 = \pi + \tilde{\phi}$

$$\ddot{\tilde{\phi}} = -\omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)\tilde{\phi}$$

We see, that for $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$ the oscillations around the upper point have the frequency

$$\tilde{\omega}^2 = \omega^2 \left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1 \right).$$

12.2.2. Validity of the approximation.

Remember, that above calculation is correct if Ω of the ξ is much larger then ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot{\xi})^2} \approx \Omega^2\xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$ is at

$$\Omega^2 > \omega^2 \frac{l^2}{\xi_0^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

12.3. Horizontal.

If ξ is horizontal, then it is convenient to redefine the angle $\phi_0 \to \pi/2 + \phi_0$, then the shake contribution changes sign and we get

$$U_{eff} = -\omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{4l^2} \cos 2\phi_0$$

The equilibrium position is found by solving

$$\frac{\partial U_{eff}}{\partial \phi_0} = \sin\phi_0 \left(\omega^2 - \frac{\overline{(\dot{\xi})^2}}{l^2} \cos\phi_0 \right) = 0.$$

and their stability is checked by checking the sign of

$$\frac{\partial^2 U_{eff}}{\partial \phi_0^2} = \omega^2 \cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{l^2} \cos 2\phi_0$$

We see, that

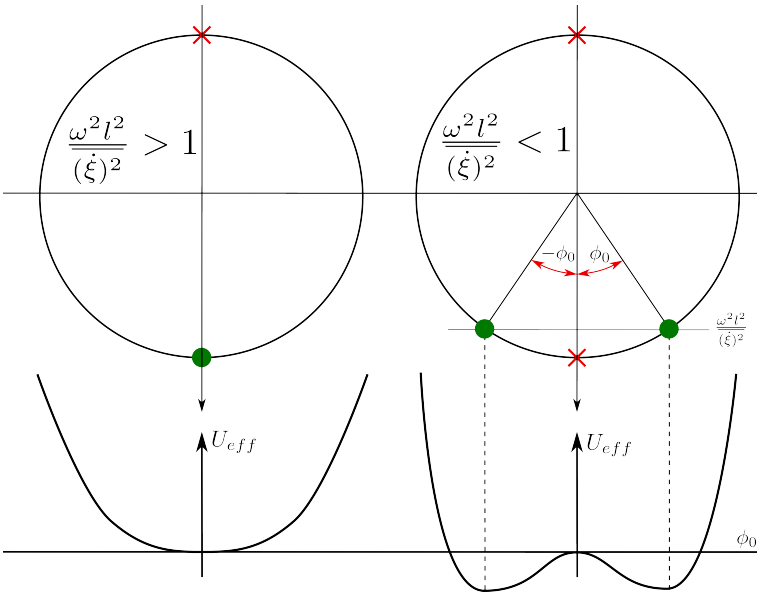
- $\phi_0 = \pi$ is always an equilibrium and it is always unstable $\frac{\partial^2 U_{eff}}{\partial \phi_0^2} = -\omega^2 - \frac{\overline{(\dot{\xi})^2}}{l^2}$.
- $\phi_0 = 0$ is always an equilibrium position. However, as in this case

  $$\frac{\partial^2 U_{eff}}{\partial \phi_0^2} = \omega^2 \left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \right),$$

  it is stable for $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} < 1$ and is unstable for $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} > 1$.
- For $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} > 1$ two new equilibrium points appear. These points are the solutions of the equation

  $$\cos\phi_0 = \frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1.$$

  These points are stable $\frac{\partial^2 U}{\partial \phi_0^2} = \frac{\overline{(\dot{\xi})^2}}{l^2} \left[1 - \left(\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} \right)^2 \right] > 0.$



12.3.1. Discussion of the horizontal case.

If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} < 1$ the angle $\phi_0 = 0$ is a stable equilibrium, so all small angle motions happen around $\phi_0 = 0$.

At $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}$ just above 1 the new equilibrium points just appear. The corresponding equilibrium angles given by $\cos\phi_0 = \frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}}$ are small. The small oscillations will be around these new equilibrium points and will also correspond to very small angles. So in both cases we only need U_{eff} for small angles.

Let's use Taylor expansion of U_{eff} up to the fourth order in small angle ϕ_0, then (dropping the constant.)

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{24}\left(4\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} - 1\right)\phi_0^4$$

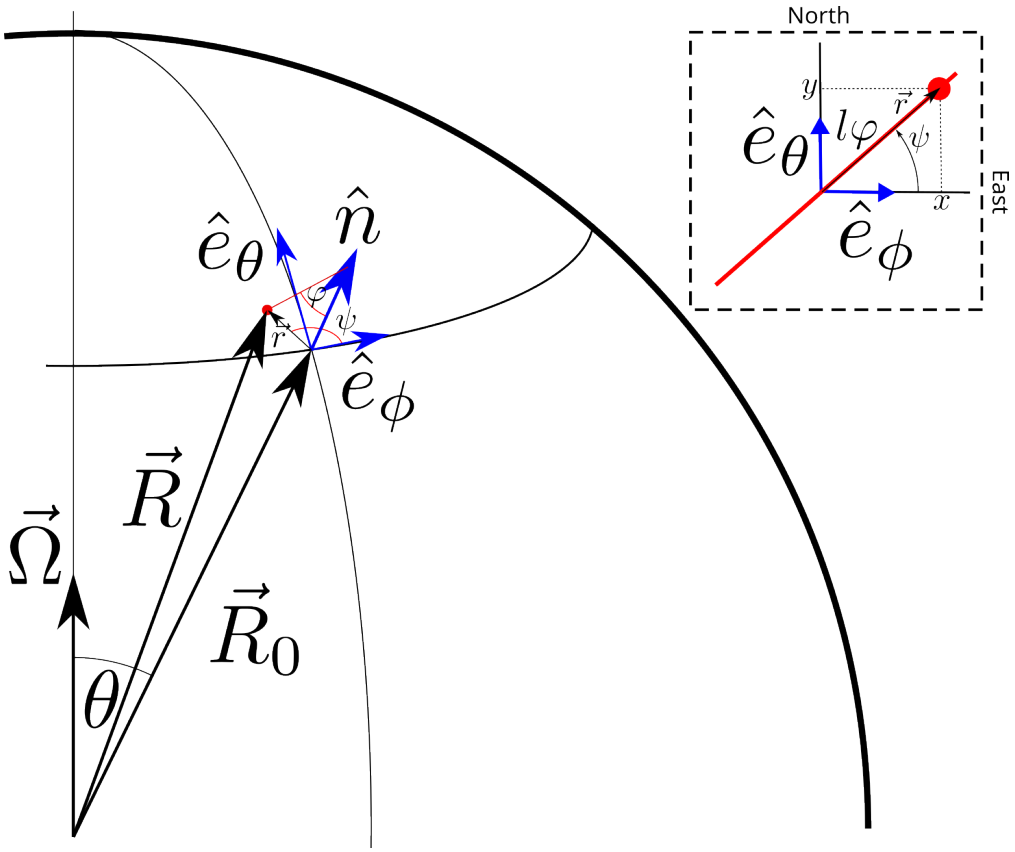
If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$, then

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{8}\phi_0^4.$$

This U_{eff} is depicted in the figure for both cases.

- Spontaneous symmetry braking.

LECTURE 13
Foucault pendulum.



The opposite situation, when the change of parameters is very slow – adiabatic approximation.

The problem we want to consider is a pendulum on Earth. What we want is to see what effect the rotation of the Earth has on a pendulum. So we are standing on the Earth next to the pendulum and are observing its oscillations. As our frame of reference is not inertial we will see "fictitious" forces in addition to the gravity. These "fictitious" forces are centripetal and Coriolis forces. We will see, that the centripetal force only changes the "effective" gravity and thus does not produce any new effects, the Coriolis force, however, produces a very interesting and universal effect.

We start by writing the Lagrangian for the pendulum on the rotating Earth.

13.1. Coordinates.

We want to describe the motion of the pendulum in our local frame of references. In this frame we observe the pendulum oscillating in some vertical plane. The two coordinates which we use are the coordinate φ of the pendulum in the plane of oscillations and the angle ψ which the plane of oscillations makes with the line of latitude.

So the first step is to write the potential and the kinetic energies in terms of φ, $\dot{\varphi}$, ψ, and $\dot{\psi}$. Then write the Lagrangian and a pair of Lagrange equations, and then analyze these equations keeping in mind that the period of the Earth rotation $2\pi/\Omega$ (24hr) is much larger than the period of oscillations $2\pi/\omega$, or $\Omega \ll \omega$.

13.2. Potential energy.

As usual for the pendulum

$$U(\varphi, \psi) = mgl(1 - \cos\varphi) \approx \frac{1}{2} mgl\varphi^2 = \frac{1}{2} ml^2\omega^2\varphi^2.$$

where $\omega^2 = g/l$. The potential energy does not depend on the coordinate ψ.

13.3. Kinetic Energy.

In our local system of coordinate (not inertial) a position vector is

$$\vec{r} = x\vec{e}_\phi + y\vec{e}_\theta.$$

So the position vector $\vec{R}$ in the inertial system of coordinates is

$$\vec{R} = \vec{R}_0 + \vec{r} = \vec{R}_0 + x\vec{e}_\phi + y\vec{e}_\theta.$$

We need the velocity of the bob in the inertial (outside) frame of references. In this frame $\vec{R}_0$, $\hat{e}_\phi$, and $\hat{e}_\theta$ as well as x and y change with time. So when we take the time derivative we must differentiate all of them $\vec{R}_0$, $\hat{e}_\phi$, $\hat{e}_\theta$, x, and y.

However, the vectors $\vec{R}_0$, $\hat{e}_\phi$, and $\hat{e}_\theta$ are constant in our rotating frame. We know

- For any vector $\vec{l}$ fixed in the rotating frame which rotates with angular velocity $\vec{\Omega}$ the following is true for an outside observer:

$$\dot{\vec{l}} = \vec{\Omega} \times \vec{l}.$$

So for $\vec{R}_0$, $\hat{e}_\phi$, and $\hat{e}_\theta$ we have

$$\dot{\vec{R}}_0 = \vec{\Omega} \times \vec{R}_0, \qquad \dot{\hat{e}}_\phi = \vec{\Omega} \times \hat{e}_\phi, \qquad \dot{\hat{e}}_\theta = \vec{\Omega} \times \hat{e}_\theta.$$

And as $\vec{v} = \dot{\vec{R}}$ we find

$$\vec{v} = \dot{\vec{R}} = \Omega \times \vec{R}_0 + \dot{x}\vec{e}_\phi + \dot{y}\vec{e}_\theta + x\vec{\Omega} \times \vec{e}_\phi + y\vec{\Omega} \times \vec{e}_\theta.$$

In order to compute the kinetic energy, we need to compute the square of the velocity $\vec{v}^2$. The expression for $\vec{v}^2$ will have the square of each term and the double of all possible square terms, the total of 15 terms. The calculation can be simplified by the following observation:

In our system of coordinates $e_\phi \perp \vec{\Omega}$, the angle between $\vec{\Omega}$ and $\vec{R}_0$ is θ, and the angle between $\vec{\Omega}$ and $\hat{e}_\theta$ is $\pi/2 - \theta$. So one can show that

$$(\Omega \times \vec{R}_0)^2 = \Omega^2 R_0^2 \sin^2\theta, \quad (\vec{\Omega} \times \vec{e}_\phi)^2 = \Omega^2, \quad (\vec{\Omega} \times \vec{e}_\theta)^2 = \Omega^2 \cos^2\theta,$$
$$[\Omega \times \vec{R}_0] \cdot \vec{e}_\phi = R_0 \Omega \sin\theta, \quad [\Omega \times \vec{R}_0] \cdot \vec{e}_\theta = 0, \quad [\Omega \times \vec{R}_0] \cdot [\vec{\Omega} \times \vec{e}_\phi] = 0,$$
$$[\Omega \times \vec{R}_0] \cdot [\vec{\Omega} \times \vec{e}_\theta] = -\Omega^2 R_0 \sin\theta\cos\theta, \quad \vec{e}_\phi \cdot [\vec{\Omega} \times \vec{e}_\theta] = -\Omega\cos\theta, \quad [\vec{\Omega} \times \vec{e}_\phi] \cdot [\vec{\Omega} \times \vec{e}_\theta] = 0$$

Now computing $\vec{v}^2$ we get

$$\vec{v}^2 = \underbrace{\Omega^2 R_0^2 \sin^2\theta}_{\text{a constant}} + \dot{x}^2 + \dot{y}^2 + x^2\Omega^2 + y^2\Omega^2\cos^2\theta + \underbrace{2\dot{x}\Omega R_0 \sin\theta}_{\text{full der.}} - \underbrace{y\Omega^2 R_0 \sin(2\theta)}_{\text{adds to gravity}} + 2\Omega(\dot{y}x - \dot{x}y)\cos\theta$$

Let's consider the $\vec{v}^2$ term by term having in mind, that $m\vec{v}^2/2$ is the first term in the Lagrangian. Then we see

- The first term $\Omega^2 R_0^2 \sin^2\theta$ can be dropped, as it is a constant.
- The term $2\dot{x}\Omega R_0 \sin\theta$ can be dropped, as it is a full derivative.
- The term $-y\Omega^2 R_0 \sin(2\theta)$ comes from the centripetal force (it is proportional $\Omega^2 R_0$) It simply changes the potential energy a little. As it is definitely a lot less than the gravity we can also drop it.

We then have

$$\vec{v}^2 \to \dot{x}^2 + \dot{y}^2 + \underbrace{x^2\Omega^2}_{\text{small}} + \underbrace{y^2\Omega^2\cos^2\theta}_{\text{small}} + 2\Omega(\dot{y}x - \dot{x}y)\cos\theta$$

Now we remember, that Ω is small. So we also drop all the terms of the order of Ω^2 (we keep the term linear in Ω, as we will need to do the experiment during one day, so $\Omega T \sim 1$.) So we have

$$\vec{v}^2 \to \dot{x}^2 + \dot{y}^2 + 2\Omega(\dot{y}x - \dot{x}y)\cos\theta$$

Now we convert the coordinates x and y into ϕ and ψ. For the small oscillations of a pendulum we have

$$x = l\varphi\cos\psi, \quad y = l\varphi\sin\psi,$$

where φ is the usual angle of the pendulum and ψ is the angle which the plane of oscillations makes with the vector $\hat{e}_\phi$ — our x coordinate. The velocities then are

$$\dot{x} = l\dot{\varphi}\cos\psi - l\varphi\dot{\psi}\sin\psi, \qquad \dot{y} = l\dot{\varphi}\sin\psi + l\phi\dot{\psi}\cos\psi$$

So

$$\dot{x}^2 + \dot{y}^2 = l^2\dot{\varphi}^2 + l^2\varphi^2\dot{\psi}^2$$
$$x\dot{y} - y\dot{x} = l^2\varphi^2\dot{\psi}$$

and

$$v^2 \to l^2\dot{\varphi}^2 + l^2\varphi^2\dot{\psi}^2 + 2\Omega l^2\varphi^2\dot{\psi}\cos\theta$$

The Lagrangian then is $\frac{m\vec{v}^2}{2} - U$

$$L = \frac{mv^2}{2} - \frac{ml^2}{2}\omega^2\varphi^2 = \frac{ml^2}{2}\left(\dot{\varphi}^2 + \varphi^2\dot{\psi}^2 + 2\Omega\varphi^2\dot{\psi}\cos\theta\right) - \frac{ml^2}{2}\omega^2\varphi^2.$$

13.4. Equations of motion.

The Lagrangian equations for the two coordinates φ and ψ are ($\omega^2 = g/l$):

$$\ddot{\varphi} = -\omega^2\varphi + \varphi\dot{\psi}^2 + 2\Omega\varphi\dot{\psi}\cos\theta$$
$$2\varphi\dot{\varphi}\dot{\psi} + \varphi^2\ddot{\psi} + 2\varphi\dot{\varphi}\Omega\cos\theta = 0$$

We will see, that $\dot{\psi} \sim \Omega$. As we have already neglected all terms of the order of Ω^2 we also must neglect all terms of the order of $\Omega\dot{\psi}$, $\dot{\psi}^2$, and $\ddot{\psi}$:

$$\ddot{\varphi} = -\omega^2\varphi$$
$$\dot{\psi} = -\Omega\cos\theta$$

The total change of the angle ψ for the period is

$$\Delta\psi = -\Omega T\cos\theta = -2\pi\cos\theta.$$

- Geometrical meaning. As ψ is an angle, we can always add 2π to it. We then have $\Delta\psi = 2\pi(1 - \cos\theta)$ — the solid angle defined by the path of the pendulum!!
- Parallel transport on a sphere:
  - What is parallel transport?
  - What is straight line?
  - Triangle is three points connected by the straight lines.
  - We can draw triangle with all angles $\pi/2$ degrees.
  - Parallel transport a vector along such a triangle.
  - Upon returning it will turn $\pi/2$.
  - The area of such a triangle is 1/8th of the area of the sphere.
  - The solid angle is $4\pi/8 = \pi/2$.

LECTURE 14
Foucault pendulum. General case.

We want to move a pendulum around along some closed trajectory Γ. The question is what angle the plane of oscillations will turn after we return back to the original point?

We assume that the earth is not rotating.

We assume that we are moving the pendulum slowly.

In order to describe the motion we need a system of coordinates. As we want to describe the motion of the pendulum in our local sustem of coordinates, we need to establish this system. But as we are moving around the sphere, we need to know how this local system of coordinates changes due to the change of our position on the sphere. The description of the system of coordinates and of the change of this system as we move is what we are going to start with.

14.1. Geometry on a sphere.

First of all we need to decide on the system of coordinates. For our simple (sphere is a simple object) case we can do it in the following way.

(a) We choose a global unit vector $\hat{z}$. The only requirement is that the z line does not intersect our trajectory.

(b) After that we can introduce the angles θ and ϕ in the usual way. (strictly speaking in order to introduce ϕ we also need to introduce a global vector $\hat{x}$, thus introducing a full global system of coordinates.)

(c) After that we can define the trajectory Γ over along which we move the pendulum as a function $\theta(\phi)$, or more precisely we can use the parametric form $\theta(t)$, $\phi(t)$, where t is just a parameter.

(d) In each point on the sphere we introduce it's own system/vectors of coordinates $\hat{e}_\phi$, $\hat{e}_\theta$, and $\hat{n}$, where $\hat{n}$ is along the radius-vector $\vec{R}$, $\hat{e}_\phi$ is orthogonal to both $\hat{n}$ and $\hat{z}$ (for this we use the fact that our sphere is embedded in flat Euclidean $3D$ space), and $\hat{e}_\theta = \hat{n} \times \hat{e}_\phi$.

The vectors $\hat{e}_\phi$, $\hat{e}_\theta$, and $\hat{n}$ are defined at any point of the sphere. At different points these vectors are different. At EVERY point of the sphere we have

$$\hat{e}_\theta^2 = \hat{e}_\phi^2 = \hat{n}^2 = 1, \qquad \hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\theta \cdot \hat{n} = \hat{e}_\phi \cdot \hat{n} = 0.$$

Let's look how the coordinate vectors change when we change a point where we are siting. So let us change our position by a small vector $d\vec{r}$. The coordinate vectors then change by

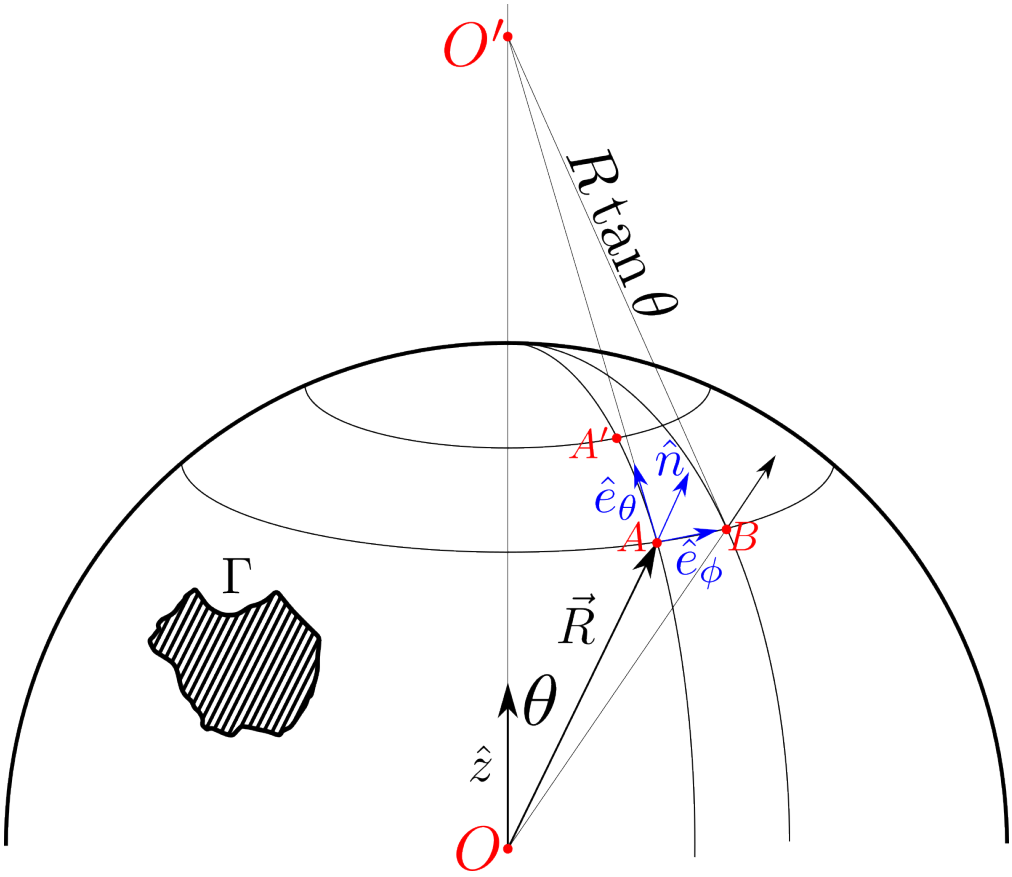


Figure 1. (Technical note: On this figure, the direction of the vector $\hat{e}_\theta$ is opposite to the standard one.)

$\hat{e}_\theta \to \hat{e}_\theta + d\hat{e}_\theta$, etc. Where $d\hat{e}_\theta$, $d\hat{e}_\phi$, and $d\hat{n}$ will be proportional to $d\vec{r}$. We then see that

$$\hat{e}_\theta \cdot d\hat{e}_\theta = \hat{e}_\phi \cdot d\hat{e}_\phi = \hat{n} \cdot d\hat{n} = 0, \qquad \hat{e}_\theta \cdot d\hat{e}_\phi + d\hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\theta \cdot d\hat{n} + d\hat{e}_\theta \cdot \hat{n} = \hat{e}_\phi \cdot d\hat{n} + d\hat{e}_\phi \cdot \hat{n} = 0.$$

These relations serve as six constraints for possible $d\hat{e}_\phi$, $d\hat{e}_\theta$, and $d\hat{n}$.

The vectors $d\hat{e}_\phi$, $d\hat{e}_\theta$, and $d\hat{n}$ are just vectors and as such can be expressed through the vectors $\hat{e}_\phi$, $\hat{e}_\theta$, and $\hat{n}$. Taking into consideration the above constraints we find

$$\begin{aligned} d\hat{e}_\theta &= a\hat{e}_\phi + b\hat{n} \\ d\hat{e}_\phi &= -a\hat{e}_\theta + c\hat{n} \\ d\hat{n} &= -b\hat{e}_\theta - c\hat{e}_\phi \end{aligned} \tag{14.1}$$

Where the three coefficients a, b, and c are linear in $d\vec{r}$.

The geometry of the sphere is described by the coefficients a, b, and c. Notice, that due to the six constraints there are only three coefficients we need to find. Without the constraints it should have been nine coefficients. So now we need to find these three coefficients. There are two ways to do that: "geometrical" and "algebraic". We consider both.

"Geometrical" way. We are interested in the vector $d\vec{r}$ which is along (tangential to) the surface of the sphere, so $d\vec{r}$ has two components along $\hat{e}_\phi$ and along $\hat{e}_\theta$.

$$d\vec{r} = (d\vec{r} \cdot \hat{e}_\phi)\hat{e}_\phi + (d\vec{r} \cdot \hat{e}_\theta)\hat{e}_\theta$$

(a) Let's take $d\vec{r}$ to be along the vector $\hat{e}_\phi$. We move from the point A to the point B on the figure. The length $|AB| = d\vec{r}\cdot\hat{e}_\phi$. Consider the triangle AOB. As the angle $\angle AOB$ is small we can write $\angle AOB = \frac{|AB|}{R} = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}$. This is also the angle by which the vector $\hat{n}$ turns in the plane AOB as we move from A to B. The magnitude by which the vector $\hat{n}$ changes is $|d\hat{n}| = |\hat{n}|\angle AOB = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}$. It is also clear from the picture of the plane AOB that the direction of the vector $d\hat{n}$ is along $\hat{e}_\phi$. So in this case $d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi$. Comparing this with the third of (14.1), we find

$$c = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}.$$

(b) Again let's take $d\vec{r}$ to be along the vector $\hat{e}_\phi$. But now consider the triangle $AO'B$. From the triangle OAO' we see, that $|AO'| = R\tan\theta$. So $\angle AO'B = \frac{|AB|}{|AO'|} = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}$. We see, that when we move from A to B the vector $\hat{e}_\theta$ is rotating in the $AO'B$ plane by the angle $\angle AO'B$, so $|d\hat{e}_\theta| = |\hat{e}_\theta|\angle AO'B = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}$. From the plane $AO'B$ it is also clear, that the direction of $d\hat{e}_\theta$ is $-\hat{e}_\phi$, so $d\hat{e}_\theta = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi$. Comparing this with the first of (14.1), we find

$$a = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}.$$

(c) Let's take $d\vec{r}$ to be along the vector $\hat{e}_\theta$, so we are moving from A to A'. It is clear that $d\hat{e}_\phi = 0$. Let's consider the change of the vector $\hat{n}$. The distance $|AA'| = d\vec{r}\cdot\hat{e}_\theta$, so from the triangle AOA' we see $\angle AOA' = \frac{|AA''|}{|OA|} = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}$. When we go from A to A' the vector $\hat{n}$ rotates in the plane AOA' by the angle $\angle AOA'$, so $|d\hat{n}| = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}$. From the plane AOA' it is also clear, that the direction of the vector $d\hat{n}$ is along $\hat{e}_\theta$, so $d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta$. Comparing this with the third of (14.1), we find

$$b = -\frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}$$

"Algebraic" way. There is an algebraic way to find the coefficients a, b, and c.

(a) Consider a vector $\vec{r} = R\hat{n}$. We have

$$d\vec{r} = Rd\hat{n} + \hat{n}dR.$$

(b) Using the last of the equations (14.1) we get

$$d\vec{r} = -Rb\hat{e}_\theta - Rc\hat{e}_\phi + \hat{n}dR.$$

(c) Multiplying this equation by $\hat{e}_\theta$ and by $\hat{e}_\phi$ we obtain

$$\hat{e}_\theta\cdot d\vec{r} = -Rb$$
$$\hat{e}_\phi\cdot d\vec{r} = -Rc$$

and find the previous results for b and c.

(d) Now we notice, that our definition of $\hat{e}_\phi$ is such, that $\hat{z}\cdot\hat{e}_\phi = 0$ at EVERY point, so

$$\hat{z}\cdot d\hat{e}_\phi = 0.$$

Using the second of the equations (14.1) we find

$$-a\hat{z}\cdot\hat{e}_\theta + c\hat{z}\cdot\hat{n} = 0.$$

As $\hat{z}\cdot\hat{n} = \cos\theta$, and $\hat{z}\cdot\hat{e}_\theta = \sin\theta$, we find $a = c/\tan\theta$.

We thus know all three coefficients a, b, and c. Collecting it all together we have

$$d\hat{e}_\theta = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi - \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{n}$$
$$d\hat{e}_\phi = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\theta - \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{n}$$
$$d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta + \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi$$

Notice, that these formulas simply describe the geometry of a sphere.

14.2. Pendulum on the sphere.

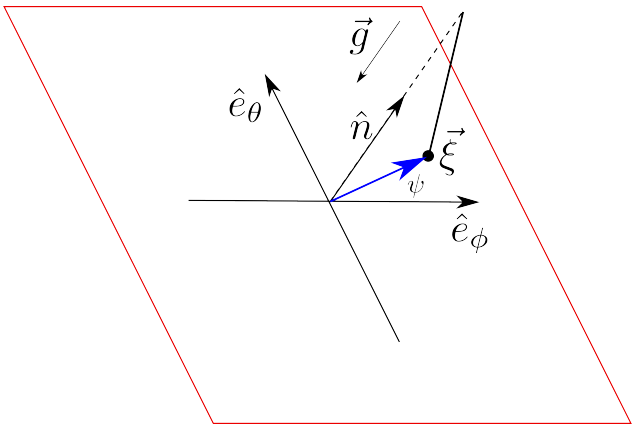


Figure 2

Now let's consider a pendulum. In our local system of coordinates it's radius vector is

$$\vec{\xi} = \xi\cos\psi\hat{e}_\phi + \xi\sin\psi\hat{e}_\theta.$$

The quantities ψ and ξ are the generalized coordinates of our pendulum.

When the pendulum oscillates, ξ changes fast. As we slowly move the pendulum along the path Γ on the sphere the vectors $\hat{e}_\phi$ and $\hat{e}_\theta$ slowly change. We expect that as the solution of the equation of motion ψ also slowly changes.

When we differentiate $\vec{\xi}$ over time we also need to differentiate $\hat{e}_\phi$, $\hat{e}_\theta$, and $\hat{n}$ over time. These vectors change with time only because the position of the pendulum changes with time. So in order to find these derivatives we use the chain rule $\frac{d\hat{e}_\theta}{dt} = \frac{\partial\hat{e}_\theta}{\partial\vec{r}}\frac{d\vec{r}}{dt}$ and $\frac{d\hat{e}_\phi}{dt} = \frac{\partial\hat{e}_\phi}{\partial\vec{r}}\frac{d\vec{r}}{dt}$. The velocity is then

$$\dot{\vec{\xi}} = \dot{\xi}(\sin\psi\hat{e}_\theta + \cos\psi\hat{e}_\phi) + \xi\dot{\psi}(\cos\psi\hat{e}_\theta - \sin\psi\hat{e}_\phi) + \xi\left(\sin\psi\frac{\partial\hat{e}_\theta}{\partial\vec{r}} + \cos\psi\frac{\partial\hat{e}_\phi}{\partial\vec{r}}\right)\frac{d\vec{r}}{dt}.$$

When we calculate $\left(\dot{\vec{\xi}}\right)^2$ we only keep terms no more than first order in $d\vec{r}/dt$

(a) The square of the first term gives $\dot{\xi}^2$.
(b) The square of the second term gives $\xi^2\dot{\psi}^2$.
(c) We drop the square of the third term, as it is proportional to $\dot{\vec{r}}^2$.

(d) The cross term between the first and the second terms is zero, as these terms are orthogonal.

(e) The cross term between the first and the third terms is zero, as these terms are orthogonal, as $\hat{e}_\theta \cdot \frac{\partial \hat{e}_\theta}{\partial \vec{r}} = \hat{e}_\phi \cdot \frac{\partial \hat{e}_\phi}{\partial \vec{r}} = 0$, and $\hat{e}_\theta \cdot \frac{\partial \hat{e}_\phi}{\partial \vec{r}} = -\hat{e}_\phi \cdot \frac{\partial \hat{e}_\theta}{\partial \vec{r}}$

(f) Let's compute the cross term between the second and the third terms

$$2\xi^2\dot{\psi}(\cos\psi\hat{e}_\theta - \sin\psi\hat{e}_\phi)\left(\sin\psi\frac{\partial\hat{e}_\theta}{\partial\vec{r}} + \cos\psi\frac{\partial\hat{e}_\phi}{\partial\vec{r}}\right)\frac{d\vec{r}}{dt} = 2\xi^2\dot{\psi}\left(-\sin^2\psi\hat{e}_\phi\frac{\partial\hat{e}_\theta}{\partial\vec{r}} + \cos^2\psi\hat{e}_\theta\frac{\partial\hat{e}_\phi}{\partial\vec{r}}\right)\frac{d\vec{r}}{dt}$$

$$= -2\xi^2\dot{\psi}\frac{\hat{e}_\phi\cdot\partial\hat{e}_\theta}{\partial\vec{r}}\left(\cos^2\psi + \sin^2\psi\right)\frac{d\vec{r}}{dt} = -2\xi^2\dot{\psi}\frac{\hat{e}_\phi\cdot\partial\hat{e}_\theta}{\partial\vec{r}}\frac{d\vec{r}}{dt}$$

where again we used $\hat{e}_\theta \cdot \frac{\partial \hat{e}_\theta}{\partial \vec{r}} = \hat{e}_\phi \cdot \frac{\partial \hat{e}_\phi}{\partial \vec{r}} = 0$ in the first line, $\hat{e}_\theta \cdot \frac{\partial \hat{e}_\phi}{\partial \vec{r}} = -\hat{e}_\phi \cdot \frac{\partial \hat{e}_\theta}{\partial \vec{r}}$ to go from the first line to the second.

Now using our relation $d\hat{e}_\theta = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi - \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{n}$ we find $\hat{e}_\phi \cdot \partial\hat{e}_\theta = -\frac{\hat{e}_\phi\cdot\partial\vec{r}}{R\tan\theta}$. Hence this cross term is

$$2\xi^2\dot{\psi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt}.$$

Collecting all terms together we get

$$\left(\dot{\vec{\xi}}\right)^2 \approx \dot{\xi}^2 + \xi^2\dot{\psi}^2 + 2\xi^2\dot{\psi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt}$$

The kinetic energy of the pendulum is given as $\frac{m\dot{\vec{\xi}}^2}{2}$.

The potential energy does not depend on ψ, so the Lagrange equation for ψ is simply $\frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}} = 0$. Taken the derivative of the Lagrangian we then get

$$\frac{\partial L}{\partial\dot{\psi}} = \frac{m}{2}\left(2\xi^2\dot{\psi} + 2\xi^2\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt}\right)$$

As ξ is fast when we take the derivative $\frac{d}{dt}$ we differentiate only ξ. Then

$$4\xi\dot{\xi}\dot{\psi} + 4\xi\dot{\xi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt} = 0,$$

or

$$\dot{\psi} = -\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt}.$$

If the position of the pendulum (θ, ϕ) changes to $(\theta + d\theta, \phi + d\phi)$, then

$$d\vec{r} = R\sin\theta d\phi\hat{e}_\phi - Rd\theta\hat{e}_\theta,$$

so $\hat{e}_\phi \cdot d\vec{r} = R\sin\theta d\phi$, and

$$\dot{\psi} = -\frac{1}{R\tan\theta}\frac{R\sin\theta d\phi}{dt} = -\cos\theta\frac{d\phi}{dt},$$

and finally,

$$d\psi = -\cos\theta d\phi.$$

So when we move the pendulum around the closed path Γ given by the function $\theta(\phi)$ the change of the angle ψ is given by

$$\Delta\psi = \oint_\Gamma (1 - \cos\theta)d\phi.$$

(If z-line does not go inside Γ, then $\oint_\Gamma d\phi = 0$, if it does go through inside of the contour, then $\oint_\Gamma d\phi = 2\pi$.)

Another way to write this is

$$\Delta\psi = \Omega_\Gamma,$$

where Ω_Γ is the solid angle "covered" by the path Γ.

LECTURE 15
Oscillations with parameters depending on time. Parametric resonance.

- Spring break. Homework due Tuesday the week after the Spring break. Exam will be out on Thursday the week after the Spring break.

15.1. Parametric resonance. Generalities.

In this lecture we consider the situation when the parameters of the oscillator depend on time and the frequency of this dependence is comparable to the frequency of the oscillator. We start from the equation

$$\ddot{x} = -\omega^2(t)x,$$

where $\omega(t)$ is a periodic function of time. The interesting case is when $\omega(t)$ is almost a constant ω_0 with a small correction which is periodic in time with period T. Then the case which we are interested in is when $2\pi/T$ is of the same order as ω_0. We are going to find the resonance conditions. Such resonance is called "parametric resonance".

First we notice, that if the initial conditions are such that $x(t=0)=0$, and $\dot{x}(t=0)=0$, then $x(t)=0$ is the solution and no resonance happens. This is very different from the case of the usual resonance — the resonance under an action of an external periodic force.

Our equation is linear second order differential equation. As such it has two linearly independent solutions. Let's assume, that we found these two linearly independent solutions $x_1(t)$ and $x_2(t)$ of the equation.

- All other solutions are just linear combinations of $x_1(t)$ and $x_2(t)$.
- If a function $x_1(t)$ is a solution, then function $x_1(t+T)$ must also be a solution, as T is a period of $\omega(t)$. It means, that the function $x_1(t+T)$ is a linear combination of functions $x_1(t)$ and $x_2(t)$. The same is true for the function $x_2(t+T)$. So we have

$$\begin{pmatrix} x_1(t+T) \\ x_2(t+T) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- We can always choose such $x_1(t)$ and $x_2(t)$ that the matrix is diagonal. In this case

$$x_1(t+T) = \mu_1 x_1(t), \qquad x_2(t+T) = \mu_2 x_2(t)$$

functions $x_1(t)$ and $x_2(t)$ as well as the numbers μ_1 and μ_2 can (and usually are) complex. μ_1 and μ_2 are the eigen values of the matrix.

- We see, that the functions $x_1(t)$ and $x_2(t)$ are multiplied by constants under the translation on one period. The most general functions that have this property are

$$x_1(t) = \mu_1^{t/T} X_1(t), \qquad x_2(t) = \mu_2^{t/T} X_2(t),$$

where $X_1(t)$, and $X_2(t)$ are periodic functions of time, $X_1(t+T) = X_1(t)$ and $X_2(t+T) = X_2(t)$.

- The numbers μ_1 and μ_2 cannot be arbitrary. In order to show that we consider the Wronskian $W(t)$.
  - The Wronskian $W(t)$ for two linearly independent solutions $x_1(t)$ and $x_2(t)$ of a linear second order differential equation is defined as

$$W(t) \equiv \dot{x}_1 x_2 - \dot{x}_2 x_1.$$

  - For our equation $\ddot{x} = -\omega^2(t)x$ we find

$$\dot{W} = \ddot{x}_1 x_2 - \ddot{x}_2 x_1 = -\omega^2(t) x_1 x_2 + \omega^2(t) x_2 x_1 = 0,$$

so

$$W(t) = \text{const}.$$

  - However, differentiating $x_{1,2}(t+T) = \mu_{1,2} x_{1,2}(t)$ with respect of t, we find $\dot{x}_{1,2}(t+T) = \mu_{1,2}\dot{x}_{1,2}(t)$, so using the definition of Wronskian we find

$$W(t+T) = \dot{x}_1(t+T)x_2(t+T) - \dot{x}_2(t+T)x_1(t+T) = \mu_1\mu_2\left(\dot{x}_1(t)x_2(t) - \dot{x}_2(t)x_1(t)\right) = \mu_1\mu_2 W(t).$$

- On one hand $W(t+T) = \mu_1\mu_2 W(t)$, on the other hand $W(t)$ must be constant and hence $W(t+T) = W(t)$. This is only possible if

$$\mu_1\mu_2 = 1.$$

- The differential equation $\ddot{x} = -\omega^2(t)x$ is real, so if $x_1(t)$ is a solution so must be $x_1^*(t)$.
  - If the function $x_1(t)$ is complex, then the function $x_1^*(t)$ is linearly independent of $x_1(t)$, so in this case our pair of linearly independent functions is $x_1(t)$ and $x_2(t) = x_1^*(t)$. In this case $\mu_2 = \mu_1^*$, and the Wronskian condition gives $|\mu_1| = |\mu_2| = 1$ — The solutions do NOT grow with time! No resonance!!
  - If the function $x_1(t)$ is real, then μ_1 must be real, then $\mu_2 = 1/\mu_1 = \mu > 1$, and

$$x_1(t) = \mu^{-t/T} X_1(t), \qquad x_2(t) = \mu^{t/T} X_2(t).$$

- We see, that in the case of real μ_1 (and μ_2) one of the solutions is unstable, it increases exponentially with time. This means, that a small initial deviation from the equilibrium will exponentially grow with time. This is the parametric resonance!

We can analyze the situation further, by considering the trace of the matrix $\text{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$. As μ_1 and μ_2 are the eigen values of this matrix we have

$$\text{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu_1 + \mu_2 = \mu + 1/\mu = 2\cosh(Ts), \qquad \mu \equiv e^{Ts}$$

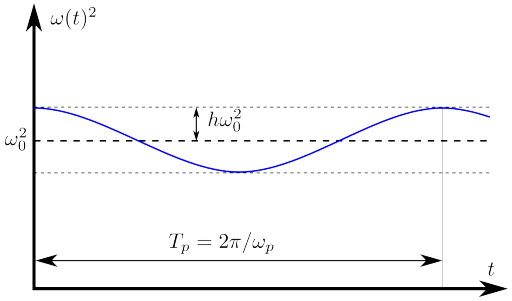
where we parametrized μ by s. The parameter s can be either real or imaginary.

It is easy to see, that if μ (and s) is real, then $\mu + 1/\mu = 2\cosh(Ts) > 2$. On the other hand, if μ is complex then s is imaginary so that then $|\mu| = 1$, and then $\mu + 1/\mu = 2\cosh(Ts) = 2\cos(T|s|) < 2$. We thus can formulate the resonance condition by

$$\text{if } a + d > 2 \text{ — resonance}$$
$$\text{if } a + d < 2 \text{ — no resonance.}$$

Moreover, in the resonance case the equation $a + d = 2\cosh(Ts)$ gives us s and hence the exponent with which the resonance solution grows, as $x_2(t) = e^{st}X_2(t)$.

15.2. Resonance.



After the general consideration we want to study an example and see how the parametric resonance actually happens.

Let's now consider the following dependence of ω on time

$$\omega^2 = \omega_0^2(1 + h\cos\omega_p t)$$

where $h \ll 1$.

- The most interesting case is when $\omega_p \sim 2\omega_0$.
- Explanation:
  - We write the equation $\ddot{x} = -\omega^2(t)x$ in the following form
  $$\ddot{x} = -\omega_0^2 x - h\omega_0^2 x\cos(\omega_p t).$$
  - Now we notice, that as $h \ll 1$ the second term in the RHS is small.
  - We then try to construct the perturbation theory and see at which ω_p it fails.
  - In order to see this, we notice that in the zeroth order in h we have an equation
  $$\ddot{x}_0 = -\omega_0^2 x_0$$
  which has a solution
  $$x_0 = \cos(\omega_0 t)$$
  - We then look for the solution in the form
  $$x = \cos(\omega_0 t) + hx_1(t) + h^2 x_2(t) + h^3 x_3(t) + \dots.$$
  where the functions $x_1(t)$, $x_2(t)$, $x_3(t)$, etc. are yet unknown functions which we want to find.
  - In order for the perturbation theory to be valid none of the functions x_1, etc must grow with time. So the perturbation theory fails when one of the functions starts to grow. Let's see at what conditions the function $x_1(t)$ grows with time.

- We substitute the perturbative solution into our equation and keep only the terms linear in h. We get
$$\ddot{x}_1 + \omega_0^2 x_1 = \omega_0^2 \cos(\omega_0 t)\cos(\omega_p t)$$
- This is the usual oscillator with a periodic force in the RHS. We see, that this periodic force has the form
$$\cos(\omega_0 t)\cos(\omega_p t) = \frac{1}{2}\cos[(\omega_p - \omega_0)t] + \frac{1}{2}\cos[(\omega_p + \omega_0)t].$$
- Notice, that if $\omega_p = 2\omega_0$, then the first term in the force is $\cos(\omega_0 t)$ — it has exactly the same frequency, as the oscillator $\ddot{x}_1 + \omega_0^2 x_1$. This is the usual resonance condition! So the solution $x_1(t)$ will grow exponentially and eventually will become larger than x_0 no matter how small h is. This means that the perturbation series brakes down.
- So in the case when $\omega_p \approx 2\omega_0$ the perturbation theory does not work and it has to be treated differently. We should expect a resonance there.

So I will take $\omega_p = 2\omega_0 + \epsilon$, where $\epsilon \ll \omega_0$. The equation of motion is
$$\ddot{x} + \omega_0^2 x[1 + h\cos(2\omega_0 + \epsilon)t] = 0$$
(Mathieu's equation) We are not exactly at the resonance. The parameter $\epsilon \ll \omega_0$ shift the system slightly off the resonance. This parameter ϵ will be called "de-tuning".

We seek the approximate solution in the form
$$x(t) = a(t)\cos[(\omega_0 + \epsilon/2)t] + b(t)\sin[(\omega_0 + \epsilon/2)t]$$
The reasons for this form are

- We expect the solution to be somewhat close to the solution of the driven oscillator, but with amplitudes a and b changing with time.
- Previous perturbative calculation showed that if we look for the solution with frequency ω, then the frequency of "driving force" is $\omega_p - \omega$, so if $\omega_p = 2\omega_0 + \epsilon$, and $\omega = \omega_0 + \epsilon/2$, then the frequency of the "driving force" $\omega_p - \omega = \omega_0 + \epsilon/2 = \omega$. So in this case the frequency of the "driving force" and the frequency of oscillations coincide! We do expect a resonance!

We need to substitute the solution in this form into our equation. We notice

- The identity
$$\cos[(\omega_0 + \epsilon/2)t]\cos[(2\omega_0 + \epsilon)t] = \frac{1}{2}\cos[3(\omega_0 + \epsilon/2)t] + \frac{1}{2}\cos[(\omega_0 + \epsilon/2)t]$$
shows that the term $x\cos[(2\omega_0+\epsilon)t]$ will have the terms with the frequency $3(\omega_0+\epsilon/2)$, and with the frequency $\omega_0 + \epsilon/2$. The terms with frequency $3(\omega_0 + \epsilon/2)$ are far away from the resonance, so we just drop them and retain only the "most dangerous" terms of frequency $\omega_0 + \epsilon/2$.
- We expect the de-tuning ϵ at which the resonance condition still holds to be of the order of h.
$$\epsilon \sim h.$$
- We assume that the functions $a(t)$ and $b(t)$ are slow functions of time and assume (and later check) that
$$\dot{a} \sim \epsilon a \quad \text{and} \quad \dot{b} \sim \epsilon b.$$

- Finally, we retain only the terms linear in h and ϵ. Notice, that if $\epsilon = h = 0$, then our approximate solution is, in fact, exact.
- In particular, this also means dropping $\ddot{a}$ and $\ddot{b}$ terms, as they are of the order of ϵ^2, but keeping $\dot{a}$ and $\dot{b}$ as they are of the order of ϵ.

The result is

$$-\omega_0\left(2\dot{a}+b\epsilon+\frac{1}{2}h\omega_0 b\right)\sin(\omega+\epsilon/2)t+\omega_0\left(2\dot{b}-a\epsilon+\frac{1}{2}h\omega_0 a\right)\cos(\omega+\epsilon/2)t=0$$

In order for this equation to be satisfied at all times the coefficients in front of $\sin(\omega+\epsilon/2)t$ and $\cos(\omega+\epsilon/2)t$ must both be zero. (sin and cos are linearly independent functions.)

So we have a pair of equations

$$2\dot{a}+b\epsilon+\frac{1}{2}h\omega_0 b=0$$
$$2\dot{b}-a\epsilon+\frac{1}{2}h\omega_0 a=0$$

These are linear first order differential equations for the functions $a(t)$ and $b(t)$ with the time independent coefficients, so we look for the solution in the form $a=a_0e^{st}$ and $b=b_0e^{st}$, then

$$\begin{array}{l}2sa_0+(\epsilon+h\omega_0/2)b_0=0,\\(-\epsilon+h\omega_0/2)a_0+2sb_0=0.\end{array}\ ,\quad\text{or}\quad\begin{pmatrix}2s & \epsilon+h\omega_0/2\\ -\epsilon+h\omega_0/2 & 2s\end{pmatrix}\begin{pmatrix}a_0\\b_0\end{pmatrix}=0$$

This is a pair of homogeneous linear equations for the numbers a_0 and b_0. In general case there is only the trivial solution $a_0=b_0=0$, unless the determinant is zero! This compatibility condition (determinant is zero) gives

$$s^2=\frac{1}{4}\left[(h\omega_0/2)^2-\epsilon^2\right].$$

- Notice, that e^{sT} is what we called μ before.
- Depending on the value of the detuning ϵ we will have s either purely imaginary or purely real.
- If $\epsilon > h\omega_0/2$, then s is imaginary and $|\mu|=|e^{sT}|=1$ – no resonance.
- The condition for the resonance is that s is real!!!

It means that the resonance happens for

$$-\frac{1}{2}h\omega_0<\epsilon<\frac{1}{2}h\omega_0$$

- The range of frequencies for the resonance depends on the amplitude h. On a swing the more you move the less precise you have to be!
- The amplification $s=\frac{1}{2}\sqrt{(h\omega_0/2)^2-\epsilon^2}$, also depends on the amplitude h and on ϵ. On a swing with the same "pumping amplitude" the more precise you tune to the correct frequency the faster your swings grow!
- Other resonances occur $\omega_0/\omega_p=n/2$, for integer n. They happen when other functions ($x_2(t)$, $x_3(t)$, etc.) in the perturbation expansion blow up. Notice, that this means that the resonance condition

  $$\omega_p=2\omega_0/n$$

  is "opposite" to the requirement for the resonance due to external force — the frequency of the force must be $n\omega_0$.

15.2.1. Adding dissipation.

What will happen if the oscillator has dissipation? We can answer this question with the following argument:

- In case of dissipation the solution acquires a decaying factor $e^{-\lambda t}$, so s should be substituted by $s - \lambda$. Then in order for the instability to occur we must have $s > \lambda$ so the range of instability is given by $\frac{1}{4}\left[(h\omega_0/2)^2 - \epsilon^2\right] > \lambda^2$:
$$-\sqrt{(h\omega_0/2)^2 - 4\lambda^2} < \epsilon < \sqrt{(h\omega_0/2)^2 - 4\lambda^2}$$
- In order for ϵ to be real we must have $h > 4\lambda/\omega_0$.
- So at finite dissipation the parametric resonance requires finite amplitude $h > 4\lambda/\omega_0$. If $h < 4\lambda/\omega_0$ there will be no resonance! On a rusty swing you cannot get a resonance if your h is too small.

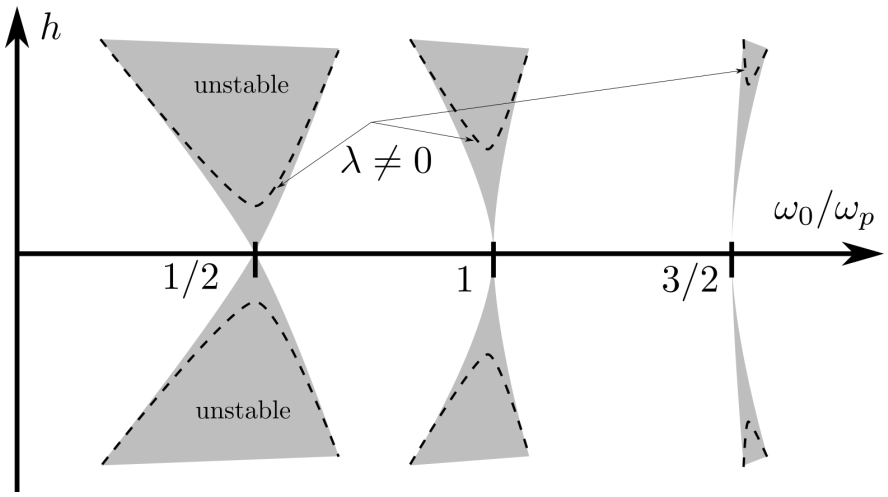


Figure 1. Parametric resonance condition.

15.2.2. Quantum particle in the periodic potential.

Let's take our equation
$$\ddot{x} + \omega_0^2 x + h\omega_0^2 x \cos(\omega_p t) = 0$$
and do the following renameing
$$x \to \psi$$
$$t \to x$$
We get
$$-\frac{\partial^2 \psi}{\partial x^2} + \left[-h\omega_0^2 \cos(\omega_p x)\right] \psi = \omega_0^2 \psi.$$
One notices, that this is stationary Schrödinger equation for a quantum mechanical particle in the periodic potential $V(x) = -h\omega_0^2 \cos(\omega_p x)$. The resonance condition then corresponds to the wave function $\psi(x)$ growing exponentially. Such finctions would be non-normalizable, so they do not correspond to quantum mechanical states. So the states are only allowed if the resonance condition is not met. As for the final amplitude h the resonance condition is met in a finite regions of ω_0 the spectrum of these states will have gaps — this is what is

called band structure. The corresponding quantum mechanical calculation is called Bloch theorem.

LECTURE 16
Oscillations of an infinite series of springs. Oscillations of a rope. Phonons.

- Physics Festival.
- Exam.
- HW.

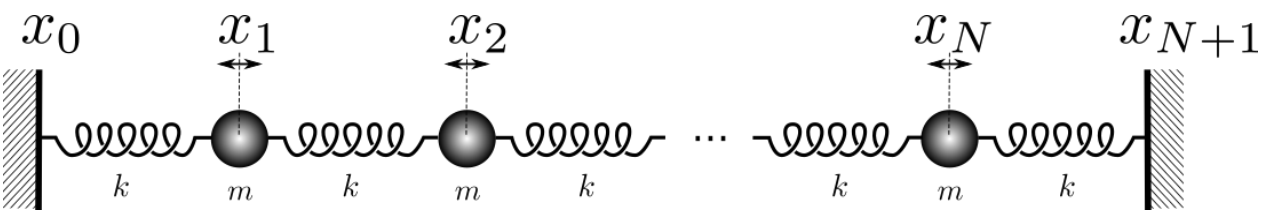
16.1. A crystal.

A monoatomic crystal is a periodic arrangements of atoms. The atoms interact with each other. We may not know the interaction exactly (we do, but it is complicated). What we do know is that at equilibrium the atoms are at certain distance from each other. This distance is called lattice constant. So when the atoms are at this distance the interaction potential energy is at minimum. If we slightly disturb the crystal the atoms will slightly shift from their equilibrium positions. If this shift is small the potential energy will be a quadratic form of all the shifts — this is just a Taylor expansion of the potential energy around the equilibrium positions of the atoms. It means that one can model the crystal by a periodic arrangement of atoms of the same mass m connected with springs of some spring strength. The springs are also arranged periodically with the same period of lattice constant.

The strength of interaction between the atoms decays strongly with the distance. So as a simplest approximation one can ignore all interactions except the interaction with the nearest neighbors.

So as the simplest model we have a $1D$ periodic arrangement of masses m connected to the nearest neighbors with the springs of strength k. As a simple exercise we will also assume that the masses can only move in the same $1D$.

16.2. Series of springs.



Consider one dimension string of N balls each of mass m connected with identical springs of spring constants k. The first and the last masses are connected by the same springs to walls. The question is what are the normal modes of such system?

- The difference between the infinite number of masses and finite, but large — zero mode.

This system has N degrees of freedom, so we must find N modes. We call x_i the displacement of the ith mass from its equilibrium position. The Lagrangian is:

$$L = \sum_{i=1}^{N} \frac{m\dot{x}_i^2}{2} - \frac{k}{2}\sum_{i=0}^{N}(x_i - x_{i+1})^2, \qquad x_0 = x_{N+1} = 0.$$

We will find the modes in two different ways.

16.2.1. First solution

This solution is more complicated and gives the normal frequencies, but not the modes. The advantage of this solution is that it requires no guesses.

The kinetic energy is $K = \sum_{i=1}^{N} \frac{m\dot{x}_i^2}{2}$ so our matrix $\hat{k}$ is simply proportional to the identity matrix

$$\hat{k} = \begin{pmatrix} m & 0 & \dots & \dots & \dots \\ 0 & m & 0 & \dots & \dots \\ \dots & 0 & m & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & m \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_N \end{pmatrix}$$

The potential energy is

$$\frac{k}{2}\sum_{i=0}^{N}(x_i - x_{i+1})^2 = \frac{k}{2}\left(2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + \cdots + 2x_{N-1}^2 - 2x_{N-1}x_N + 2x_N^2\right)$$

so our matrix $\hat{u}$ is

$$\hat{u} = \begin{pmatrix} 2k & -k & 0 & \dots & \dots \\ -k & 2k & -k & 0 & \dots \\ 0 & -k & 2k & -k & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -k & 2k \end{pmatrix}$$

The matrix $-\omega^2\hat{k} + \hat{u}$ is

$$-\omega^2\hat{k} + \hat{u} = \begin{pmatrix} -m\omega^2 + 2k & -k & 0 & \dots & \dots \\ -k & -m\omega^2 + 2k & -k & 0 & \dots \\ 0 & -k & -m\omega^2 + 2k & -k & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -k & -m\omega^2 + 2k \end{pmatrix}$$

This is $N \times N$ matrix. We need to find its determinant and then find for which ωs this determinant is zero.

Let's call the determinant of the $N \times N$ matrix D_N. First we notice that

$$D_1 = -m\omega^2 + 2k, \qquad D_2 = (-m\omega^2 + 2k)^2 - k^2.$$

Then we see

$$D_N = (-m\omega^2 + 2k)D_{N-1} - k^2 D_{N-2}.$$

This is a second order linear difference equation with constant coefficients. The solution should be of the form $D_N = a^N$. Then we have

$$a^2 = (-m\omega^2 + 2k)a - k^2, \qquad a_\pm = \frac{-m\omega^2 + 2k \pm i\sqrt{m\omega^2(4k - m\omega^2)}}{2}.$$

The two values a_+ and a_- are just complex conjugated to each other, we will call them $a_+ = a$ and $a_- = \bar{a}$. Notice, that $|a|^2 = k^2$.

As our difference equation is linear and homogeneous, the general solution is a linear combination of the two solutions which we found with arbitrary coefficients.

$$a^N \qquad \text{and} \qquad \bar{a}^N$$

However, D_N must be real, so the coefficients must also be complex conjugated.

$$D_N = Aa^{N-1} + \bar{A}\bar{a}^{N-1}$$

where A is so far arbitrary complex number.

In addition for $N = 1$ and $N = 2$ we must reproduce D_1 and D_2. Thus the general solution and the initial conditions are

$$A + \bar{A} = -m\omega^2 + 2k = a + \bar{a}, \qquad Aa + \bar{A}\bar{a} = (-m\omega^2 + 2k)^2 - k^2 = (a + \bar{a})^2 - a\bar{a} = a^2 + a\bar{a} + \bar{a}^2.$$

The solution is

$$A = \frac{a^2}{a - \bar{a}}, \qquad \bar{A} = \frac{\bar{a}^2}{\bar{a} - a}.$$

So the determinant D_N of the $N \times N$ matrix $-\omega^2\hat{k} + \hat{u}$ is

$$D_N = \frac{a^2}{a - \bar{a}}a^{N-1} - \frac{\bar{a}^2}{a - \bar{a}}\bar{a}^{N-1}, \qquad a = \frac{-m\omega^2 + 2k + i\sqrt{m\omega^2(4k - m\omega^2)}}{2}$$

Now in order to find the normal frequencies we need to solve the following equation for ω.

$$D_N = \frac{a^2}{a - \bar{a}}a^{N-1} - \frac{\bar{a}^2}{a - \bar{a}}\bar{a}^{N-1} = 0, \qquad \text{or} \qquad \left(\frac{a}{\bar{a}}\right)^{N+1} = 1.$$

As $|a|^2 = k^2$, we present a as

$$a = ke^{i\phi}, \qquad \cos\phi = \frac{-m\omega^2 + 2k}{2k},$$

then our equation reads

$$e^{2i\phi(N+1)} = 1, \qquad 2\phi(N + 1) = 2\pi n, \qquad \text{where} \quad n = 1 \dots N.$$

So we have $\phi = \frac{\pi n}{N+1}$ and thus

$$\frac{-m\omega_n^2 + 2k}{2k} = \cos\phi = \cos\frac{\pi n}{N + 1}, \qquad \omega_n^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N + 1)}.$$

- We found the normal/eigen frequencies ω_n, but we do not have the modes.

16.2.2. Second solution.

From the Lagrangian we find the equations of motion

$$\ddot{x}_j = -\frac{k}{m}(2x_j - x_{j+1} - x_{j-1}), \qquad x_0 = x_{N+1} = 0.$$

We look for the solution in the form

$$x_j = \sin(\beta j)\sin(\omega t).$$

With this form the condition $x_{j=0} = 0$ is satisfied automatically. In order to satisfy the condition $x_{j=N+1} = 0$ we must demand that β is such, that

$$\sin\beta(N+1) = 0.$$

Substituting this guess into the equation we get

$$-\omega^2\sin(\beta j) = -\frac{k}{m}\left(2\sin(\beta j) - \sin\beta(j+1) - \sin\beta(j-1)\right)$$
$$= -\frac{k}{m}\Im\left(2e^{ij\beta} - e^{i(j+1)\beta} - e^{i(j-1)\beta}\right) = -\frac{k}{m}\Im e^{ij\beta}\left(2 - e^{i\beta} - e^{-i\beta}\right) = \frac{k}{m}\Im e^{ij\beta}\left(e^{i\beta/2} - e^{-i\beta/2}\right)^2$$
$$= -4\frac{k}{m}\Im e^{ij\beta}\sin^2(\beta/2) = -4\frac{k}{m}\sin(j\beta)\sin^2(\beta/2).$$

So we have

$$\omega^2 = 4\frac{k}{m}\sin^2(\beta/2),$$

but β must be such that $\sin\beta(N+1) = 0$, so $\beta = \frac{\pi n}{N+1}$, and we have the frequencies of the normal/eigen modes numbered by n.

$$\omega_n^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N+1)}, \qquad n = 1, \ldots, N$$

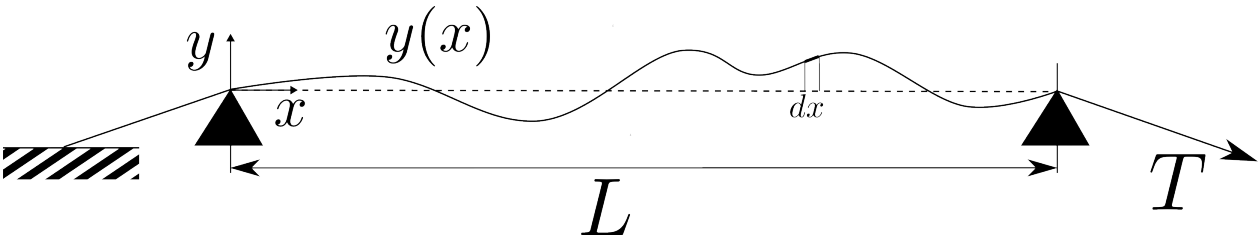
and the modes themselves

$$x_j^n = \sin\left(\frac{\pi n j}{N+1}\right)\sin(\omega_n t).$$

This formula shows the position of the ball number j in nth mode as a function of time.

- So we found both the normal/eigen frequencies ω_n and the corresponding normal/eigen modes x_j^n.
- Notice, that $\omega_1 = 2\sqrt{\frac{k}{m}}\sin\frac{\pi}{2(N+1)} \to 2\sqrt{\frac{k}{m}}\frac{\pi}{2(N+1)} \to 0$ when $N \to \infty$.
- So for any finite N the lowest frequency is not zero. However, at the limit $N \to \infty$ a zero mode emerges.

16.3. A guitar string.



Here we consider a guitar/piano/harp/violin/viola. . . string. It has a tension and it has a linear mass density. Gravity plays no role. To simplify the problem I will consider only the string in 2D.

At each moment of time t the string has a shape $y(x)$. This shape changes with time, so we need to consider the functions $y(x,t)$ as a time-dependent shape of the string.

The potential energy of a (2D) string at tension T (gravity is neglected) of shape $y(x)$ is $T\int_0^L \sqrt{1+y'^2}dx \approx \frac{T}{2}\int_0^L y'^2 dx$. The kinetic energy is $\int_0^L \frac{\rho}{2}\dot{y}^2 dx$, so the Lagrangian is

$$\mathcal{L} = \int_0^L \left(\frac{\rho}{2}\dot{y}^2 - \frac{T}{2}y'^2\right)dx, \qquad y(x=0,t) = y(x=L,t) = 0, \quad \forall t.$$

In order to find the normal modes we need to decide on the coordinates in our space of functions $y(x,t)$. We will use the standard Fourier basis $\sin kx$ and write any function as

$$y(x,t) = \sum_k A_k(t)\sin kx.$$

The constants $A_k(t)$ are the coordinates in the Fourier basis.

Notice that with this form the boundary condition $y(x=0,t)=0$ is satisfied automatically (this is why I excluded the cosines). However, in order to satisfy $y(x=L,t)=0$ the values of k must only be such that

$$\sin kL = 0.$$

Substituting this function into the Lagrangian and using

$$\int_0^L \sin(kx)\sin(k'x) = \int_0^L \cos(kx)\cos(k'x) = \frac{L}{2}\delta_{k,k'},$$

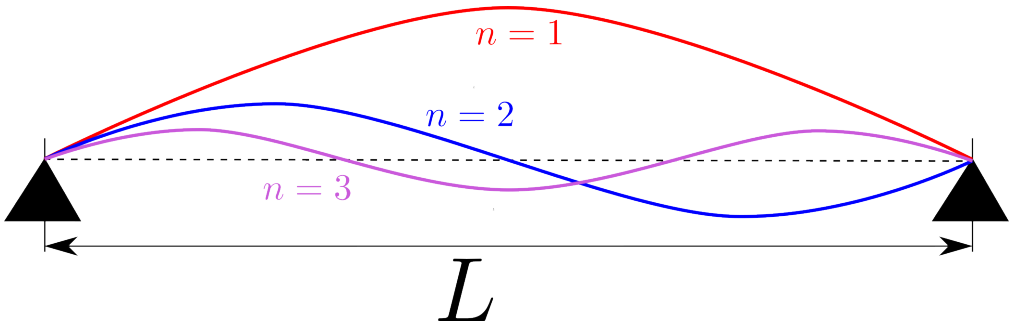
find the following form of the Lagrangian

$$\mathcal{L} = \frac{L}{2}\sum_k \left(\frac{\rho}{2}\dot{A}_k^2 - \frac{T}{2}k^2 A_k^2\right).$$

We see, that it is just a set of decoupled harmonic oscillators and k just enumerates them. The normal frequencies are

$$\omega_k^2 = \frac{T}{\rho}k^2, \qquad \omega = \sqrt{\frac{T}{\rho}}k.$$

- We also see, that the wavelength $\lambda = 2\pi/k$. So using $\omega = 2\pi f$ we find that $\lambda f = \omega/k = \sqrt{T/\rho}$. So the speed of these waves is
$$c^2 = T/\rho.$$
- The condition $\sin kL = 0$ gives the discritization of k
$$k_n = \frac{\pi}{L}n, \qquad \omega_n = \frac{\pi}{L}\sqrt{\frac{T}{\rho}}n, \qquad \lambda_n = \frac{2L}{n}$$


$n = 1$
$n = 2$
$n = 3$
L

LECTURE 17
Motion of a rigid body. Kinematics.

17.1. Kinematics.

We want to describe the a motion of a rigid body. We want to be able to write a Lagrangian for this motion. In order to do that we need to write the kinetic energy of the body.

The rigid body generally moves as a whole and rotates. During such a motion each point of the body has different velocity. The velocity of each point also changes with time. We need the kinetic energy at one (arbitrary) instant of time. In order to do that we need to split the body into infinitesimally small pieces, find the kinetic energy of each piece, and sum all these infinitesimal contributions up. So we need to find a way to describe the velocity of each point of the body at one (arbitrary) instant of time.

It turns out, that in order to do that for the rigid body we only need to know two! vectors at each instant of time: The linear velocity $\vec{V}$ of an arbitrary point of the body and the vector of angular velocity $\vec{\Omega}$.

I emphasize, that these two vectors $\vec{V}$ and $\vec{\Omega}$ depend on time. We are considering only some particular instant of time.

17.1.1. Vector of angular velocity.

We will use two different system of coordinates XYZ — fixed, or external inertial system of coordinates, and xyz the moving, or internal (non-inertial) system of coordinates which is attached to the body itself and moves with it.

- The system xyz is convenient to describe the body.
- The system XYZ is needed to describe the motion.

Before we start I want to remind that for any vector $\vec{r}$ which is fixed in the *internal* xyz frame of references and thus rotates *with* the body with the angular velocity $\vec{\Omega}$ we have

$$\dot{\vec{r}} = \vec{\Omega} \times \vec{r},$$

where $\vec{\Omega}$ is the vector of angular velocity *at this instant of time*.

- Again, I emphasize, that the vector $\vec{r}$ is fixed in the xyz frame. The time evolution written above is in XYZ frame. Vector $\vec{r}$ fixed in xyz is rotating with the body, so it is changing if viewed from the XYZ frame.

Multiplying this equation by dt — time interval — we get

$$d\vec{r} = d\vec{\phi} \times \vec{r}$$

- Notice, that ϕ is not a vector, while $d\vec{\phi}$ is.

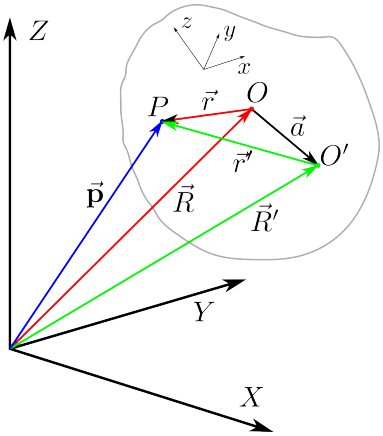


Figure 1

We want to find the velocity of an arbitrary point P of the body. This point has a position vector $\vec{\mathbf{p}}$ with respect to the external XYZ frame. The velocity of the point P is $\dot{\vec{\mathbf{p}}}$. As the body is rigid, the velocity of the point P cannot be independent from the velocities of other points of the body.

Let's $\vec{R}$ be the position vector of some arbitrary point O of a body with respect to the external XYZ frame of reference, $\vec{r}$ be the position vector of any point P of the body with respect to the point O, and $\vec{\mathbf{p}}$ the position vector of the point P with respect to the external XYZ frame of reference: $\vec{\mathbf{p}} = \vec{R} + \vec{r}$, see figure. Notice, that vector $\vec{r}$ is a vector between the two points O and P of the body, so it is fixed in the internal xyz frame. For any infinitesimal displacement $d\vec{\mathbf{p}}$ of the point P we have

$$d\vec{\mathbf{p}} = d\vec{R} + d\vec{r} = d\vec{R} + d\vec{\phi} \times \vec{r}.$$

Or dividing by dt we find the velocity of the point P as

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}, \quad \text{where} \quad \vec{v} = \frac{d\vec{\mathbf{p}}}{dt}, \quad \vec{V} = \frac{d\vec{R}}{dt}, \quad \vec{\Omega} = \frac{d\vec{\phi}}{dt}.$$

What we see here is

- $\vec{v}$ is the velocity of the point P in the external frame of reference.
- $\vec{V}$ is the velocity of the point O in the external frame of reference.
- Vector $\vec{r}$ is the position of the point P with respect to point O *in the body*.
- $\vec{\Omega}$ is the *vector* of the angular velocity.

However, as was defined above the vector $\vec{\Omega}$ may depend on the choice of the point O, as the definition used the point O. So to make it transparent we need to call this vector $\vec{\Omega}_O$ to remind us that this is vector of angular velocity defined at the point O.

- In order to call the vector of angular velocity $\vec{\Omega}_O$ a vector of angular velocity of the BODY $\vec{\Omega}$, we must prove, that the vector of angular velocity $\vec{\Omega}_O$ is independent of the choice of the point O.

In the previous calculation the point O was an arbitrary point of the body. Let's take any other point of the body O' with a position vector $\vec{R}' = \vec{R} + \vec{a}$, where $\vec{a}$ is a vector from O to O'. The vector $\vec{a}$ is fixed *in the body*. According to the previous calculation the velocity $\vec{v}$ of the point P of the body in the *external* frame XYZ is given by

$$\vec{v} = \vec{V}' + \vec{\Omega}_{O'} \times \vec{r}',$$

where $\vec{r}'$ is the vector *in the body* from the point O' to the point P, $\vec{\Omega}_{O'}$ is the vector of angular velocity measured with respect to the point O', and $\vec{V}'$ is the velocity of the point O' in the external frame XYZ.

From the picture we see that $\vec{r} = \vec{r}' + \vec{a}$. Now substituting $\vec{r} = \vec{r}' + \vec{a}$ into $\vec{v} = \vec{V} + \vec{\Omega}_O \times \vec{r}$ we get

$$\vec{v} = \vec{V} + \vec{\Omega}_O \times \vec{a} + \vec{\Omega}_O \times \vec{r}'.$$

The velocity $\vec{v}$ in both calculations is the velocity of the of the *same* point P of the body measured in the same external frame of reference XYZ. Moreover, point P is an *arbitrary* point of the body. So both calculations must give the same result for any $\vec{r}'$! We then conclude that

$$\vec{V}' = \vec{V} + \vec{\Omega}_O \times \vec{a}, \qquad \vec{\Omega}_{O'} = \vec{\Omega}_O.$$

- The last equation shows, that the vector of angular velocity is the same and does not depend on the choice of the point O. So $\vec{\Omega}$ can be called the angular velocity of the body.
- In particular, this means that one can deal with vector $\vec{\Omega}$ as with any other vector.
- Notice, that the velocity of *any* point of the body at some instant of time is completely defined by just two vectors $\vec{V}$ and $\vec{\Omega}$ at this instant of time. (The vector $\vec{r}$ simply defines which point *in the body* we are considering.)

17.1.2. Instantaneous axis of rotation.

From the equation

$$\vec{V}' = \vec{V} + \vec{\Omega} \times \vec{a}$$

we see that at any moment of time $\vec{\Omega} \cdot \vec{V} = \vec{\Omega} \cdot \vec{V}'$. So if at some instant the vectors $\vec{V}$ and $\vec{\Omega}$ are perpendicular for some choice of O, then they will be perpendicular for any other O' at the SAME instant. In this case, it is possible to find a set of points (a line) such that their velocity is zero in the *external* frame XYZ.

- It is important, to recognize, that the velocity of this set of points is zero *at this instant* of time. In the next instant the velocity of this set generally will not be zero.

Let's assume, that we have chosen some point of the body O. Its velocity at this instant is $\vec{V}$, the angular velocity at this instant is $\vec{\Omega}$. We want to find ALL points of the body O', such that their velocity is zero $\vec{V}' = 0$ at the same instant of time. So, in order to find this set of points O' we need to solve the equation for $\vec{a}$

$$\vec{V} + \vec{\Omega} \times \vec{a} = 0.$$

These are three inhomogeneous linear equations for the components of the vector $\vec{a}$.

If we take the dot product of the above equation with $\vec{\Omega}$, we find that $\vec{\Omega} \cdot \vec{V} = 0$, this is the requirement that the above equation has a solution (rhs must be orthogonal to the eigen vectors with zero eigen values).

If we take the cross product of the above equation with $\vec{V}$ we find

$$0 = \vec{V} \times [\vec{\Omega} \times \vec{a}] = \vec{\Omega}(\vec{V} \cdot \vec{a}) - \vec{a}(\vec{V} \cdot \vec{\Omega}) = \vec{\Omega}(\vec{V} \cdot \vec{a}),$$

as $\vec{\Omega} \neq 0$ we must have $\vec{V} \cdot \vec{a} = 0$. This means, that the vector $\vec{a}$ has zero component along the vector $\vec{V}$. So the vector $\vec{a}$ has only components along vectors $\vec{\Omega}$ and $\vec{\Omega} \times \vec{V}$. So it must have the form

$$\vec{a} = \alpha \vec{\Omega} \times \vec{V} + \beta \vec{\Omega}.$$

Substituting this form of $\vec{a}$ into the main equation we get

$$0 = \vec{V} + \alpha \vec{\Omega} \times [\vec{\Omega} \times \vec{V}] + \beta \vec{\Omega} \times \vec{\Omega} = \vec{V} + \alpha \vec{\Omega}(\vec{\Omega} \cdot \vec{V}) - \alpha \vec{V} \Omega^2 = \vec{V}\left(1 - \alpha \Omega^2\right)$$

So we have $\alpha \Omega^2 = 1$ and β can be any number.

$$\vec{a} = \frac{\vec{\Omega} \times \vec{V}}{\Omega^2} + \beta \vec{\Omega}.$$

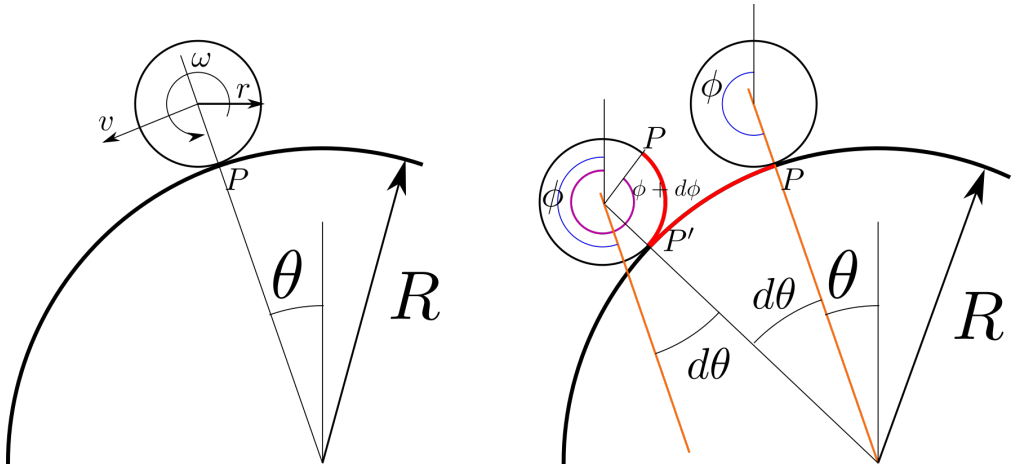
This equation describes a line (as β is arbitrary) parallel to $\vec{\Omega}$. This line/axis (it may be outside of the body) has zero velocity *at this particular instant of time*. The distribution of the velocities of the rest of the body *at this particular instant of time* looks simply like rotation of the body around this axis with the angular velocity Ω.

This line is called "instantaneous axis of rotation". (In the general case (when $\vec{\Omega} \cdot \vec{V} \neq 0$) the body is rotating around the instantaneous axis and moves parallel to this axis.)

- At any *instant* the velocities of all points of a body can be described as rotation with respect to some *instantaneous* axis of rotation plus translation along this axis.
- In general both the magnitude and the direction of $\vec{\Omega}$ are changing with time, so the "*instantaneous axis of rotation*" is also changing with time!
- If one finds two points which have zero velocity at this *instant*, then the *instantaneous* axis of rotation is the straight line through these two points. This way one figures out the direction of the vector $\vec{\Omega}$ *at this instant of time*.

17.1.3. Examples of the use of *instantaneous* axis of rotation

- First example: cylinder on cylinder, $2D$, no slipping.



Let's consider a cylinder of radius r moving without slipping on top of the cylinder of radius R (for shortness I will call the cylinders cylinder r and cylinder R). We want to use the angle θ as a coordinate. We thus need to express the angular velocity ω of the cylinder r through $\dot{\theta}$. There are two ways to approach this problem:

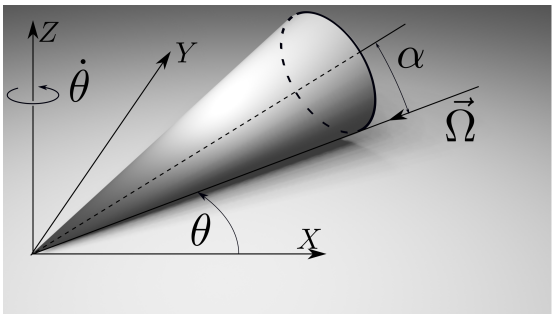
- The easy way. Look at the figure on the left. Due to non-slipping condition the point P of the cylinder r is not moving. So the *instantaneous* axis of rotation goes through this point. Then the velocity of the center of the cylinder r is $v = \omega r$. The same velocity is also $v = \dot{\theta}(R + r)$ as the center is moving on a circle of radius $R + r$. So we have $v = \omega r = \dot{\theta}(R + r)$, or

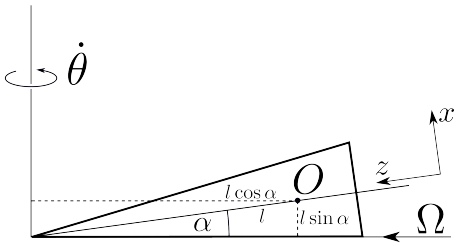
$$\omega = \dot{\theta}\frac{R + r}{r}.$$

- The hard way. Look at the figure on the right. What is depicted is the change of the position of the cylinder r after time interval dt. The angle θ is changed by $d\theta$. The cylinder r also rotated by the angle $d\phi$. It is *very* important to fix the direction from which the angle ϕ is measured! On the figure it is measured

from the vertical line. The non-slipping condition means that the length of the red arc PP' on the cylinder r must be equal to the length of the red arc PP' on the cylinder R. The red arc on the cylinder R has length $Rd\theta$. In order to find the length of the red arc on the cylinder r we draw the orange line parallel to the initial orange line. Now from the figure it is clear, that the angle of the red arc on the cylinder r is $d\phi - d\theta$, so that the length of this arc is $r(d\phi - d\theta)$. So no slipping condition gives $Rd\theta = r(d\phi - d\theta)$. Or $d\phi = d\theta\frac{R+r}{r}$. Dividing this by dt we get the previous result.

- Second example: a cone on a plane, $3D$.





A cone of angle α is moving on a plane without slipping. We want to use the angle θ as the coordinate. So we want to find the vector $\vec\Omega$ (both, direction and the magnitude) in terms of θ and $\dot\theta$.

- The instantaneous axis of rotation is the line where the cone touches the plane.
- So the vector Ω has the direction shown on the figure.
- Take any point O on the axis of the cone. Let's say this point is a distance l from the top of the cone.
- On the one hand the velocity of this point is $\Omega l\sin\alpha$. On the other hand the velocity of this very point is $\dot\theta l\cos\alpha$.
- As these must be equal we conclude

$$\Omega = \dot\theta\frac{\cos\alpha}{\sin\alpha}.$$

- And thus the components of the vector $\vec\Omega$ in the external frame XYZ AT THIS INSTANT OF TIME (instant at which the angle is θ) are

$$\Omega_X = -\dot\theta\frac{\cos\alpha}{\sin\alpha}\cos\theta, \qquad \Omega_Y = -\dot\theta\frac{\cos\alpha}{\sin\alpha}\sin\theta, \qquad \Omega_Z = 0.$$

- The components of the vector $\vec\Omega$ in the internal frame xyz AT THIS INSTANT OF TIME (instant at which the axis x is in the vertical plane, as the cone rotates and xyz is attached to the cone, in the next instant the axis x will NOT be in the vertical plane) are

$$\Omega_x = \dot\theta\frac{\cos\alpha}{\sin\alpha}\sin\alpha = \dot\theta\cos\alpha, \qquad \Omega_y = 0, \qquad \Omega_z = \dot\theta\frac{\cos^2\alpha}{\sin\alpha}.$$

- Notice that the vector $\vec\Omega$ changes with time in both external XYZ and internal xyz frames!!!!

17.2. Kinetic energy.

The total kinetic energy of a body is the sum of the kinetic energies of its parts. Lets take the origin of the moving system of coordinates to be in the center of mass. Then

$$K = \frac{1}{2}\sum m_\alpha v_\alpha^2 = \frac{1}{2}\sum m_\alpha \left(\vec{V} + \vec{\Omega}\times\vec{r}_\alpha\right)^2 = \frac{1}{2}\sum m_\alpha \vec{V}^2 + \sum m_\alpha \vec{V}\cdot\vec{\Omega}\times\vec{r}_\alpha + \frac{1}{2}\sum m_\alpha\left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$
$$= \frac{MV^2}{2} + \vec{V}\cdot\vec{\Omega}\times\sum m_\alpha\vec{r}_\alpha + \frac{1}{2}\sum m_\alpha\left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$

For the center of mass $\sum m_\alpha\vec{r}_\alpha = 0$ and we have

$$K = \frac{MV^2}{2} + \frac{1}{2}\sum m_\alpha\left(\vec{\Omega}^2\vec{r}_\alpha^2 - (\vec{\Omega}\cdot\vec{r}_\alpha)^2\right) = \frac{MV^2}{2} + \frac{I_{ij}\Omega^i\Omega^j}{2},$$

where

$$I_{ij} = \sum m_\alpha\left(\delta_{ij}\vec{r}_\alpha^2 - r_\alpha^i r_\alpha^j\right).$$

I_{ij} is the tensor of inertia. This tensor is symmetric and positive definite.

- The combination $I_{ij}\Omega^i\Omega^j$ does not depend on which coordinate system you use as long as you use THE SAME coordinate system for both $\hat{I}$ and $\vec{\Omega}$.
- We typically compute $\hat{I}$ in the internal coordinate system xyz. if this is what we use, then the vector Ω must also be computed in the internal frame xyz.
- If we use $\vec{\Omega}$ in the external frame XYZ, then the tensor of inertia $\hat{I}$ must also be expressed in the external frame XYZ. On then must keep in mind, that as the body rotates $\hat{I}$ will change with time.

17.3. Angular momentum

The origin is at the center of mass. So we have

$$\vec{M} = \sum m_\alpha\vec{r}_\alpha\times\vec{v}_\alpha = \left(\sum m_\alpha\vec{r}_\alpha\right)\times\vec{V} + \sum m_\alpha\vec{r}_\alpha\times(\vec{\Omega}\times\vec{r}_\alpha) = \sum m_\alpha\left(r_\alpha^2\vec{\Omega} - \vec{r}_\alpha(\vec{r}_\alpha\cdot\vec{\Omega})\right)$$

Writing this in components we have

$$M_i = \sum m_\alpha\left(\delta_{ij}\vec{r}_\alpha^2 - r_\alpha^i r_\alpha^j\right)\Omega^j$$

or

$$M_i = I_{ij}\Omega^j.$$

- In general the direction of angular momentum $\vec{M}$ and the direction of the angular velocity $\vec{\Omega}$ do not coincide.

17.4. Tensor of inertia.

Tensor of inertia is a symmetric tensor of rank two. As any such tensor it can be reduced to a diagonal form by an appropriate choice of the moving axes. Such axes are called the principal axes of inertia. The diagonal components I_1, I_2, and I_3 are called the principal moments of inertia.

- Notice, that these axes are "attached" to the body and thus rotate with the body.

In this axes the kinetic energy is simply

$$K = \frac{I_1\Omega_1^2}{2} + \frac{I_2\Omega_2^2}{2} + \frac{I_3\Omega_3^2}{2}.$$

(a) If all three principal moments of inertia are different, then the body is called "asymmetrical top".
(b) If two of the moments coincide and the third is different, then it is called "symmetrical top".
(c) If all three coincide, then it is "spherical top".

For any plane figure if z is perpendicular to the plane, then $I_1 = \sum m_\alpha y_\alpha^2$, $I_2 = \sum m_\alpha x_\alpha^2$, and $I_3 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = I_1 + I_2$. If symmetry demands that $I_1 = I_2$, then $\frac{1}{2} I_3 = I_1$. Example: a disk, a square.

If the body is a line, then (if z is along the line) $I_1 = I_2$, and $I_3 = 0$. Such system is called "rotator".

LECTURE 18
Rotation of a symmetric top. Euler angles.

18.1. Tensor of inertia.

Last semester we introduces tensor of inertia

$$I_{ij} = \sum m_\alpha \left(\delta_{ij}\vec{r}_\alpha^2 - r_\alpha^i r_\alpha^j\right) = \int_V \left(\delta_{ij}\vec{r}^2 - r^i r^j\right)\rho(\vec{r})dV$$

The tensor of inertia is the *property of the body*. It is computed using the internal (attached to the body) frame of references xyz. For any object one can compute its tensor of inertia. If we take the origin of the internal xyz frame in the center of mass of a body and compute the tensor of inertia I_{ij} of that body in this frame, then the kinetic energy of the body as viewed from the external XYZ frame is given by

$$K = \frac{mV^2}{2} + \frac{I_{ij}\Omega_i\Omega_j}{2}$$

(Einstein notation is assumed) where m is the mass of the body, V is the velocity of the center of mass with respect to XYZ frame, and Ω_i are the components of the vector of angular velocity $\vec{\Omega}$ in the internal xyz frame.

We can also compute the vector of angular momentum $\vec{M}$ of the body

$$M_i = I_{ij}\Omega_j$$

(Einstein notation is assumed) In this formula both Ω_i and M_i are the components of the corresponding vectors in the internal xyz frame.

- Notice, that the directions of vectors $\vec{\Omega}$ and $\vec{M}$ in general do not coincide!

Tensor of inertia is a symmetric tensor of rank two. As any such tensor it can be reduced to a diagonal form by an appropriate choice of the axes of the internal (moving) frame xyz. Such axes are called the principal axes of inertia. The diagonal components I_{xx}, I_{yy}, and I_{zz} (we will also denote them as I_x, I_y, and I_z) are called the principal moments of inertia.

- Notice, that these axes are "attached" to the body and thus rotate and move with the body.

In these specific axes the kinetic energy is simply

$$K = \frac{mV^2}{2} + \frac{I_x\Omega_x^2}{2} + \frac{I_y\Omega_y^2}{2} + \frac{I_z\Omega_z^2}{2},$$

and the angular momentum

$$M_x = I_x\Omega_x, \qquad M_y = I_y\Omega_y, \qquad M_z = I_z\Omega_z, \quad \text{or} \quad \vec{M} = I_x\Omega_x\hat{x} + I_y\Omega_y\hat{y} + I_z\Omega_z\hat{z}$$

where Ω_x, Ω_y, and Ω_z as well as M_x, M_y, and M_z are projections of the vector $\vec{\Omega}$ and $\vec{M}$ on axes of *internal* frame xyz.

Terminology:

(a) If all three principal moments of inertia are different, then the body is called "asymmetrical top".

(b) If two of the moments coincide and the third is different, then it is called "symmetrical top". If $I_x = I_y$, then the axes x and y can be chosen arbitrarily (but orthogonal to each other) in the plane orthogonal to the axis z! This leads to great simplifications!

(c) If all three principle moments coincide, then the body is called "spherical top". (A uniform sphere is just one example of such a top.)

18.2. Kinematics.

We consider a body freely (no forces applied) moving and rotating. As there are no forces the center of mass must move with a constant velocity. So if we consider the motion in the external frame of references which moves with the same velocity, in this frame of references the center of mass will not be moving, or its velocity will be zero. It means that at every instant the instantaneous axis of rotation will go through the center of mass.

So if there are no forces one can always consider just a rotation of a body around an axis which goes through the center of mass.

As there are no forces acting on the body we can immediately state that

- Kinetic energy is conserved.
- Angular momentum is conserved.

Let's consider the motion in more detail in some specific cases.

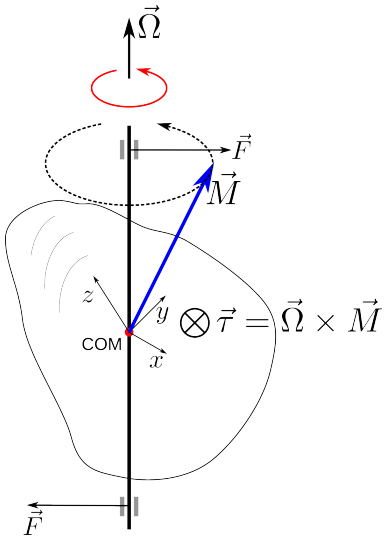


Figure 1

- Spherical top: $I_x = I_y = I_z = I$ and $I^{ij} = I\delta^{ij}$ for *any* choice of internal axes. So for any direction we have $\vec{M} = I\vec{\Omega}$. So if $\vec{M}$ is conserved, then $\vec{\Omega}$ is a constant. The body simply rotates around the axis given by $\vec{M}$.
- Arbitrary top rotating around one of its principal axes, say x, Then $\vec{\Omega} = \Omega\vec{x}$ and $\vec{M} = I_x\Omega\hat{x}$. The axis x is not moving $\dot{\hat{x}} = \vec{\Omega}\times\hat{x} = 0$. So if $\vec{M}$ is conserved, then $\vec{\Omega}$ is conserved, and the body keeps rotating around the axis x which is also not changing.
- Arbitrary top rotating around a FIXED arbitrary axes which goes through the center of mass, fig. 1. As the axis is fixed it is also an instantaneous axis of rotation at any instant. The directions of angular velocity vector $\vec{\Omega}$ is along this axis. The angular momentum vector $\vec{M} = \hat{I}\vec{\Omega}$ is NOT along the axis of rotation. Let's chose the INTERNAL axes along the principle axes of inertia. In that INTERNAL frame xyz The components of the angular momentum are $M_x = I_{xx}\Omega_x$, $M_y = I_{yy}\Omega_y$, and $M_z = I_{zz}\Omega_z$ and they are

constant. So the vector $\vec{M}$ is a constant vector in the INTERNAL frame. So looking from the EXTERNAL frame we have $\dot{\vec{M}} = \vec{\Omega} \times \vec{M}$. Or angular momentum is NOT conserved. It means that there MUST be torque acting on the top $\vec{\tau} = \dot{\vec{M}} = \vec{\Omega} \times \vec{M}$. This torque is perpendicular to the vector Ω and thus perpendicular to the axis of rotation. The power supplied by such a torque $P = \vec{\tau} \cdot \vec{\Omega} = 0$, so the kinetic energy is conserved. This torque is applied to the top by whoever/whatever keeps the axis fixed. As COM is not moving, the total force on the top must be zero. So whatever holds the axis fixed must apply forces as shown on the figure (the direction of the forces at any instant is in the plane defined by the axis and $\vec{M}$, this plane is rotating, so the direction of the forces is changing.)

- Symmetric top with no forces applied to it. Free symmetric top.

The last case requires some work.

18.3. Free symmetric top.

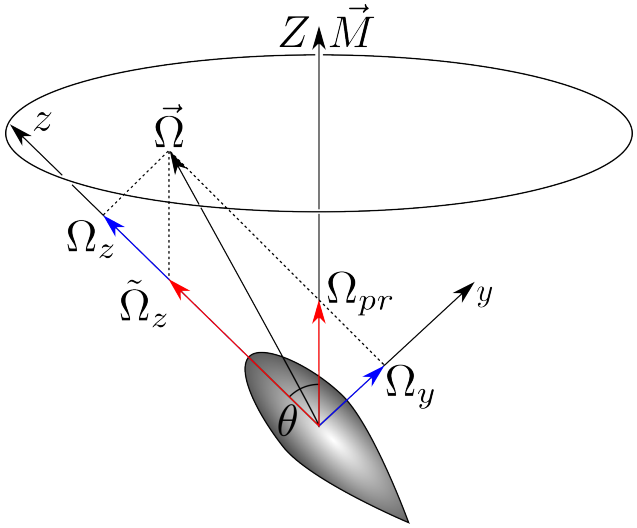


Figure 2

A symmetric top is moving and no forces are applied to it. The center of mass of the top is moving at a constant velocity. So we switch to the frame which moves with the same velocity. In this frame the center of mass is not moving, but the top is rotating. As there are no forces acting on the top both kinetic energy K and angular momentum vector $\vec{M}$ are conserved.

We consider a free rotation of a symmetric top $I_x = I_y \neq I_z$, where x, y, and z are the principal (internal) axes. The direction of the angular momentum does not coincide with the direction of any principle axes. Let's say, that the angle between $\vec{M}$ and the moving axes z at some instant is θ. As the top is symmetric, we can chose the axes x and y to be any orthogonal axes in the plane perpendicular to the z axes. We chose the axis y to be in the plane with the two vectors $\vec{M}$ and $\hat{z}$.

During the motion the total angular momentum (both magnitude and direction!) is conserved. So we chose the external axis Z to be along the direction of $\vec{M}$. The unit vector $\hat{Z}$ then is given by $\hat{Z} = \vec{M}/M$, where $M = |\vec{M}|$.

The whole motion can be thought of as two rotations one the rotation of the body around the axes z and the other, called precession, is the rotation of the axis z around the axis Z.

We can think of vector $\vec{\Omega}$ in two different ways

$$\vec{\Omega} = \hat{z}\Omega_z + \hat{y}\Omega_y \tag{18.1}$$

$$\vec{\Omega} = \frac{\vec{M}}{M}\Omega_{pr} + \hat{z}\tilde{\Omega}_z \tag{18.2}$$

- Notice, that Ω_z is not the same as $\tilde{\Omega}_z$.

and angular momentum

$$\vec{M} = \Omega_y I_y \hat{y} + \Omega_z I_z \hat{z}, \tag{18.3}$$

Multiplying (18.3) by $\hat{z}$ (at this instant of time) we get ($\hat{z} \cdot \vec{M} = M_z = M\cos\theta$)

$$\Omega_z = \frac{M_z}{I_z} = \frac{M}{I_z}\cos\theta.$$

In order to find the angular velocity of precession we multiply (18.2) and (18.3) by $\hat{y}$ and get

$$\Omega_y = \frac{\hat{y}\cdot\vec{M}}{M}\Omega_{pr} \qquad \text{and} \qquad \hat{y}\cdot\vec{M} = I_y\Omega_y$$

- From the second equation above we see, that $\Omega_y = \frac{M_y}{I_y} = \frac{M}{I_y}\sin\theta$.

Substituting $\hat{y}\cdot\vec{M}$ from the second equation into the first

$$\Omega_y = \frac{I_y}{M}\Omega_y\Omega_{pr}.$$

This equation has two solutions $\Omega_y = 0$ – which corresponds to $\vec{M} \parallel \hat{z}$, or if $\Omega_y \neq 0$

$$\Omega_{pr} = \frac{M}{I_y}.$$

Also multiplying (18.2) by $\hat{z}$ we find

$$\tilde{\Omega}_z = \Omega_z - \frac{M_z}{M}\Omega_{pr} = M\left(\frac{1}{I_z} - \frac{1}{I_y}\right)\cos\theta.$$

- Notice, that as there are no forces/torques acting on the top, the vector of angular momentum $\vec{M}$ is constant.
- However, despite there are no forces/torques acting on the top, the vector of angular velocity $\vec{\Omega}$ is NOT constant. It rotates around the axis Z with Ω_{pr}.

18.4. Euler's angles

The purpose of this section is to construct and learn to work with a specific set of generalized coordinates suitable to describe rotations.

A rotation is the change of the orientation of a body with time. The orientation of a rigid body is described by three angles. There are different ways to parametrize orientation. Here we consider one particular way which is called Euler's angles.

The external (fixed) coordinates are XYZ, the internal coordinates are xyz. The plane xy intersects the plane XY along the line ON called the line of nodes (see figure 3).

The angle θ is the angle between the Z and z axes. The angle ϕ is the angle between the X axes and the line of nodes, and the angle ψ is the angle between the x axes and the line of nodes.

Let's consider what we need to do to orient a body according to the given Euler angles.

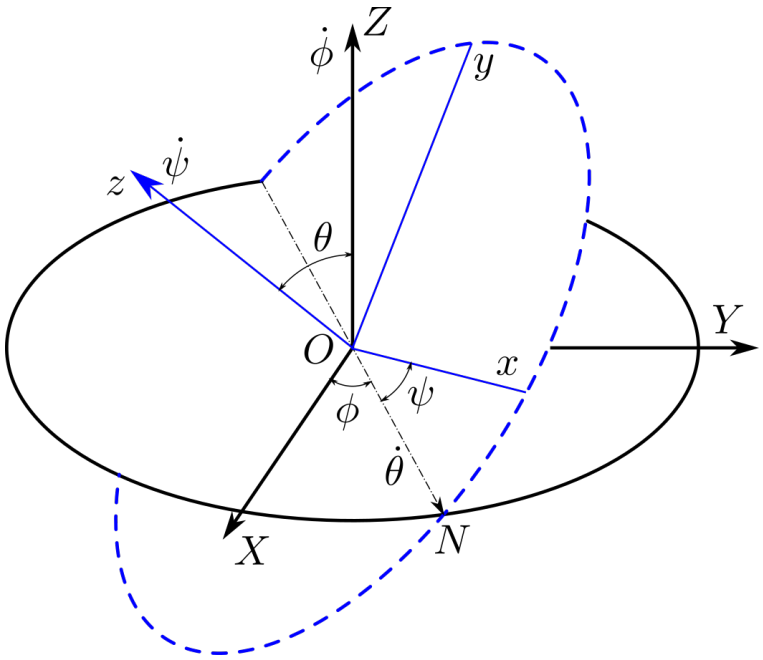


Figure 3

Initially the axes XYZ and xyz coincide. Let's denote $\hat{O}_{\hat{\xi}}(\alpha)$ a rotation around a unit vector $\hat{\xi}$ on the angle α. Then in order to get the orientation on the picture we need to perform three separate rotations

$$\hat{O}_{\hat{z}}(\psi) \circ \hat{O}_{\hat{Z}}(\phi) \circ \hat{O}_{\hat{X}}(\theta)|\text{initial state}\rangle$$

What it means is that one needs (read above from right to left — it is an operator. Also remember, that xyz is attached to the body)

- In the initial state the frames XYZ and xyz coincide.
- Rotate the body around axis X by angle θ.

  After that the axes Z and z have an angle θ between them. The axes X, x, and the line of nodes coincide.
- Rotate the body around axis Z by angle ϕ.

  After that the axes Z and z still have an angle θ between them. The axis x and the line of nodes still coincide, but there is an angle ϕ between the axis X and the line of nodes.
- Rotate the body around axis z by angle ψ.

  After that the axes Z and z still have an angle θ between them. The angle between the axis X and the line of nodes is still ϕ. But there is an angle ψ between the axis x and the line of nodes.

The angle θ is from 0 to π, the ϕ and ψ angles are from 0 to 2π.

$$\theta \in [0, \pi], \qquad \phi \in [0, 2\pi], \qquad \psi \in [0, 2\pi].$$

Another way of looking at Euler angles is to realize, that the angles θ and ϕ are our normal angles of spherical coordinates. They determine the direction of the axis z. After that we rotate the body around the axis z by angle ψ.

These three Euler angles θ, ϕ, and ψ completely determine the orientation of a body.

- So in order to describe rotations Euler's angles are good generalized coordinates.

By knowing all three angles θ, ϕ, and ψ we completely know the orientation of the body and can compute the potential energy of the body (if it depends on the angles).

18.4.1. Angular velocity and kinetic energy through Euler's angles.

In order to get the full use of the Euler's angles we also need to express the kinetic energy through our generalized coordinates (θ, ϕ, and ψ) and their generalized velocities ($\dot\theta$, $\dot\phi$, and $\dot\psi$).

If xyz are the principle axes of inertia, then the kinetic energy is simply $K = \frac{1}{2}I_x\Omega_x^2 + \frac{1}{2}I_y\Omega_y^2 + \frac{1}{2}I_z\Omega_z^2$. Where Ω_x, Ω_y, and Ω_z are projections of the vector $\vec\Omega$ on the *internal* axes xyz. So we need to express Ω_x, Ω_y, and Ω_z through θ, ϕ, and ψ and their velocities $\dot\theta$, $\dot\phi$, and $\dot\psi$.

Let's consider changing the angle θ to $\theta + d\theta$. It is clear, that we need to rotate the body by an angle $d\theta$ around the line of nodes ON. So the vector $d\vec\theta$ has a length $d\theta$ and is directed along the line of nodes ON. Then the vector $\dot{\vec\theta}$ has a length $\dot\theta$ and is directed along the line of nodes ON.

The same way the vector $d\vec\phi$ has a length $d\phi$ and is directed along Z. Then the vector $\dot{\vec\phi}$ has the length $\dot\phi$ and is directed along Z. The vector $d\vec\psi$ has a length $d\psi$ and is directed along z. Then the vector $\dot{\vec\psi}$ has the length $\dot\psi$ and is directed along z.

It is important to realize, that vector $\vec\Omega$ is simply the sum

$$\vec\Omega = \dot{\vec\theta} + \dot{\vec\phi} + \dot{\vec\psi}.$$

So in order to find the components Ω_x, Ω_y and Ω_z we need to write the components of the vectors $\dot{\vec\theta}$, $\dot{\vec\phi}$, and $\dot{\vec\psi}$ in the internal frame xyz.
Look at the figure 3:

(a) The **vector** $\dot{\vec\theta}$ has length $\dot\theta$ and is directed along the line of nodes ON, so its components along $\hat x$, $\hat y$, and $\hat z$ are (I'll write the z component first just for convenience)

$$\begin{aligned}\dot\theta_z &= 0\\ \dot\theta_x &= \dot\theta\cos\psi\\ \dot\theta_y &= -\dot\theta\sin\psi\end{aligned}$$

(b) The **vector** $\dot{\vec\phi}$ has length $\dot\phi$ and is directed along the Z direction, so its component along $\hat z$, $\hat x$, and $\hat y$.

$$\begin{aligned}\dot\phi_z &= \dot\phi\cos\theta\\ \dot\phi_x &= \dot\phi\sin\theta\sin\psi\\ \dot\phi_y &= \dot\phi\sin\theta\cos\psi\end{aligned}$$

(c) The **vector** $\dot{\vec\psi}$ has length $\dot\psi$ and is directed along the z direction, so its component along $\hat z$, $\hat x$, and $\hat y$.

$$\begin{aligned}\dot\psi_z &= \dot\psi\\ \dot\psi_x &= 0\\ \dot\psi_y &= 0\end{aligned}$$

We now collect all angular velocities along each axis as $\Omega_x = \dot\theta_x + \dot\phi_x + \dot\psi_x$ etc. and find

$$\begin{aligned}\Omega_x &= \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi\\ \Omega_y &= -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi\\ \Omega_z &= \dot\phi\cos\theta + \dot\psi\end{aligned} \tag{18.4}$$

These equations allow us:

- To express the kinetic energy $K = \frac{1}{2}I_x\Omega_x^2 + \frac{1}{2}I_y\Omega_y^2 + \frac{1}{2}I_z\Omega_z^2$ in terms of the coordinates θ, ϕ, ψ and generalized velocities $\dot\theta$, $\dot\phi$, $\dot\psi$. Or
- first solve problem in the moving system of coordinates, find Ω_x, Ω_y, and Ω_z, and then calculate $\dot\theta$, $\dot\phi$, and $\dot\psi$.

18.5. Free symmetric top again.

Consider the symmetric top again $I_y = I_x \neq I_z$. We take Z to be the direction of the angular momentum. The axis z is along the z principle axis of inertia of the top. We can take the axis x coincide with the line of nodes. Then $\psi = 0$ (but $\dot\psi \neq 0$!), and from (18.4) we have

$$\begin{aligned}\Omega_x &= \dot\theta\\ \Omega_y &= \dot\phi\sin\theta\\ \Omega_z &= \dot\phi\cos\theta + \dot\psi\end{aligned}$$

Looking at the figure 2 we identify $\Omega_{pr} = \dot\phi$ and $\tilde\Omega_z = \dot\psi$.

The components of the angular momentum are

$$\begin{aligned}M_x &= I_x\Omega_x = I_x\dot\theta\\ M_y &= I_y\Omega_y = I_y\dot\phi\sin\theta\\ M_z &= I_z\Omega_z\end{aligned}$$

On the other hand at this instant (look at the figure 2)

$$\begin{aligned}M_x &= 0\\ M_y &= M\sin\theta\\ M_z &= M\cos\theta\end{aligned}$$

Comparing those we find

$$\dot\theta = 0, \qquad \Omega_{pr} = \dot\phi = \frac{M}{I_y}, \qquad \Omega_z = \frac{M}{I_z}\cos\theta.$$

Now using $\dot\psi = \Omega_z - \dot\phi\cos\theta$ and the results for $\dot\phi$ and for Ω_z we find

$$\tilde\Omega_z = \dot\psi = M\left(\frac{1}{I_z} - \frac{1}{I_y}\right)\cos\theta.$$

18.6. Free symmetric top motion viewed from the top itself.

Previously we described the motion of the free top from the outside observer. The top spins around it's internal z axis with the frequency $\tilde\Omega_z$ and the z axis precesses around Z axis with frequency Ω_{pr}.

Now if we are sitting on the spinning top. What we see is the top is, naturally, motionless, but the world around us is rotating with angular velocity Ω around some axis. Moreover, as we will discuss later, this axis around which the world rotates is itself rotates around our z axis with some particular frequency.

LECTURE 19
Symmetric top in gravitational field.

- Physics Festival.

19.1. What we have learned so far.

- A rigid body is characterized by the tensor of inertia — 3×3 symmetric positive definite matrix.
- This matrix can be diagonalized. This procedure will give us the principle axes of inertia x, y, z (these axes are the property of the body, they are "attached" to the body) and the principle moments of inertia I_x, I_y, and I_z.
- The rotation is described by the vector of angular velocity $\vec{\Omega}$.
- At any moment of time this vector $\vec{\Omega}$ can be projected on the principle axes of the body. We then obtain the components Ω_x, Ω_y, and Ω_z of the vector $\vec{\Omega}$ in the *internal* frame x, y, z.
- We then can compute the kinetic energy of the body

$$K = \frac{I_x \Omega_x^2}{2} + \frac{I_y \Omega_y^2}{2} + \frac{I_z \Omega_z^2}{2}.$$

- We can also compute the components of the angular momentum $M_x = I_x \Omega_x$, $M_y = I_y \Omega_y$ and $M_z = I_z \Omega_z$ in the *internal* frame x, y, z.

One way to describe the rotation of the rigid body is to have the Euler angles as functions of time $\theta(t)$, $\phi(t)$, $\psi(t)$. If we know the motion (coordinates as functions of time) of a point like particle of mass m we know the force that acts on it $\vec{F} = m\vec{a}$, the same way if we know $\theta(t)$, $\phi(t)$, $\psi(t)$, we can compute all the torques, but the procedure is more convoluted.

- If we know $\theta(t)$, $\phi(t)$, $\psi(t)$ with respect to an external frame X, Y, Z, then we can compute $\Omega_x(t)$, $\Omega_y(t)$, and $\Omega_z(t)$ — the components of the vector $\vec{\Omega}$ in the *internal* frame x, y, z — at the same moment of time

$$\begin{aligned}
\Omega_x &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\
\Omega_y &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\
\Omega_z &= \dot{\phi}\cos\theta + \dot{\psi}.
\end{aligned}$$

- We then can compute $M_x(t)$, $M_y(t)$ and $M_z(t)$ — the components of the angular momentum $\vec{M}$ in the *internal* frame x, y, z at the same moment of time.

- As we know the orientation of the internal frame xyz with respect to the external frame XYZ (this orientation is given by the Euler angles) at time t, we can find the components of the angular momentum $\vec{M}$ in the *external* (inertial) frame XYZ.
- This way we will know the vector $\vec{M}(t)$ in the *external* (inertial) frame XYZ.
- The torque acting on the body is then given by

$$\vec{\tau} = \dot{\vec{M}}.$$

- It is very important, that one must take the time derivative in the *external* (inertial) frame XYZ.

19.2. Symmetric top in gravitational field.

We want to consider the motion of the symmetric top ($I_x = I_y \neq I_z$) whose lowest point is fixed in the gravitation field g pointing down.

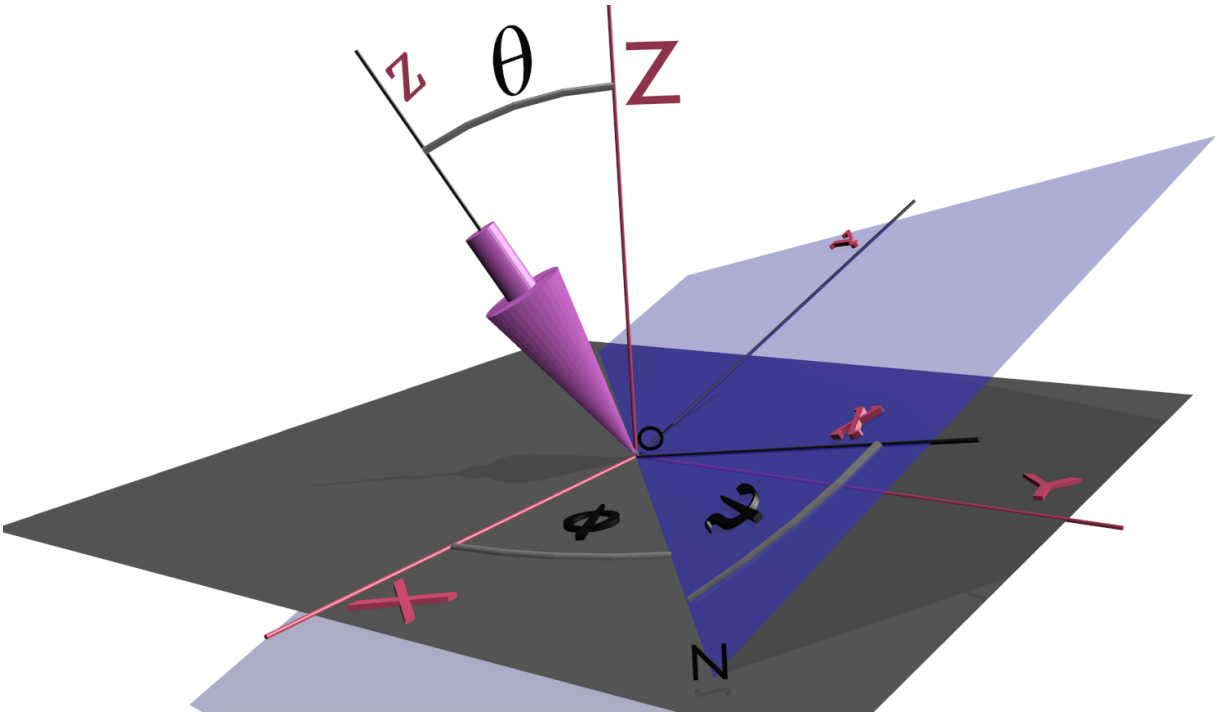


Figure 1

19.2.1. The Lagrangian.

We call the fixed point O.

- As point O is fixed, the instantaneous axis of rotation goes through it.

It then makes sense to chose the coordinate frames as shown in the figure with the origin at point O.

The Euler angles θ, ϕ, and ψ fully describe the orientation of the top. The angles are unconstrained and change $0 < \theta < \pi$, $0 < \psi, \phi < 2\pi$. So the Euler angles are good generalized coordinates

- The line of nodes is an intersection between XY and xy planes. It is the line ON on the figure.

Instead of defining the tensor of inertia with respect to the center of mass, we now need to define it with respect to the point O. The principal axes through this point are parallel to the ones through the center of mass. The principal moment I_z does not change under such shift, the principal moment with respect to the axes x and y become $I = I_x + ml^2$, where l is the distance from the point O to the center of mass.

Now we can simply use the formulas derived in the previous lecture to express the components of the vector $\vec{\Omega}$ in the internal xyz frames of reference through the our generalized coordinates (Euler angles) and their velocities.

$$\begin{aligned}\Omega_x &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi\\ \Omega_y &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi\\ \Omega_z &= \dot{\phi}\cos\theta + \dot{\psi}\end{aligned}$$

The kinetic energy of the symmetric top is

$$K = \frac{I_z}{2}\Omega_z^2 + \frac{I}{2}\left(\Omega_x^2 + \Omega_y^2\right) = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

- Notice that the kinetic energy is greatly simplified due to the fact that the principle moments of inertia along x and y equal to each other.
- It is because of that symmetry the kinetic energy only depends on the combination $\Omega_x^2 + \Omega_y^2$.
- Which leads to the fact, that the kinetic energy does not depend on ψ. (It does depend on $\dot{\psi}$.)

The potential energy is simply $mgl\cos\theta$, so the Lagrangian is

$$L = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta.$$

19.2.2. The solution.

In principle we can write the equations of motion from the Lagrangian. These will be three second order nonlinear differential equations. These equations are needed to be supplemented by six initial conditions:

$$\begin{aligned}\theta(t=0) &= \theta_0, \qquad & \dot{\theta}(t=0) &= \dot{\theta}_0\\ \phi(t=0) &= \phi_0, \qquad & \dot{\phi}(t=0) &= \dot{\phi}_0\\ \psi(t=0) &= \psi_0, \qquad & \dot{\psi}(t=0) &= \dot{\psi}_0\end{aligned}$$

However, there is a simpler way!

We see that the Lagrangian does not depend on ϕ and on ψ. Notice the origin of this independence.

- The independence of the Lagrangian of ϕ comes from the rotational invariance with respect to rotations around Z axis. This is a general situation.
- However, the independence of the Lagrangian of ψ comes from the rotational invariance with respect to rotations around z axis. This is only correct for a symmetric top!

The corresponding momenta $M_Z = \frac{\partial L}{\partial \dot{\phi}}$ and $M_3 = \frac{\partial L}{\partial \dot{\psi}}$ are conserved.

$$M_3 = \frac{\partial L}{\partial \dot{\psi}} = I_z(\dot{\psi} + \dot{\phi}\cos\theta), \qquad M_Z = \frac{\partial L}{\partial \dot{\phi}} = (I\sin^2\theta + I_z\cos^2\theta)\dot{\phi} + I_z\dot{\psi}\cos\theta.$$

It is instructive to see what symmetries are responsible for these conservation laws.

- The conservation of M_Z is due to the symmetry of the system under the rotation around Z axis.
- The conservation of M_3 is due to the symmetry of the system under the rotation around z axis – it only exists for the symmetric top!

So the fact that the top is symmetric provides us with an additional conservation law!

The Lagrangian has no explicit time dependence, so the energy is also conserved

$$E = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta.$$

- The energy conservation is due to time translation invariance.

As usual:

- All three M_Z, M_3, and E are conserved, so we can find them from the initial conditions using their definitions and the initial values of θ_0, ϕ_0, ψ_0 and $\dot{\theta}_0$, $\dot{\phi}_0$, $\dot{\psi}_0$

We have three unknown functions $\theta(t)$, $\phi(t)$, and $\psi(t)$. All equations of motion are second order differential equations. So the total rank of the equations is $2 \times 3 = 6$. However, we have three conserved quantities, this decreases the total rank by three. As the result the total rank is $6 - 3 = 3$. So we will have three first order differential equations.

- I again want to emphasize, that we have one extra conservation law because the top is symmetric. Without this symmetry the total rank of equations would be $6-2 = 4$ (total angular momentum and total energy are still conserved) So we would have two first order non-linear differential equations and one second order nonlinear differential equation.

The equations for M_Z and M_3 can be considered as two linear equations for two velocities $\dot{\phi}$ and $\dot{\psi}$. We can solve these equations and express $\dot{\phi}$ and $\dot{\psi}$ through M_Z, M_3, and θ.

$$\dot{\phi} = \frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$
$$\dot{\psi} = \frac{M_3}{I_z} - \cos\theta\frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$

We then substitute the values of the $\dot{\phi}$ and $\dot{\psi}$ into the expression for the energy. As there is no dependence on ϕ and ψ anywhere, the resulting energy depends only on θ and $\dot{\theta}$!!!! Moreover, the only place where $\dot{\theta}$ appears in energy is the term $\frac{1}{2}I\dot{\theta}^2$, so the energy can be written as

$$E' = \frac{1}{2}I\dot{\theta}^2 + U_{eff}(\theta),$$

where

$$E' = E - \frac{M_3^2}{2I_z} - mgl, \qquad U_{eff}(\theta) = \frac{(M_Z - M_3\cos\theta)^2}{2I\sin^2\theta} - mgl(1-\cos\theta).$$

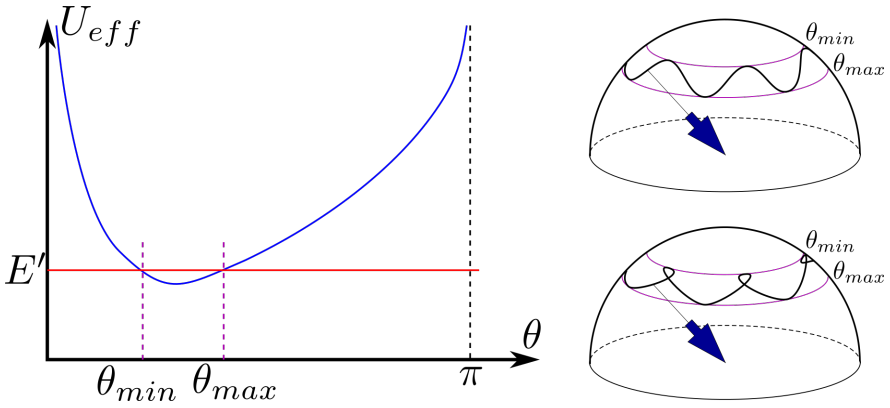


Figure 2. On the left the sketch of the effective potential energy $U_{eff}(\theta)$ is shown. On the right the nutations are show for two cases: when $\dot{\phi}$ does not change sign (top), and when $\dot{\phi}$ does change sign (bottom).

This is the standard equation for a $1D$ motion which we have solved before!!! So we get

$$t = \sqrt{\frac{I}{2}} \int_{\theta_0}^{\theta(t)} \frac{d\theta'}{\sqrt{E' - U_{eff}(\theta')}}.$$

where θ_0 is given by the initial condition $\theta(t = 0) = \theta_0$. The integral written above is an elliptic integral. Taking this integral (and inverting) we will find $\theta(t)$. Knowing this function we will find $\dot{\phi}(t)$ and $\dot{\psi}(t)$. Integrating these velocities over time (and using the initial conditions $\phi(t = 0) = \phi_0$ and $\psi(t = 0) = \psi_0$) we will find $\phi(t)$ and $\psi(t)$. So we solve the problem.

The result will be expresses in some elliptic functions. However, analyzing, the motion as we usually analyze the $1D$ motion reveals all the features of the solution without much work.

For $M_Z \neq M_3$ the effective potential energy goes to infinity when $\theta \to 0, \pi$. So for any E' there will be two turning points θ_{min} and θ_{max} — the solutions of the equation $U_{eff}(\theta) = E'$. The function $\theta(t)$ oscillates between θ_{min} and θ_{max}. These oscillations generally are NOT harmonic. These oscillations (look at the figure to see what they mean) are called *nutations*. As $\dot{\phi} = \frac{M_Z - M_3 \cos\theta}{I \sin^2\theta}$ the motion depends on whether $M_Z - M_3 \cos\theta$ changes sign in between θ_{min} and θ_{max}.

19.2.3. Stability of rotation around Z axis.

We can find a condition for the stable rotation about the Z axes. For such rotation $M_3 = M_Z = M$, so the effective potential energy is

$$U_{eff} = \frac{M^2}{2I} \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} - 2mgl \sin^2(\theta/2) \approx \left(\frac{M^2}{8I} - \frac{1}{2} mgl \right) \theta^2,$$

where the last is correct for small θ. We see, that the rotation is stable if $M^2 > 4Imgl$, or, as $M = I_z \Omega_z$

$$\Omega_z^2 > \frac{4Imgl}{I_z^2}.$$

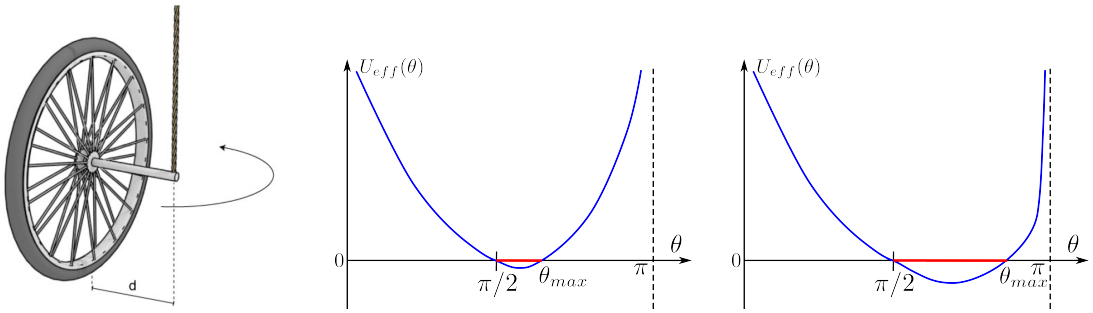


Figure 3. First panel: Spinning wheel demo. Second and third panels are sketches of $U_{eff}(\theta)$ for the fast and slow spinning wheel.

19.2.4. Spinning wheel

Let's now consider the standard physics demo shown on the figure. We have a symmetric top spinning around its z axis and supported by only one point off the center on the axis.

As we expect θ to be close to $\pi/2$ we will use the equations in the following form

$$E' = \frac{1}{2}I\dot{\theta}^2 + U_{eff}(\theta), \qquad E' = E - \frac{M_3^2}{2I_z}, \qquad U_{eff}(\theta) = \frac{(M_Z - M_3\cos\theta)^2}{2I\sin^2\theta} + mgl\cos\theta.$$

(These are the same equations as before, I simply added mgl to both sides.)

The initial conditions — this is how we start the motion — are

$$\theta_0 = \pi/2, \quad \dot{\theta}_0 = 0, \qquad \dot{\phi}_0 = 0, \qquad \dot{\psi}_0 = \Omega_0$$

(The initial values of ϕ and ψ will not matter, they simply define the initial position of the top in horizontal plane and in its own plane.)

Using these initial values and the definitions of M_Z, M_3, and E we find

$$M_Z = 0, \qquad M_3 = I_z\Omega_0, \qquad E = \frac{I_z\Omega_0^2}{2}.$$

and

$$E' = 0, \qquad U_{eff}(\theta) = \frac{M_3^2}{2I}\cot^2\theta + mgl\cos\theta.$$

Notice, that for all values of parameters $U_{eff}(\pi/2) = 0$. The equation $U_{eff}(\theta) = 0$ has another solution

$$\cos\theta_{max} = -\left[\left(\frac{M_3^2}{4Imgl}\right)^2 + 1\right] + \frac{M_3^2}{4Imgl}$$

We see, that during the motion θ will be confined to the interval $[\pi/2, \theta_{max}]$

19.2.4.1. *Fast spinning.* Let's first consider the case of the fast spinning. What it means is the coefficient in front of $\cot^2\theta$ is much larger than the coefficient in front of $\cos\theta$.

$$\frac{M_3^2}{2I} \gg mgl, \quad \text{or} \quad \Omega_0^2 \gg \frac{2Imgl}{I_z^2}.$$

In this case the potential energy has a form shown on the second panel on the figure. .

For fast spinning we expect θ_{max} to be close to $\pi/2$. So let's denote $\theta = \pi/2 + \alpha$ and use the Taylor expansion of U_{eff} for small α up to the second order.

$$U_{eff} \approx \frac{M_3^2}{2I}\alpha^2 - mgl\alpha.$$

Now solving $U_{eff} = 0$ we find that α is in the interval $[0, \alpha_{max}]$, where $\alpha_{max} = \frac{2Imgl}{I_z^2\Omega_0^2} \ll 1$. Moreover, as U_{eff} is quadratic in α the motion $\alpha(t)$ will be a simple harmonic motion with frequency

$$\omega^2 = \frac{1}{I}\left.\frac{\partial^2 U_{eff}}{\partial\alpha^2}\right|_{\alpha=\alpha_{eq}} = \frac{M_3^2}{I^2}, \qquad \text{or} \qquad \omega = \frac{I_z}{I}\Omega_0$$

, with the amplitude $\alpha_{max}/2$ about a point $\alpha_{eq} = \alpha_{max}/2$ so (the initial conditions $\alpha(t = 0) = 0$, $\dot\alpha(t = 0) = 0$)

$$\alpha(t) \approx \frac{\alpha_{max}}{2}(1 - \cos(\omega t)).$$

We now can compute $\dot\phi$ and $\dot\psi$. We use $\theta = \pi/2 + \alpha(t)$ and Taylor expansion for small α.

$$\dot\phi(t) \approx \frac{M_3}{I}\alpha(t) \approx \frac{mgl}{I_z\Omega_0}(1 - \cos(\omega t))$$
$$\dot\psi = \Omega_0 + \frac{M_3}{I}\alpha^2(t).$$

Notice, $\dot\phi$ is NOT constant. However, $\omega \sim \Omega_0$ is large, so the oscillations are fast. If we average $\dot\phi$ over time, we get (average of $\cos(\omega t)$ is zero)

$$\bar{\dot\phi} = \frac{mgl}{I_z\Omega_0}.$$

This is exactly the result you might have obtained in the introductory class.

19.2.4.2. *Slow spinning.* One should ask what happens if we do not spin the wheel that fast. In the limiting case when we do not spin it at all, it should fall down and oscillate as a simple pendulum.

For the slow spinning top, $M_3/2I$ will be small, and U_{eff} will look like shown on the third panel of the figure. θ will change over the large interval, so Taylor expansion and approximation by SHM is not valid. However, we see right away, that as M_3 decreases, θ_{max} will be getting closer and closer to π — the direction vertically down

$$\theta_{max} \approx \pi - \frac{M_3}{\sqrt{2Imgl}}.$$

For any nonzero Ω_0 it will never be exactly down, but will always miss this point deviating slightly to the side as $\dot\phi \neq 0$.

LECTURE 20

Self study. Rolling coin. An example of the rigid body dynamics.

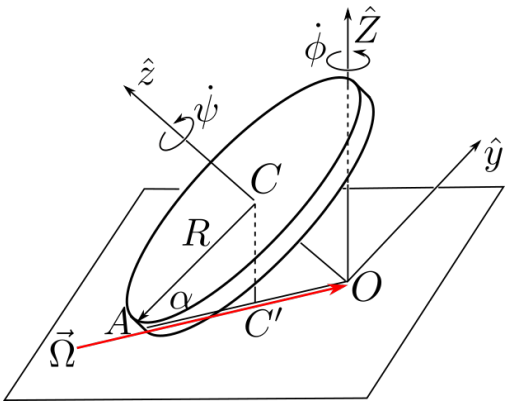


Figure 1

In this lecture we consider the dynamics of the following object:

A uniform thin disc (a coin) of mass m and radius R rolls without slipping on a horizontal plane. The disc makes an angle α with the plane, the center of the disc C moves around the horizontal circle with a constant speed v. The axis of symmetry of the disc CO intersects the horizontal plane in the point O. The point O is stationary.

- Intuitively clear, that this problem is overdetermined. We want to see why.

First, some geometrical facts

$$|OC| = R\tan\alpha, \qquad |OA| = \frac{R}{\cos\alpha}, \qquad |OC'| = |OC|\sin\alpha = R\frac{\sin^2\alpha}{\cos\alpha}, \qquad |CC'| = R\sin\alpha.$$

This is a symmetric top, so we can chose the internal x and y axes anywhere in the plane orthogonal to the internal z axis.

We chose the internal system of coordinates xyz as shown on the figure. In this system the principal moments of inertia are

$$I_z = \frac{1}{2}mR^2, \qquad I_y = I_x = \frac{1}{4}mR^2 + m|OC|^2 = mR^2\left(\frac{1}{4} + \tan^2\alpha\right)$$

20.1. Kinematics.

Simple way. According to the problem statement the points O and A are stationary at this instant of time. So the are on the *instantaneous* axis of rotation. It means that the vector $\vec{\Omega}$ is along this axis.

The point C has a velocity v. For any point $\vec{r}$ of a rotating body the velocity is $\vec{v} = \vec{\Omega}\times\vec{r}$. So we see, that $v = \Omega|CC'|$, or

$$\Omega = \frac{v}{R}\frac{1}{\sin\alpha}.$$

So we know both the direction and the magnitude of the vector $\vec{\Omega}$.

In the internal system of coordinates xyz we then have

$$\Omega_z = -\Omega\sin\alpha = -\frac{v}{R}, \qquad \Omega_y = \Omega\cos\alpha = \frac{v}{R}\frac{\cos\alpha}{\sin\alpha}, \qquad \Omega_x = 0.$$

Euler angles. We can find the same result from the Euler angles. As this is symmetric top, we can set $\psi = 0$, but we need to know θ, $\dot\theta$, $\dot\phi$, and $\dot\psi$. According to the figure the Euler angle $\theta = \alpha$, so $\dot\theta = 0$. Our equation for Ω_x, Ω_y, and Ω_z read

$$\begin{aligned}\Omega_x &= 0\\ \Omega_y &= \dot\phi\sin\alpha\\ \Omega_z &= \dot\phi\cos\alpha + \dot\psi\end{aligned}$$

The vector $\vec\Omega = \Omega_x\hat x + \Omega_y\hat y + \Omega_z\hat z$.

We know the velocities of points C and A: $\vec v_C = v\hat x$ and $\vec v_A = 0$ — this is no-slipping condition. According to the general formula $\dot{\vec r} = \vec\Omega\times\vec r$ we find $\vec v_C = \vec\Omega\times\vec{OC}$ and $\vec v_A = \vec\Omega\times\vec{OA}$. Using $\vec{OC} = |OC|\hat z$ and $\vec{OA} = |OC|\hat z - R\hat y$ we compute

$$\vec v_C = \left(\hat y\dot\phi\sin\alpha + \hat z(\dot\phi\cos\alpha + \dot\psi)\right)\times|OC|\hat z = \hat x|OC|\dot\phi\sin\alpha = v\hat x$$

$$\vec v_A = \left(\hat y\dot\phi\sin\alpha + \hat z(\dot\phi\cos\alpha + \dot\psi)\right)\times(|OC|\hat z - R\hat y) = \hat x\left(|OC|\dot\phi\sin\alpha + R(\dot\phi\cos\alpha + \dot\psi)\right) = 0$$

From the first equation we find

$$\dot\phi = \frac{v}{|OC|\sin\alpha} = \frac{v}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

From the second equation (remember, this is our no-slipping condition) we find

$$\dot\psi = -\dot\phi\left(\frac{|OC|}{R}\sin\alpha + \cos\alpha\right) = -\frac{\dot\phi}{\cos\alpha} = -\frac{v}{R}\frac{1}{\sin^2\alpha}$$

(the $-$ sign is important here!)

Using these relations for Ω_x, Ω_y, and Ω_z we find

$$\begin{aligned}\Omega_x &= 0\\ \Omega_y &= \frac{v}{R}\frac{\cos\alpha}{\sin\alpha}\\ \Omega_z &= -\frac{v}{R}\end{aligned}$$

These are exactly the results we got earlier.

20.2. Dynamics.

The main dynamic equations are $\frac{d\vec M}{dt} = \vec\tau$ and $\vec F = m\vec a$. It is very important to recognize that these equations must be written in the external/inertial frame of reference.

Let's start with the angular momentum.

In the internal system of coordinates the angular momentum at this instant of time is

$$\vec M = I_z\Omega_z\hat z + I_y\Omega_y\hat y.$$

From the external point of view, this is a vector of constant magnitude which rotates around the $\hat Z$ axis with angular velocity $\dot\phi$. So we write

$$\dot{\vec M} = \dot\phi\hat Z\times\vec M.$$

Now the expression $\hat{Z} \times \vec{M}$ has no differentiation over time, it is simply the vector product of two vectors. So it can be computed in any frame of references. We compute the RHS at this instant of time

$$\dot{\vec{M}} = \dot{\phi}\hat{Z} \times (I_z\Omega_z\hat{z} + I_y\Omega_y\hat{y}) = \hat{x}\dot{\phi}\left(-I_y\Omega_y\cos\alpha + I_z\Omega_z\sin\alpha\right)$$

Using our kinematic relations we get

$$\dot{\vec{M}} = -\frac{v^2}{R^2}\frac{\cos\alpha}{\sin^3\alpha}\left(I_z\sin^2\alpha + I_y\cos^2\alpha\right)\hat{x}.$$

There are three forces that act on the coin: the gravity mg applied to the point C, pointing down; the normal force N applied to the point A and pointing up; and the friction force F applied to the point A and pointing towards point O. As the center of mass does not move in the Z direction, the normal force and the gravity must compensate each other, so $N = mg$ (N is up).

We want to compute the total torque with respect to point O acting on the coin at this instant of time. The torque of a force $\vec{F}$ applied at point $\vec{r}$ is $\vec{\tau} = \vec{r} \times \vec{F}$. So the torque of the friction force is zero. The torque of the gravity is $\vec{\tau}_g = -\vec{OC} \times mg\hat{Z} = \hat{x}|OC'|mg = \hat{x}Rmg\frac{\sin^2\alpha}{\cos\alpha}$. The torque of the normal force is $\vec{\tau}_N = -mg|OA|\hat{x}$. So the total torque is

$$\vec{\tau} = Rmg\left(\frac{\sin^2\alpha}{\cos\alpha} - \frac{1}{\cos\alpha}\right)\hat{x} = -mgR\cos\alpha\hat{x}.$$

Notice, that this result would be much easier to obtain if we simply computed the torques with respect to point A, but this is not a trivial statement, as point A is not inertial.

Thus we have

$$\frac{v^2}{R^2}\frac{\cos\alpha}{\sin^3\alpha}\left(I_z\sin^2\alpha + I_y\cos^2\alpha\right) = mgR\cos\alpha$$

Substituting here the values of I_y and I_z we get

$$\frac{1}{4}\frac{v^2}{R}\frac{\cos\alpha}{\sin^3\alpha}\left(1 + 5\sin^2\alpha\right) = g\cos\alpha.$$

- Notice, that $\alpha = \pi/2$ (or $\cos\alpha = 0$) is a solution for any v and R – as expected.

For $\cos\alpha \neq 0$ we get

$$\frac{1}{4}\frac{v^2}{R}\left(1 + 5\sin^2\alpha\right) = g\sin^3\alpha.$$

- So v, R, and α cannot be arbitrary!!!

20.3. Friction force.

The center of mass of the coin moves around the circle of radius $|OC'|$ with velocity v, so its acceleration is $\frac{v^2}{|OC'|} = \frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}$ The force that provides this acceleration is the friction force, so

$$F = m\frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

However, this force cannot be larger than $\mu N = \mu Mg$, so we have

$$\mu g > \frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

Using the previous result $\frac{v^2}{Rg} = \frac{4\sin^3\alpha}{1+5\sin^2\alpha}$ we get

$$\frac{4\cos\alpha\sin\alpha}{1+5\sin^2\alpha} < \mu$$

20.4. Lagrangian approach to the rolling coin.

This part of the lecture is for self-study/fun.

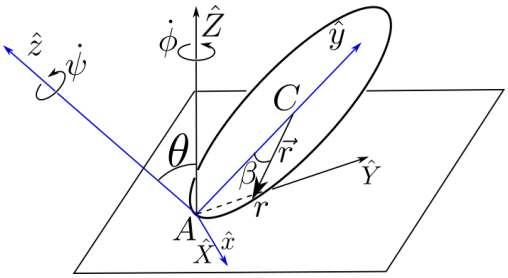


Figure 2

In order to study the full Lagrangian dynamics of the coin we first must assume, that the non-slipping condition is valid at all times – otherwise we will have dissipation and the Lagrangian method would not work. Second we should not assume that the angle θ (or α) is constant in time, so the kinetic energy term will have $\dot{\theta}$.

As point A has zero velocity I want to use it as the coordinate origin (the instantaneous axis of rotation goes through it). So instead of the figure 1 I will use figure 2.

If a coin has the principle moment of inertia around the perpendicular axes through the center of mass I (for a uniform disc it is $I = mR^2/2$) then for our internal axes

$$I_x = I/2 + mR^2$$
$$I_y = I/2$$
$$I_z = I + mR^2$$

I will assume, that at this particular point of time $\psi = 0$ and $\phi = 0$, but naturally $\dot{\psi} \neq 0$, and $\dot{\phi} \neq 0$ (putting $\phi = 0$ has no consequences, but it simplifies the figure.) According to our relations

$$\Omega_x = \dot{\theta}$$
$$\Omega_y = \dot{\phi}\sin\theta$$
$$\Omega_z = \dot{\phi}\sin\theta + \dot{\psi}$$

The most tricky and nontrivial part is to use the non-slipping condition. Naively it is automatically satisfied, as the point A is the origin then if we compute its velocity it will be zero. This is very misleading.

Lets compute the velocity of a point r (see figure) on the rim. This point is given by the vector $R\hat{y} + \vec{r}$ — see figure. The velocity is

$$\vec{v}_r = \vec{\Omega} \times (R\hat{y} + \vec{r})$$

The vector $\vec{r} = -R\cos\beta\hat{y} + R\sin\beta\hat{x}$, so we have

$$\vec{v}_r = (\Omega_x\hat{x} + \Omega_y\hat{y} + \Omega_z\hat{z}) \times (R(1-\cos\beta)\hat{y} + R\sin\beta\hat{x})$$
$$= R\left[\dot{\theta}(1-\cos\beta) - \dot{\phi}\sin\theta\sin\beta\right]\hat{z} - R(1-\cos\beta)(\dot{\phi}\cos\theta + \dot{\psi})\hat{x} + R\sin\beta(\dot{\phi}\cos\theta + \dot{\psi})\hat{y}$$

Now I want to project this velocity on the external XYZ axes, by simply $v_X = \hat{X} \cdot \vec{v}_r$ etc. The result is

$$v_Z = R\dot{\theta}(1-\cos\beta)\cos\theta + R\dot{\psi}\sin\beta\sin\theta$$
$$v_X = R(1-\cos\beta)(\dot{\phi}\cos\theta + \dot{\psi})$$
$$v_Y = -R\dot{\theta}(1-\cos\beta) + R\sin\beta\left[\dot{\phi} + \dot{\psi}\cos\theta\right]$$

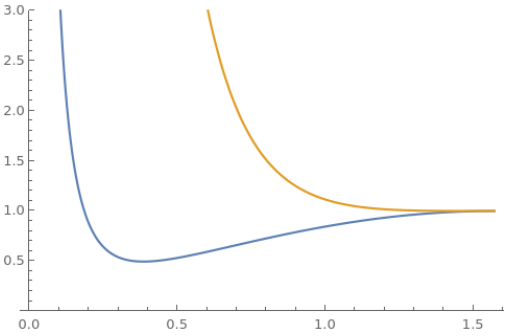


Figure 3

Let's consider very small β. No slipping means that the coin does not scraping on the plane. It means that the projection of the velocity of a point on a rim which is very close to A on the plane must not have a component of the order of β at small β. This requirement gives

$$\dot{\phi} + \dot{\psi}\cos\theta = 0$$

Notice, that it is the same relation that we had before! Using $\dot{\psi} = -\frac{\dot{\phi}}{\cos\theta}$ we find

$$\Omega_x = \dot{\theta}$$
$$\Omega_y = \dot{\phi}\sin\theta$$
$$\Omega_z = -\dot{\phi}\frac{\sin^2\theta}{\cos\theta}$$

- Notice, that $\frac{-\Omega_z}{\Omega_y} = \tan\theta$. So the vector $\vec{\Omega}$ is lying in the table's plane.

Now we can write the kinetic energy. And as the potential energy is simply $mgR\sin\theta$ we get

$$L = \frac{1}{2}\left[\dot{\phi}^2\left((I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}\right)\sin^2\theta + \dot{\theta}^2\left(\frac{I}{2} + mR^2\right)\right] - mgR\sin\theta.$$

This Lagrangian does not depend on ϕ! It also conserves energy. So we have two conserved quantity

$$M = \frac{\partial L}{\partial\dot{\phi}} = \dot{\phi}\left((I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}\right)\sin^2\theta$$
$$E = \frac{1}{2}\left[\dot{\phi}^2\left((I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}\right) + \dot{\theta}^2\left(\frac{I}{2} + mR^2\right)\right] + mgR\sin\theta.$$

Expressing $\dot{\phi}$ from the first equation

$$\dot{\phi} = \frac{1}{\sin^2\theta}\frac{M}{(I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}}$$

and substituting it to the second, we find

$$E = \frac{1}{2}\dot{\theta}^2\left(\frac{I}{2} + mR^2\right) + \frac{1}{2}\frac{1}{\sin^2\theta}\frac{M^2}{(I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}} + mgR\sin\theta.$$

This is just $1D$ motion in the effective potential

$$U_{eff}(\theta) = \frac{1}{2}\frac{1}{\sin^2\theta}\frac{M^2}{(I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}} + mgR\sin\theta$$

One can simplify it a bit if one uses $I = mR^2/2$. In this case

$$U_{eff}(\theta) = \frac{2M^2}{mR^2}\frac{\cos^2\theta}{\sin^2\theta}\frac{1}{1+5\sin^2\theta} + mgR\sin\theta$$

A plot of this function for small M (blue for $2M^2/m^2gR^3 = 3$) and for large M (orange for $2M^2/m^2gR^3 = 1/30$) is shown on figure 3. One can clearly see, that for large initial M the coin will straighten up, and for small it will wobble (provided the energy in the right range.)

If we solve for $\theta(t)$ we will know $\vec{\Omega}$, hence, as the velocity of the point A is zero we will know the velocity of the center of mass. Knowing this velocity we will be able to find the coin's trajectory as it rolls on the plane.

LECTURE 21
Euler equations. Stability of asymmetric top.

21.1. Euler equations.

A few lectures before (see section 18.3) we considered free motion of a symmetric top. Due to the additional symmetry this motion can be exactly solved. In this section we will derive the equation of motion for a free asymmetric top.

As there no forces acting on the top, we can always think that it rotates around an axis which goes through the center of mass (there is always an external frame of reference where the center of mass does not move.) As there are no torques acting on the top the main equation is simply the angular momentum conservation law.

$$\dot{\vec{M}} = 0,$$

where $\vec{M}$ is the vector of angular momentum. However, one has to be careful with this equation. Namely, one has to be careful with the time differentiation. In this equation it is assumed that the time derivative should be taken in the external/inertial frame of references.

Let's write the vector $\vec{M}$ in the following form

$$\vec{M} = I_x\Omega_x\hat{x} + I_y\Omega_y\hat{y} + I_z\Omega_z\hat{z},$$

where I_x, I_y, and I_z are principal moments of inertial, $\hat{x}$, $\hat{y}$, and $\hat{z}$ are principle axes of inertia, and Ω_x, Ω_y, and Ω_z are the projections of the vector of angular velocity $\vec{\Omega}$ on the principle axes at this particular instant of time.

In the next instant of time the vector $\vec{\Omega}$ itself will change, but also the orientation of the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$ will also change – remember we are observing the motion from the external frame of references. So when taking the time derivative we need to differentiate both the components Ω_x, Ω_y, and Ω_z, and the vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$.

$$\dot{\vec{M}} = I_x\dot{\Omega}_x\hat{x} + I_y\dot{\Omega}_y\hat{y} + I_z\dot{\Omega}_z\hat{z} + I_x\Omega_x\dot{\hat{x}} + I_y\Omega_y\dot{\hat{y}} + I_z\Omega_z\dot{\hat{z}}$$

All three vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ are constant vectors in the internal frame of references – they are the internal frame of references. As for any such vector for the vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ we can write

$$\dot{\hat{x}} = \vec{\Omega} \times \hat{x}, \qquad \dot{\hat{y}} = \vec{\Omega} \times \hat{y}, \qquad \dot{\hat{z}} = \vec{\Omega} \times \hat{z}.$$

(after all, this is the definition of $\vec{\Omega}$)

So we have:

$$0 = \dot{\vec{M}} = I_x\dot{\Omega}_x\hat{x} + I_y\dot{\Omega}_y\hat{y} + I_z\dot{\Omega}_z\hat{z} + I_x\Omega_x\vec{\Omega}\times\hat{x} + I_y\Omega_y\vec{\Omega}\times\hat{y} + I_z\Omega_z\vec{\Omega}\times\hat{z}.$$

Multiplying (dot product) the above equation by $\hat{x}$, and using $\vec{a}\cdot[\vec{b}\times\vec{c}] = \vec{b}\cdot[\vec{c}\times\vec{a}]$ will find

$$0 = I_x\dot{\Omega}_x + I_y\Omega_y\vec{\Omega}\cdot[\hat{y}\times\hat{x}] + I_z\Omega_z\vec{\Omega}\cdot[\hat{z}\times\hat{x}],$$

or, as $[\hat{y}\times\hat{x}] = -\hat{z}$ and $[\hat{z}\times\hat{x}] = \hat{y}$ – remember the principle axes are orthogonal to each other, we can write

$$0 = I_x\dot{\Omega}_x - I_y\Omega_y\left(\vec{\Omega}\cdot\hat{z}\right) + I_z\Omega_z\left(\vec{\Omega}\cdot\hat{y}\right)$$

so finally

$$I_x\dot{\Omega}_x = (I_y - I_z)\Omega_y\Omega_z.$$

Analogously multiplying by $\hat{y}$ and $\hat{z}$ (or simply using the circular permutation of indexes), and we get the set of three the Euler equations:

$$\begin{aligned} I_x\dot{\Omega}_x &= (I_y - I_z)\Omega_y\Omega_z \\ I_y\dot{\Omega}_y &= (I_z - I_x)\Omega_z\Omega_x \\ I_z\dot{\Omega}_z &= (I_x - I_y)\Omega_x\Omega_y \end{aligned}$$

(these equation turn into each other under cyclic permutation of x, y, and z indexes.) These three equation are called the Euler equations.

One can immediately see, that the energy is conserved. In order to do that one simply multiplies the first equation by Ω_x, the second equation by Ω_y, and the third equation by Ω_z. Then one sums the equations up. The sum of the right hand sides (after the corresponding multiplication) is zero. So we have

$$I_x\dot{\Omega}_x\Omega_x + I_y\dot{\Omega}_y\Omega_y + I_z\dot{\Omega}_z\Omega_z = 0,$$

which is the same as

$$\frac{d}{dt}\left(\frac{I_x\Omega_x^2}{2} + \frac{I_y\Omega_y^2}{2} + \frac{I_z\Omega_z^2}{2}\right) = 0.$$

So the kinetic energy is constant.

- The Euler equations are three non-linear coupled first order differential equations.
- As such one needs only three initial conditions for the complete solution.
- We know, however, that the complete solution must depend on six initial conditions (three degrees of freedom, each requires initial position and initial velocity). The question is: where do the other three initial conditions go?
- Imaging, that we have solved the Euler equations with some initial conditions for Ω_x, Ω_y, and Ω_z. Then we will know $\Omega_x(t)$, $\Omega_y(t)$, and $\Omega_z(t)$ — the components of the angular velocity in the internal frame as functions of time.
- But in order to describe the motion in the external frame we need to know how the (external) coordinates (Euler angles, for example) depend on time.
- If we use the Euler angles as the coordinates, then we need to solve three more first order non-linear coupled differential equations for $\theta(t)$, $\phi(t)$, and $\psi(t)$:

$$\begin{aligned} \Omega_x(t) &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\ \Omega_y(t) &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\ \Omega_z(t) &= \dot{\phi}\cos\theta + \dot{\psi}. \end{aligned}$$

In the left had side are now the known functions: $\Omega_x(t)$, $\Omega_y(t)$, and $\Omega_z(t)$. The full solution will require three more initial conditions.

21.1.1. Symmetric top again.

The symmetric top is a particular case of the asymmetric one. For a symmetric top, taking $I_y = I_x \equiv I$ we find

$$I\dot{\Omega}_x = -(I_z - I)\Omega_y\Omega_z$$
$$I\dot{\Omega}_y = (I_z - I)\Omega_z\Omega_x$$
$$I_z\dot{\Omega}_z = 0$$

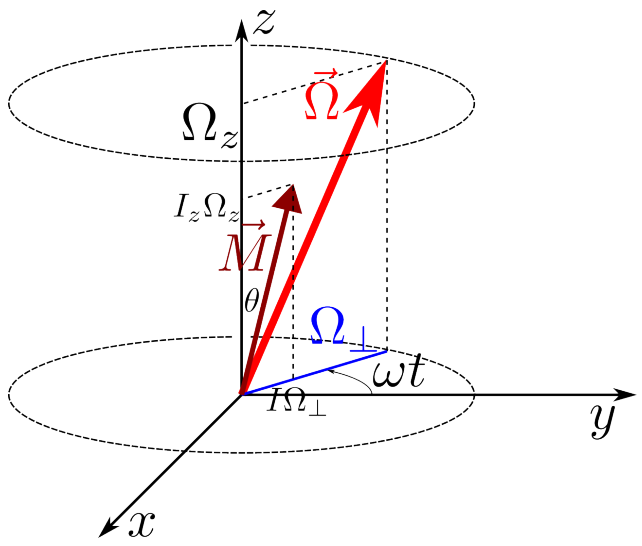
So we have $\Omega_z = \text{const.}$, then denoting $\omega = \Omega_z\frac{I_z - I}{I}$ — this is just a constant — we get

$$\dot{\Omega}_x = -\omega\Omega_y$$
$$\dot{\Omega}_y = \omega\Omega_x$$

The solution is

$$\Omega_x = A\cos\omega t, \qquad \Omega_y = A\sin\omega t, \qquad A^2 = \Omega_x^2 + \Omega_y^2 = \Omega_\perp^2 = \text{const.}$$

So the vector $\vec{\Omega}$ rotates around the z axis with the frequency ω.



Notice, that the vector $\vec{\Omega}$ rotates in the internal frame of reference! So if you are sitting on top of this top you will have the axis $\hat{z}$ which is motionless for you.

You will see the world around you spinning around some instantaneous axes given by $\vec{\Omega}$, but this instantaneous axis around which the world is spinning will also rotate around the $\hat{z}$ axis (this is the axis in your frame of references) with the frequency ω.

If $\omega \ll \Omega$ (or $|I_z - I| \ll I$), then the motion of the instantaneous axis is much slower than the rotation of the world around you. For the times much shorter than $2\pi/\omega$, but much larger than $2\pi/\Omega$ you might notice that there is one point in the sky (the point where $\vec{\Omega}$ is pointing at at this moment) which is not spinning. However, if you observe this motion for times longer than $2\pi/\omega$, you will notice, that this "notmoving" point is moving on a circle with frequency ω.

The vector $\vec{M}$ will also rotate with frequency ω in the internal frame.

This is the picture of the rotation as seen from a person on top of the top. Previously, we described the rotation as it is viewed from the outside. Let's check that these two pictures describe the same motion.

Using Euler angles we can write with respect to some arbitrary external system of coordinates.

$$\Omega_x(t) = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi = A\cos\omega t$$
$$\Omega_y(t) = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi = A\sin\omega t$$
$$\Omega_z(t) = \dot{\phi}\cos\theta + \dot{\psi} = \Omega_z$$

Multiplying the first equation by $\cos\psi$, the second by $\sin\psi$ and subtracting one from another we get

$$\dot{\theta} = A\cos(\psi + \omega t).$$

This is written in arbitrary external frame of references. Previously, we have used a specific external frame in which the angular momentum was along the $\hat{Z}$ axis. In order to compare the two motions we need to use the same external frame in both cases.

It is not obvious how to do that. But we remember, that previously we obtained θ to be a constant. So let's chose such an external frame where $\dot{\theta} = A\cos(\psi + \omega t) = 0$. We do not know *a priory* that such frame exists. We simply guess that it does. At the end we must check that this guess is consistent with the rest of the full set of equations.

The requirement $\dot{\theta} = 0$ means

$$\psi = \pi/2 - \omega t.$$

Plugging $\psi = \pi/2 - \omega t$ and $\dot{\theta} = 0$ into the first two equations we obtain the same relation

$$\dot{\phi}\sin\theta = A$$

and the third equation gives

$$\dot{\phi}\cos\theta = \Omega_z + \omega = \Omega_z\frac{I_z}{I},$$

so we see, that from these two equations $\tan\theta = \frac{AI}{\Omega_z I_z}$ is indeed a constant, so our guess $\dot{\theta} = 0$ is consistent with all of the equations. Moreover, as $A = \sqrt{\Omega_x^2 + \Omega_y^2} = \Omega_\perp$ we see, that $\tan\theta = \frac{I\Omega_\perp}{I_z\Omega_z} = \frac{M_\perp}{M_z}$ as it should be, because $\vec{M}$ is constant vector at angle θ to the $\hat{z}$ direction.

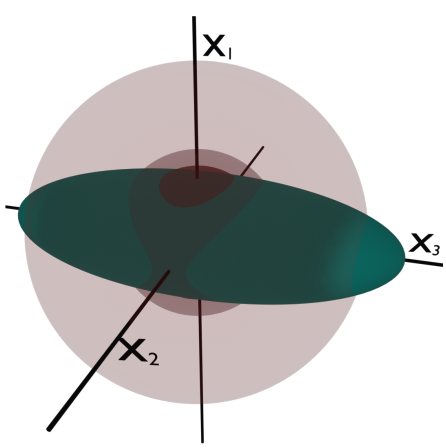
As $M_z = M\cos\theta$), we see that

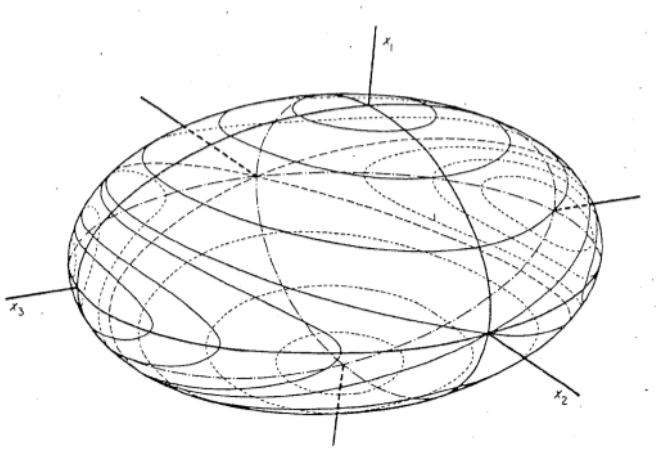
$$\dot{\psi} = -\omega = \Omega_z\frac{I - I_z}{I} = \frac{M_z}{I_z}\frac{I - I_z}{I} = M\left(\frac{1}{I_z} - \frac{1}{I}\right)\cos\theta$$
$$\Omega_{pr} = \dot{\phi} = \frac{\Omega_z I_z}{\cos\theta}\frac{1}{I} = \frac{M_z}{\cos\theta}\frac{1}{I} = \frac{M}{I}$$

Which are the same results, that we had before.

21.2. Stability of the free rotation of a asymmetric top.

- Different meaning of stability. Static stability and dynamic stability.





As the top is free, there are no forces acting on it and both the energy E and the vector of angular momentum $\vec{M}$ are conserved. Conservation of energy and the magnitude of the total angular momentum in terms of the components of the angular velocity $\vec{\Omega}$ in the internal frame xyz read

$$\frac{I_x\Omega_x^2}{2}+\frac{I_y\Omega_y^2}{2}+\frac{I_z\Omega_z^2}{2}=E$$
$$I_x^2\Omega_x^2+I_x^2\Omega_x^2+I_x^2\Omega_x^2=M^2$$

Writing the same equations, but in terms of the components of the angular momentum $\vec{M}$ (in the internal frame xyz) $M_x=I_x\Omega_x$, etc. we get

$$\frac{M_x^2}{2I_x}+\frac{M_y^2}{2I_y}+\frac{M_z^2}{2I_z}=E$$
$$M_x^2+M_y^2+M_z^2=M^2$$

The first equation describes an ellipsoid with the semiaxes $\sqrt{2I_xE}$, $\sqrt{2I_yE}$, and $\sqrt{2I_zE}$. The second equation describes a sphere of a radius M. The initial conditions give us E and M, the true solution must satisfy the conservation lows at all times. So the vector $\vec{M}$ will lie on the lines of intersection of the ellipsoid, and sphere. Notice, how different these lines are.

- If the radius of the sphere close to the smallest semi-axis of the ellipsoid, then the lines of intersection are paths in the neighborhood of the corresponding "tip" of the ellipsoid.
- If the radius of the sphere close to the largest semi-axis of the ellipsoid, then the lines of intersection again are paths in the neighborhood of the correspondin "tip" of the ellipsoid.
- If the radius of the sphere close to the middle semi-axis of the ellipsoid, then the lines of intersection go far from the corresponding tip. In fact they go from one "corresponding tip" to another.

The angular momentum is conserved. So in order to vector of angular momentum in the internal frame to go from one "tip" to another, the body must flip from the point of view of external observer.

Watch the video on the effect `https://youtu.be/NJLdW4DHRcA`.

LECTURE 22
Statics.

22.1. What we know.

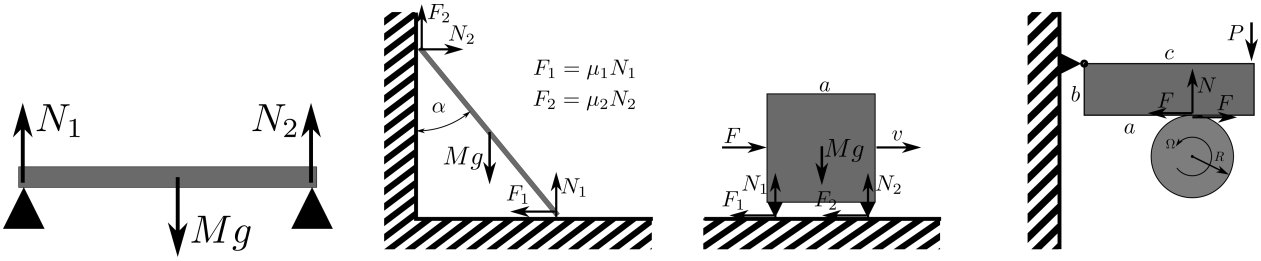
Static conditions:

- Sum of all forces is zero. $\sum \vec{F}_i = 0$.
- Sum of all torques is zero: $\sum \vec{r}_i \times \vec{F}_i = 0$.

If the sum of all forces is zero, then the torque condition is independent of where the coordinate origin is.

$$\sum (\vec{r}_i - \vec{a}) \times \vec{F}_i = \sum \vec{r}_i \times \vec{F}_i - \vec{a} \times \sum \vec{F}_i = \sum \vec{r}_i \times \vec{F}_i$$

22.2. Examples.



22.2.1. A uniform rigid bar on two supports.

The force equation

$$-Mg + N_1 + N_2 = 0.$$

The torque equation (with respect to the support 1)

$$Mgl/2 - N_2 l = 0.$$

So the solution is $N_1 = N_2 = Mg/2$.

If we write the torque equation with respect to the support 2. We get

$$-Mgl/2 + N_1 l = 0$$

Together with the force equation it gives the same result $N_1 = N_2 = Mg/2$ Notice, that we get the third equation by simply multiplying the first equation by l and adding to it the second equation. So the third equation is not independent equation, it is a consequence of the first two.

22.2.2. A uniform ladder of a length l in a corner.

$$\begin{aligned}
\text{Force equation, } x\text{-component:} &\qquad -F_1 + N_2 = 0 \\
\text{Force equation, } y\text{-component:} &\qquad -Mg + F_2 + N_1 = 0 \\
\text{Torque equation with respect to point 1:} &\qquad Mg\frac{l}{2}\sin\alpha - N_2 l\cos\alpha - F_2 l\sin\alpha = 0 \\
\text{Friction force:} &\qquad F_2 = \mu_2 N_2, \quad F_1 = \mu_1 N_1
\end{aligned}$$

We have five equations and only 4 unknowns F_1, F_2, N_1, and N_2. The system is over-determined! It is no surprise, as in reality the friction force conditions are inequalities. We then can pose the following question: At which angle α_c the ladder starts to slide? As the ladder is sliding, the friction forces must be at maximum and we can use $F_2 = \mu_2 N_2$ and $F_1 = \mu_1 N_1$. But then we have one more unknown – α_c. So we have the set of five equations for five unknowns α_c, F_1, F_2, N_1, and N_2.

$$\begin{aligned}
&-F_1 + N_2 = 0 \\
&-Mg + F_2 + N_1 = 0 \\
&Mg\frac{l}{2}\sin\alpha_c - N_2 l\cos\alpha_c - F_2 l\sin\alpha_c = 0 \\
&F_2 = \mu_2 N_2 \\
&F_1 = \mu_1 N_1
\end{aligned}$$

This is just a set of (almost) linear equations. The solution is

$$\begin{aligned}
N_2 &= Mg\frac{\mu_1}{1+\mu_1\mu_2} \\
N_1 &= \frac{Mg}{1+\mu_1\mu_2} \\
\tan\alpha_c &= \frac{2\mu_1}{1-\mu_1\mu_2}
\end{aligned}$$

- Again, one can chose any other point to write the torque equation.
- Notice, that if $\mu_1\mu_2 > 1$, then $\tan\alpha_c$ is negative! This obviously makes no sense. What happens?

22.2.3. A block with two legs on the floor with μ_1 and μ_2 coefficients of friction.

The question is: what force F we need to apply to the center of the side of the square block of mass M and side a supported by two legs in the front and in the back in order for the block to move with constant velocity.

As the velocity is constant everything works exactly the same as for the static case.

$$\begin{aligned}
\text{Force equation, } x\text{-component:} &\qquad F - F_1 - F_2 = 0 \\
\text{Force equation, } y\text{-component:} &\qquad -Mg + N_2 + N_1 = 0 \\
\text{Torque equation with respect to point 1:} &\qquad F\frac{a}{2} + Mg\frac{a}{2} - N_2 a = 0 \\
\text{Friction force:} &\qquad F_2 = \mu_2 N_2, \quad F_1 = \mu_1 N_1
\end{aligned}$$

we have 5 equations and 5 unknowns: F, F_1, F_2, N_1, and N_2. Solving the equations we find

$$F = \frac{\mu_1 + \mu_2}{2 + \mu_1 - \mu_2} Mg, \qquad N_1 = \frac{1 - \mu_2}{2 + \mu_1 - \mu_2} Mg.$$

- Notice, that if $\mu_2 > 1$, then N_1 is negative. This is impossible. What it means is that if $\mu_2 > 1$ the leg 1 will lift off the floor and the box will tip over — the front leg "gets stuck".
- Notice, that this effect would be missed if we did not compute N_1, even though we were only asked to compute F.
- Also, this resolves the problem of negative F in case $2 + \mu_1 - \mu_2 < 0$. In order for our solution to be valid we must have $\mu_2 < 1$, so $2 + \mu_1 - \mu_2$ cannot be negative.

22.2.4. A brake.

We neglect the block's weight. Consider the equilibrium of the block. I do not know the forces on the hinge (I can compute them, but I will not need to) I will write the torque equation with respect to the hinge

$$aN = Fb + Pc.$$

Also $F = \mu N$, so we have

$$N = \frac{c}{a - \mu b} P, \qquad F = \frac{c}{a - \mu b} \mu P$$

Notice, that

- for ANY μ, at some ratio of a and b the forces change sign. It is obviously impossible.
- Notice, that right before that the force N diverges. So does the force F. So the wheel cannot rotate.
- Notice, that this divergence happens at any non-zero P no matter how small. However, for $P = 0$ the result is obviously $N = F = 0$.
- There is a huge difference between $P = 0$ and $P \to 0$ cases.

22.2.5. A problem for students in class.

- A uniform horizontal rigid bar on three supports.

22.3. Elastic deformations.

- Continuous media. Scales.
- Small, only linear terms.
- No nonelastic effects.
- Static. We will not consider the dynamic effects. What it means is that we will write only the potential energy. If one also writes the kinetic energy, then one will be able to write the Lagrangian and use all the Lagrangian machinery to study the dynamics.
- Isothermal.

Definition of derivatives.

LECTURE 23
Strain.

When we deform an object there are a bunch of internal forces that appears inside the object. We need to write the relation/equation between the deformation and these forces. In order to do that we need to first figure out how to describe the deformation and how to describe the internal forces.

The deformation and the internal forces are described by the strain and the stress tensors respectively.

In this lecture we define both tensors and discuss their meaning.

23.1. Einstein notations

First we start with some preliminary mathematics.

- Einstein notations — summation over repeated indexes. Differential, divergence etc.
  - Differential. If we have a function $u(\vec{r})$, then the differential of this function is
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = \frac{\partial u}{\partial x_i}dx_i.$$
  - If we have a vector field $\vec{u}(\vec{r})$ with components $u_i(\vec{r})$, then the components of the differential of this vector field is
$$du_i = \frac{\partial u_i}{\partial x_j}dx_j.$$
  - The divergence of the vector field $\vec{u}(\vec{r})$ is
$$\mathrm{div}\vec{u}(\vec{r}) = \frac{\partial u_i}{\partial x_i}$$
- Difference between u_{ii}^2 and u_{ij}^2.
- We only work in Euclidean space, so there is no reason to distinguish co- and contravariant indexes.

A word on tensors.

- A tensor of a second rank can be thought of as a "coefficient" which linearly relates one vector to another (again, it is that simple only in Euclidean space). For example the tensor of inertia $\hat{I}$ relates the vector of angular velocity $\vec{\Omega}$ to the vector of angular momentum $\vec{M}$: $\vec{M} = \hat{I}\vec{\Omega}$.

- As vectors are "objects" — they do not depend on the coordinate system one uses, so are the tensors themselves. A tensor is an "object" it does not depend on the system of coordinates one uses.
- However, if one wants to write a tensor in components (the same as vector components), then one has to first choose the system of coordinates, and then the components do depend on the system of coordinates chosen. For example, the relation between the vector of the angular velocity and the vector of the angular momentum in components reads $M_i = I_{ij}\Omega_j$ (remember Einstein notations). In order for this expression to have a meaning one has to know what system of coordinates one uses.

23.1.1. Example.

Consider an arbitrary 3×3 tensor u_{ij} (this is just 3×3 matrix). It can be written as

$$u_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right) + \frac{1}{2}\left(u_{ij} - u_{ji}\right) = u^s_{ij} + u^a_{ij}$$

The first term is a symmetric tensor, the second term is antisymmetric tensor. Any antisymmetric 3×3 tensor (a bit more precise language would be any antisymmetric second rank tensor in $3D$) can be written as

$$u^a_{ij} = \epsilon^{ijk} l_k$$

For some vector $\vec{l}$. It is easy to prove simply by multiplying the above equation by ϵ^{ijn} (remember! I use Einstein notation. This multiplication also involves summation over repeated indexes.)

$$\epsilon^{ijn} u^a_{ij} = \epsilon^{ijn}\epsilon^{ijk} l_k = 2\delta_{nk} l_k = 2 l_n.$$

So any 3×3 tensor can be expressed as

$$u_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right) + \epsilon^{ijk} l_k.$$

- Notice, that if we take the antisymmetric part and construct a vector by $u^a_{ij} dx_j = \epsilon^{ijk} dx_j l_k = -[\vec{l} \times d\vec{r}]_i$, then this vector represent the change of $d\vec{r}$ under rotation around the vector $\vec{l}$. So the antisymmetric part corresponds to "rotations".

A side note:

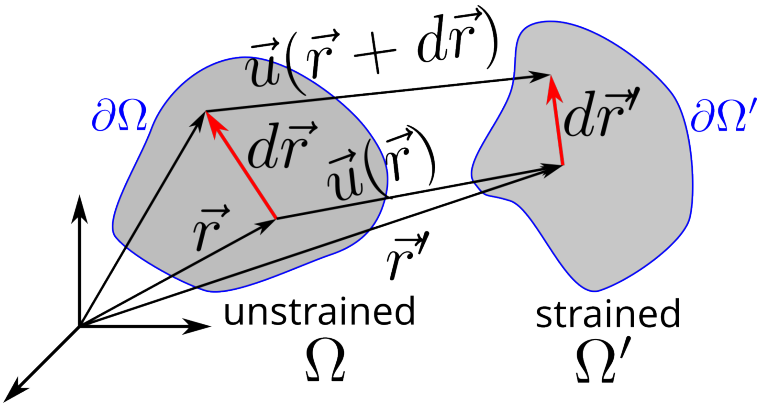
- This construction works only in $3D$. Only in $3D$ the number of independent elements of a 3×3 antisymmetric tensor is 3 – exactly the number of components of a vector.
- In $4D$ the number of independent components of a 4×4 antisymmetric tensor is 6. So if we want to present it using $3D$ vectors, we would need two such vectors. In electrodynamics for the electromagnetic tensor (which is antisymmetric) these are vectors of electric and magnetic fields.

23.2. Strain

- Strain (strain tensor) describes the deformation of a body.

We need do find a way to describe strain in a body. If no forces act on a body, then there is a certain distance between different points of a body. Strain appears when this distance changes. In particular, no strain appears if

- we uniformly translate a body;
- we uniformly rotate a body.



We can describe the change of the position of a point of a body by a vector field $\vec{u}(\vec{r})$. However, this vector field in general will also describe the parallel translations and rotations of the body. We need to find a way to exclude these contributions.

Let's then do it in a systematic way. Look at the figure. We consider two nearby points $\vec{r}$ and $\vec{r}+d\vec{r}$ in the unstrained object. After strain they will move to the points $\vec{r}' = \vec{r}+\vec{u}(\vec{r})$ and $\vec{r}'+d\vec{r}' = \vec{r}+d\vec{r}+\vec{u}(\vec{r}+d\vec{r})$ respectively. Subtracting, we see, that $d\vec{r}' = d\vec{r}+\vec{u}(\vec{r}+d\vec{r})-\vec{u}(\vec{r}) = d\vec{r}+d\vec{u}$. Or in components

$$dx_i' = dx_i + du_i.$$

The distance dl between two points in the unstrained object is given by $dl^2 = (d\vec{r})^2 = dx_i^2$. The distance dl'^2 between two points in the strained object is given by

$$\begin{aligned} dl'^2 &= (d\vec{r}')^2 = dx_i'^2 = (dx_i + du_i)^2 = dx_i^2 + 2dx_j du_j + du_i^2 \\ &= dl^2 + 2\frac{\partial u_j}{\partial x_k}dx_j dx_k + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}dx_j dx_k = dl^2 + \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right)dx_j dx_k \\ &= dl^2 + 2u_{jk}dx_j dx_k, \end{aligned} \tag{23.1}$$

where

$$u_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right) \tag{23.2}$$

Notice,

- The strain tensor u_{ij} is dimensionless, it has no units, as its terms are length differentiated over length.
- The tensor u_{ij} gives the change of distance between two nearby points – so it does describe the deformation.
- We specifically defined the tensor u_{ij} to be symmetric, so the rotations are excluded (the tensor has no antisymmetric part).
- The tensor u_{ij} may be different in every point of the body. It then defines a tensor field $u_{ij}(\vec{r})$.

Let me explain why we need a tensor to describe strain. The strain describes the shift of the relative position of two nearby points. The shift of the relative position is a vector, but the nearby points are also described by a vector. So the strain must connect two vectors — it, thus, must be a second rank tensor.

23.2.1. Linear approximation.

Normally we will take only the case of small strains and consider only linear approximation. Small strains do not mean, that $\vec{u}$ is small, but it does mean that the derivatives $\partial u_i/\partial x_j$ are small. So for small deformations we can use the *linear* approximation

$$u_{ik} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \tag{23.3}$$

In the *linear* approximation we can also write

$$dx'_i = (\delta_{ij} + u_{ij})\, dx_j.$$

This can be checked by using the above to compute dl'^2 and see that the result coincides with (23.1) in the *linear* approximation: $(dl')^2 = (dx'_i)^2 = (\delta_{ij} + u_{ij})\, dx_j\, (\delta_{ik} + u_{ik})\, dx_k \approx \delta_{ij}\delta_{ik}dx_i dx_j + u_{ij}\delta_{ik}dx_j dx_k + \delta_{ij}u_{ik}dx_j dx_k = (dx_i)^2 + 2u_{jk}dx_j dx_k = (dl)^2 + 2u_{jk}dx_j dx_k$. I used the smallness of u_{ij} in order to neglect the term $u_{ij}u_{ik}dx_j dx_k$.

23.2.2. Volume change in linear approximation

We can locally (at some point $\vec{r}$) diagonalize the real symmetric u_{ik}, and get orthogonal basis set. In that local frame $(1, 2, 3)$ (the local frame will be different for different points of the object) we have $dx'_1 = dx_1(1 + u_{11})$, etc, where u_{11}, u_{22}, and u_{33} are the eigen values of the tensor u_{ij} at point $\vec{r}$. These eigen values will also be different at different points of the object. The element of volume $dV = dx_1 dx_2 dx_3$ at point $\vec{r}$ of the unstrained objects under the strain turns to the new volume element $dV' = dx'_1 dx'_2 dx'_3$ of the strained object which is related to dV by (remember, we keep only terms up to linear order in u_{ij})

$$\begin{aligned} dV' = dx'_1 dx'_2 dx'_3 &\approx dx_1 dx_2 dx_3 (1 + u_{11} + u_{22} + u_{33}) \\ &= dV(1 + u_{ii}), \end{aligned} \tag{23.4}$$

where u_{ii} is the trace of the tensor. From the linear algebra we know that the trace is invariant to the coordinate system used. Hence the fractional change in the volume is given by

$$\frac{\delta(dV)}{dV} \equiv \frac{dV' - dV}{dV} = u_{ii}. \tag{23.5}$$

So if the unstrained body Ω of volume V is strained with a strain tensor $u_{ij}(\vec{r})$, then the total volume V' of the strained body is given by (in the *linear* approximation)

$$V' = V + \int_\Omega u_{ii} dV.$$

23.2.3. Deformation u_i from strain tensor u_{ij} in linear approximation.

We also see, that in the *linear* approximation

$$du_i = dx'_i - dx_i = u_{ij}dx_j,$$

or

$$u_i(\vec{r}) = \int_{\vec{r}_0,\Gamma}^{\vec{r}} u_{ij}dx_j.$$

where $\vec{r}_0$ is some point which we think as not moving under the deformation (Such point can always be found, as we can always add a parallel translation and rotation to any deformation) and Γ is an arbitrary path from the point $\vec{r}_0$ to the point $\vec{r}$ in the unstrained object. This formula will give the shift $\vec{u}(\vec{r})$ of the point $\vec{r}$ of the unstrained object.

In particular, if we are only interested in the shape of the strained body, then taking $\vec{r}$ to span all the points of the boundary $\partial\Omega$ of the unstrained body Ω we will find all the points $\vec{r}' = \vec{r} + \vec{u}(\vec{r})$ of the boundary of the strained body.

- If one uses this formula to find the shape of the strained body, one has to remember that this will give the shape up to a uniform rotation and uniform translation.
- If the components of the strain tensor u_{ij} are not small, then we have to consider the full non-linear definition (23.2). In this case in order to find $u_i(\vec{r})$ from the strain tensor $u_{ij}(\vec{r})$ we will have to consider (23.2) as a set of non-linear differential equations for $u_i(\vec{r})$.

23.3. Strain, summary

If a deformation of a body is described by a vector field $\vec{u}(\vec{r})$ — a point $\vec{r}$ of an unstrained body Ω is shifted by vector $\vec{u}(\vec{r})$ — then, in the *linear* approximation the strain tensor is given by

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

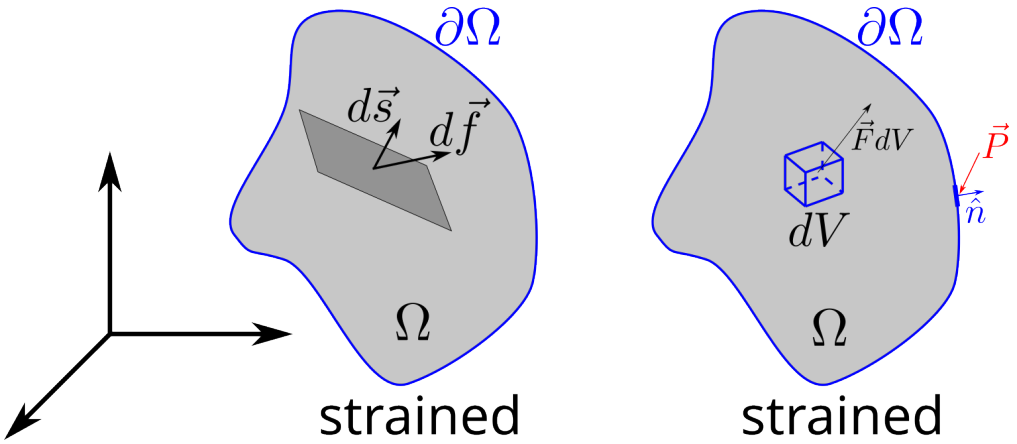
The strain is different in different points of the body, so strain is given by a tensor *field* $u_{ij}(\vec{r})$.

Linear approximation means that $u_{ij}(\vec{r})$ is small at all points $\vec{r}$, the vector of deformation $\vec{u}(\vec{r})$ does not have to be small. In fact one can always add an arbitrary constant vector to $\vec{u}(\vec{r})$, or constant uniform rotation.

In the *linear* approximation, the strain tensor has the following properties

(a) $u_{ij} = u_{ji}$ – this is correct for non-linear also.
(b) $\frac{\delta(dV)}{dV} = u_{ii}$, where $u_{ii} = \mathrm{tr}\hat{u}$ — only in linear approximation.
(c) $u_i(\vec{r}) = \int_{\vec{r}_0,\Gamma}^{\vec{r}} u_{ij}dx_j$ — only in linear approximation.
(d) In particular, if we apply the above formula for all the points of the boundary $\partial\Omega$ of the unstrained object Ω, then we will know the shape of the strained object.

LECTURE 24
Stress.



- Stress (stress tensor) describes the forces inside a deformed body and on the surface of that body.

24.1. Forces.

If we take a body and split it on very many small (infinitesimal) volumes, these volumes will interact with each other, exerting forces on each other. These forces are described by another 3×3 tensor called stress tensor.

- The forces are considered to be short range.

Let me explain why we need a tensor to describe these forces. The idea is to consider an infinitesimal square inside a body and think about the force $d\vec{f}$ (infinitesimal) which acts on it from one side, see left drawing on the Figure. The force is a vector. But this vector depends on the orientation (and area) of the square, which is given by another vector $d\vec{s}$ (this is true only in $3D$). The force vector $d\vec{f}$ must be proportional to the area vector $d\vec{s}$. So the stress connects two vectors, $d\vec{s}$ and $d\vec{f}$ – it, thus, must be a second rank tensor. Namely, in components

$$df_i = \sigma_{ij} ds_j.$$

- This is the force which acts from one side on a small plane piece of surface of area $d\vec{s}$ inside the object.

- On the same piece the force acting from the other side MUST by equal and opposite to the one shown here. As if it is not, the piece of surface which has zero mass would have an infinite acceleration.

The 3×3 tensor σ_{ij} is called stress tensor.

Consider a body Ω with the volume V which is embedded in a large deformed media. We want to compute a force which acts on a small piece of volume dV inside the object Ω, see right drawing on the Figure.

- There are forces which are applied by the media to the boundary/surface $\partial\Omega$ of the object Ω. This results in the total force $\vec{\mathcal{F}}$ acting on the whole object Ω.
- We are considering one particular moment of time. At this moment we do not assume, that the body is at equilibrium. So it is accelerating as a whole and deforming.
- Consider an infinitesimally small piece of volume dV inside the body. As the body is NOT in equilibrium, at this moment of time this piece is moving with some finite acceleration $\vec{a}$, which is different in different points of the body.
- The mass of this piece is ρdV, where ρ is the mass density. So the force which acts on this piece is $\vec{a}\rho dV$.
- The only fact that we need from this is that the force that acts on the volume dV is proportional to the volume dV.
- So we write, that the force $d\vec{f}$, that acts on the volume dV can be written as $d\vec{f} = \vec{F}dV$. ($\vec{F}$ is not the force, it is "density of force". The force is $d\vec{f}$, or $\vec{F}dV$.)
- The total force $\vec{\mathcal{F}}$ which acts on the whole body Ω is just the vector sum of all forces which act on all the infinitesimal pieces.

So we write (in components)

$$\mathcal{F}_i = \int_\Omega df_i = \int_\Omega F_i dV. \tag{24.1}$$

The meaning of the formula is that the total force $\mathcal{F}_i$ which acts on the volume V is the sum of the forces $F_i dV$ which act on all small pieces of volume dV of the body Ω.

However, because the forces are short-range it should also be possible to write the total force as sum of all forces acting on the boundary $\partial\Omega$ of the body Ω, or as an integral over the surface elements $ds_i = n_i ds$ of the surface $\partial\Omega$ of the body Ω, where $\hat{n}$ is the outward normal (L&L use df_i for the surface element). Thus we expect that

$$\mathcal{F}_i = \int_{\partial\Omega} \sigma_{ij} ds_j \tag{24.2}$$

for some σ_{ij}. (The notation $\partial\Omega$ just denotes the surface of a volume Ω.) Thinking of it as a set of three vectors (labeled by i) with vector index j, we can apply Gauss's Theorem to rewrite this as

$$\mathcal{F}_i = \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j} dV, \tag{24.3}$$

so comparison of the two volume integrals (24.1) and (24.3) gives

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{24.4}$$

What we found is the following. If we know the stress tensor $\sigma_{ij}(\vec{r})$ — it is stress tensor field — then

- The force acting on a small volume dV at position $\vec{r}$ is given by
$$df_i = \frac{\partial \sigma_{ij}}{\partial x_j} dV.$$
- The force which acts on a infinitesimal surface $d\vec{s}$ from one side is given by
$$df_i = \sigma_{ij} ds_j.$$
- The stress tensor σ_{ij} has units of pressure: $\frac{\text{force}}{\text{area}}$.

(The fact that the force on the piece of volume dV is given by $\frac{\partial \sigma_{ij}}{\partial x_j} dV$ can also be understood by considering a small cube with sides dx, dy, and dz and computing the total force acting on it through the sides.)

24.2. Torques.

Just as the total force due to the internal stresses should be written as a surface integral, so should be the total torque.

Again, the object Ω is NOT in equilibrium. A force $df_i = F_i dV$ acts on a small piece of volume dV at point $\vec{r}$ of the object. The torque of this force is $d\vec{\tau} = \vec{r} \times d\vec{f}$. Or in components we write $d\tau^l = -\epsilon^{lik} F_i x_k dV = -\frac{1}{2}\epsilon^{lik} (F_i x_k - F_k x_i) dV$, where $F_i = \frac{\partial \sigma_{ij}}{\partial x_j}$. The total torque acting on the body Ω is the vector sum of all torques, so

$$\begin{aligned}
\tau^l &= -\frac{1}{2}\epsilon^{lik} \int_\Omega (F_i x_k - F_k x_i) dV = -\frac{1}{2}\epsilon^{lik} \int_\Omega \left(\frac{\partial \sigma_{ij}}{\partial x_j} x_k - \frac{\partial \sigma_{kj}}{\partial x_j} x_i \right) dV \\
&= -\frac{1}{2}\epsilon^{lik} \int_\Omega \left(\frac{\partial (\sigma_{ij} x_k)}{\partial x_j} - \frac{\partial (\sigma_{kj} x_i)}{\partial x_j} - (\sigma_{ik} - \sigma_{ki}) \right) dV \\
&= -\frac{1}{2}\epsilon^{lik} \int_{\partial\Omega} (\sigma_{ij} x_k - \sigma_{kj} x_i) ds_j + \frac{1}{2}\epsilon^{lik} \int_\Omega (\sigma_{ik} - \sigma_{ki}) dV. \qquad (24.5)
\end{aligned}$$

We see, that in general there are two contributions, one from the surface – the first term, and the other is from the bulk – the second term. But all the forces are short range, so the total torque MUST only come from the surface. To eliminate the volume term we require that

$$\sigma_{ik} = \sigma_{ki}. \qquad (24.6)$$

So the stress tensor must be symmetric.

24.3. Equilibrium.

In equilibrium when only the internal stresses act, the total force acting on any piece dV must be zero, so the equilibrium condition is

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0. \qquad (24.7)$$

with constraint that σ_{ij} is symmetric.

If there is a long-range force, such as gravity, acting on a small volume dV with force $F_i^g dV = \rho g_i dV$, where ρ is the mass density and g_i is the gravitational field, then in equilibrium $F_i + F_i^g = 0$, or $\frac{\partial \sigma_{ij}}{\partial x_j} = -\rho g_i$ (where all $\frac{\partial \sigma_{ij}}{\partial x_j}$, ρ, and g_i must be taken at the same point $\vec{r}$).

The equation (24.7) is a set of three partial differential equations with constraint that $\sigma_{ij} = \sigma_{ji}$ for the stress tensor σ_{ij} in equilibrium. As any such equations they must be supplied with boundary conditions. These boundary conditions are given by the forces on the boundary of the body.

When there is a surface force P_i per unit area on the boundary $\partial\Omega$ of the body Ω, then the fact that the surface must be in equilibrium determines the stress force $\sigma_{ij}|_{\partial\Omega}\,\hat{n}_j$ on the boundary, so

$$P_i = \sigma_{ij}|_{\partial\Omega}\,\hat{n}_j \tag{24.8}$$

If we know the force P_i per unit area of the boundary of our object (it can be different in different points of the boundary, it also does not have to be perpendicular to the surface), then this equation provides us with the boundary condition for the equation (24.7).

24.3.1. Simple examples.

- No force on the boundary:

  When no force is applied to any point of the surface $\partial\Omega$ of the body Ω the stress at the surface is zero so $\sigma_{ij}|_{\partial\Omega} = 0$ and if there is no long range forces, then $\sigma_{ij} = 0$ is the solution of the equation (24.7).
- Hydrostatic pressure:

  If the surface force is a hydrostatic pressure, then $P_i = -P\hat{n}_i = \sigma_{ij}\hat{n}_j$. The only way this can be true for any $\hat{n}$ is if

$$\sigma_{ij} = -P\delta_{ij}. \tag{24.9}$$

  Notice that this tensor is also a solution of the equation (24.7). So for just hydrostatic pressure applied to the body the above equation gives the stress tensor in equilibrium everywhere.

24.4. Stress tensor, summary.

Here is what we know about the stress tensor σ_{ij} of the body Ω in equilibrium with the force $P_i(\vec{r}_{\partial\Omega})$ per unit area applied to the point $\vec{r}_{\partial\Omega}$ of the surface $\partial\Omega$ of the body Ω.

- The stress tensor $\sigma_{ij}(\vec{r})$ must be symmetric at every point of the body Ω.

$$\sigma_{ij}(\vec{r}) = \sigma_{ji}(\vec{r}).$$

- At every point $\vec{r}_{\partial\Omega}$ of the boundary $\partial\Omega$ of the body Ω the stress tensor must satisfy

$$P_i(\vec{r}_{\partial\Omega}) = \sigma_{ij}(\vec{r}_{\partial\Omega})\hat{n}_j(\vec{r}_{\partial\Omega}),$$

  where $\hat{n}$ is a unit vector perpendicular to the surface $\partial\Omega$ at the point $\vec{r}_{\partial\Omega}$.
- Inside the body Ω in the absence of long-range forces the stress tensor must satisfy the set of three partial differential equations

$$\frac{\partial\sigma_{ij}}{\partial x_j} = 0.$$

  If there are long-range forces the right hand side of this equation is that force per unit volume instead of 0.

LECTURE 25
Work, Stress, and Strain.

- Students evaluation 04-19-2023 until 05-03-2023.

25.1. Summary of Strain and Stress.

If a deformation of a body is described by a vector field $\vec{u}(\vec{r})$ – a point $\vec{r}$ of an unstrained body is shifted by vector $\vec{u}$ – then, in the *linear* approximation the strain tensor is given by

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In the *linear* approximation, it has the following properties

(a) $u_{ij} = u_{ji}$ – this is correct for non-linear also.
(b) $\frac{\delta(dV)}{dV} = u_{ii}$, where $u_{ii} = \mathrm{tr}\hat{u}$ — only in linear approximation.
(c) $u_i(\vec{r}) = \int_{\vec{r}_{0,\Gamma}}^{\vec{r}} u_{ij}dx_j$ — only in linear approximation.

The stress tensor σ_{ij} describes the internal elastic forces.

(a) $\sigma_{ij} = \sigma_{ji}$.
(b) The force on the internal volume element dV is $df_i = \frac{\partial \sigma_{ij}}{\partial x_j}dV$.
(c) The force on the surface element $d\vec{s}$ is $df_i = \sigma_{ij}ds_j$.

So, when the linear approximation is sufficient

- if we know strain tensor $u_{ij}(\vec{r})$ at every point we know the deformation $u_i(\vec{r})$ (in the linear approximation),

$$u_i(\vec{r}) = \int_{\vec{r}_{0,\Gamma}}^{\vec{r}} u_{ij}(\vec{r}')dx'_j.$$

Thus we know the shape of the body.

If we know the deformation u_i we know the strain tensor.

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

- If we know stress tensor σ_{ij} we know all the forces on any piece of volume dV:

$$df_i = \frac{\partial \sigma_{ij}}{\partial x_j}dV$$

or any piece of surface (from one side) $d\vec{s}$:

$$df_i = \sigma_{ij} ds_j.$$

(σ_{ij} has the units of pressure.) In particular, we know all the forces which are applied to the surface of our object.

On the other hand, if we know all the forces per unit area $P_i(\vec{r}_{\partial\Omega})$ on the surface $\partial\Omega$ of an object Ω, and the object is at equilibrium, then we need to solve the equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \qquad \sigma_{ij} = \sigma_{ji}, \qquad P_i = \sigma_{ij}|_{\partial\Omega}\, \hat{n}_j$$

where the last equality is the boundary condition for the partial differential equations (the first equality) which states that $\sigma_{ij} ds_j$ gives the known forces on the surface of the object. Then by knowing the forces on the boundary we will know the stress tensor field $\sigma_{ij}(\vec{r})$ everywhere.

25.2. Relation between Strain and Stress tensors.

What we now need to complete the problem is the connection between the strain u_{ij} and stress σ_{ij} tensors. If we know this connection, then by knowing deformation u_i of the object, we will know what forces on the surface of the object which are required for such a deformation. Also if we know what forces are acting on the surface we will know the deformation of the object. Let's assume, that we know the connection

$$u_{ij}(\vec{r}) \longleftrightarrow \sigma_{ij}(\vec{r})$$

then we can use the following strategies to solve all the static problems (*in linear approximation*)

- If we know the deformation $\vec{u}(\vec{r})$ of the object Ω then we can find what forces we need to apply to the boundary $\partial\Omega$ of this object

  $$u_i(\vec{r}) \longrightarrow u_{ij}(\vec{r}) \longrightarrow \sigma_{ij}(\vec{r}) \longrightarrow [\text{forces needed to be applied to the surface}]$$

- If we know the forces that are applied to the surface $\partial\Omega$ of the object Ω, then we can find the deformation $\vec{u}(\vec{r})$ of the object and, thus, its shape.

$$\left[\text{forces applied to the surface and } \frac{\partial \sigma_{ij}}{\partial x_j} = 0,\ \sigma_{ij} = \sigma_{ji}\right] \longrightarrow \sigma_{ij}(\vec{r}) \longrightarrow u_{ij}(\vec{r}) \longrightarrow u_i(\vec{r}).$$

- Strictly speaking the forces are applied to the boundary $\partial\Omega'$ of the strained object Ω', not to the boundary $\partial\Omega$ of an unstrained object Ω, but in the *linear* approximation this difference can be ignored.

Different materials deform differently under the same forces, so the connection between two tensors of strain and stress is material dependent.

We are working in a linear and local approximation, so the connection must be linear and local and have a form

$$u^{ij}(\vec{r}) = D^{ijkl}(\vec{r})\sigma^{kl}(\vec{r}),$$

where D^{ijkl} is a material dependent local tensor of the fourth rank. The same way as the liner relation between two vectors is described by a tensor of rank 2, the linear relation between two tensors of rank 2 is described by a tensor of rank 4.

If the material is uniform, than the tensor $D^{ijkl}(\vec{r})$ must not depend on $\vec{r}$ and we have

$$u^{ij}(\vec{r}) = D^{ijkl}\sigma^{kl}(\vec{r}),$$

- An arbitrary fourth rank tensor in $3D$ has $3^4 = 81$ independent elements.
- In our case the tensor D^{ijkl} must be symmetric with respect to exchange $i \leftrightarrow j$ and $k \leftrightarrow l$. This reduces the number of independent elements to 36 (each pair of indexes has $1 \cdot 2 \cdot 3 = 6$ possibilities, two pairs of indexes then has $6^2 = 36$ independent components.).
- However, any crystal has additional symmetries which will reduce the number of independent elements even further.
- It is known how to write any tensor which respects any crystal symmetry.
- The higher the symmetry, the more restrictions we have, the simpler the tensor will be.
- The simplest tensor appears when the symmetry is the highest — isotropic material.
- Crystals are not isotropic, but a lot of materials (amorphous, multicrystals etc.) are isotropic.

In addition, the requirement of stability of the equilibrium (when $u^{ij} = 0$) must lead to some inequalities between the elements of the tensor D. As it is a tensor of the fourth rank it is very tedious to analyze. So instead we will work around it.

25.3. Work against Internal Stresses

Let's imagine an experiment when we want to slightly change the field $\vec{u}(\vec{r})$ while keeping the shape of the object intact.

- We assume that we have full control and can apply any forces we want to any infinitesimal piece of volume inside the object Ω. We also can apply any forces to the boundary of the object $\partial\Omega$.
- Initially the deformation was given by the vector field $u_i(\vec{r})$. This deformation is the result of some long-range (not elastic) forces which act on all volume elements. The object is at equilibrium at this moment.
- As the object is deformed the force $F_i dV$, where $F_i = \frac{\partial \sigma_{ij}}{\partial x_j}$, from the internal stress acts on every infinitesimally small piece of volume dV.
- As the deformed object is in equilibrium, the force applied by us to an infinitesimal piece of volume dV must be $-F_i dV$.
- Now we want to adiabatically slowly change the deformation $u_i(\vec{r}) \to u_i(\vec{r}) + \delta u_i(\vec{r})$, where $\delta u_i(\vec{r})$ is infinitesimally small and $\delta u_i|_{\partial\Omega} = 0$ — we are not changing the shape of the object — this way we exclude the work of any "outside" forces.
- As $\delta u_i(\vec{r})$ is very small, the forces $-F_i dV$ do not change during the change of deformation.
- So the work against forces of internal stress which we need to do to shift a piece of volume dV by δu_i is $\delta R_{our} = -F_i \delta u_i dV = -\frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i dV$.

Hence the total work done by us in order to change the deformation by δu_i is

$$\begin{aligned}\delta W_{our} &= \int \delta R_{our} = -\int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i dV = -\int_\Omega \frac{\partial(\sigma_{ij}\delta u_i)}{\partial x_j}dV + \int_\Omega \sigma_{ij}\frac{\partial(\delta u_i)}{\partial x_j}dV \\ &= -\int_{\partial\Omega}\sigma_{ij}\delta u_i ds_i + \int_\Omega \sigma_{ij}\frac{\partial(\delta u_i)}{\partial x_j}dV = \int_\Omega \sigma_{ij}\delta\frac{\partial u_i}{\partial x_j}dV = \int_\Omega \sigma_{ij}\delta\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)dV \\ &= \int_\Omega \sigma_{ij}\delta u_{ji}dV\end{aligned} \tag{25.1}$$

Where we first used Gauss's theorem to transform one of the volume integral to a surface integral. Then we used $\delta u_i|_{\partial\Omega} = 0$ — the deformation δu_i is zero on the boundary, as we fixed the boundary, we do not change the shape of the object. Then we used the symmetry of σ_{ij} to symmetrize $\frac{\partial u_i}{\partial x_j} \to \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$. And finally we used the fact that in *linear* approximation $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) = u_{ji}$.

So the work done on volume dV by us in order to change the deformation is

$$\delta R_{our} = \sigma_{ik}\delta u_{ki}dV. \tag{25.2}$$

This work is done by us at constant temperature, so it goes into the change of free energy density F_F.

$$dF_F = \delta R_{our} = \sigma_{ik}\delta u_{ki}$$

25.3.1. Thermodynamics

We now assume the system to be in thermodynamic equilibrium. Using the energy density ϵ and the entropy density s, the first law of thermodynamics gives (it is dR/dV because we write everything in terms of densities.)

$$d\epsilon = Tds - \frac{dR}{dV} = Tds + \sigma_{ij}du_{ji}. \tag{25.3}$$

Defining the free energy density $F_F = \epsilon - Ts$ we have

$$dF_F = -sdT + \sigma_{ij}du_{ji}.$$

In the next section we consider the form of the free energy density as a function of T and u_{ij}: $F_F(T, u_{ij})$. Then we will use

$$dF_F = \sigma_{ij}du_{ji}, \quad \text{if } dT = 0, \text{ or temperature is fixed.} \tag{25.4}$$

$dT = 0$ — this is exactly what we need, as we consider only isothermal processes

25.4. Elastic Energy

The elastic equations must be linear, as this is the accuracy which we work with. The free energy density then must be quadratic in the strain tensor. We thus need to construct a scalar out of the strain tensor in the second order. If we assume that the body is isotropic, then the only way to do that is:

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ik}^2. \tag{25.5}$$

- Notice, that in this approach instead of working with the tensor of the fourth rank we are working with a scalar – free energy.

Here λ and μ are the only parameters (in the isotropic case). These parameters are different for different materials.

- The elastic properties (in linear order) of any isotropic material are characterized by only two (temperature dependent) parameters!

These parameters λ and μ are called *Lamé coefficients,* and in particular μ is called the *shear modulus* or *modulus of rigidity.* Note that u_{ii} is associated with a volume change, by (23.5).

- The two terms in (25.5) are not independent, so in order to take the derivative, we must rewrite it as independent terms.

The quantity

$$\tilde{u}_{ik} = u_{ik} - \frac{1}{3}\delta_{ik}u_{jj} \tag{25.6}$$

satisfies $\tilde{u}_{ii} = 0$, and is said to describe a pure shear.

With this definition we have

$$u_{ik} = \tilde{u}_{ik} + \frac{1}{3}\delta_{ik}u_{jj} \tag{25.7}$$

$$u_{ik}^2 = \tilde{u}_{ik}^2 + \frac{2}{3}\tilde{u}_{ii}u_{kk} + \frac{1}{3}u_{jj}^2 = \tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2. \tag{25.8}$$

Hence (25.5) becomes

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu(\tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2) = F_0 + \frac{1}{2}Ku_{ii}^2 + \mu\tilde{u}_{ik}^2. \qquad (K \equiv \lambda + \frac{2}{3}\mu) \tag{25.9}$$

In this form the two elastic terms are independent of one another.

- For the elastic energy to correspond to a stable system, each of them must be positive, so $K > 0$ and $\mu > 0$.

25.4.1. Stress$\longleftrightarrow$Strain.

Now we have the free energy as a function of *independent* variables u_{ii} and $\tilde{u}_{ij}$. So we can take the variation of the Free energy with respect to these independent variables.

On varying u_{ik} at fixed T the free energy of (25.9) changes by

$$\begin{aligned} dF =& Ku_{jj}du_{kk} + 2\mu\tilde{u}_{ik}d\tilde{u}_{ik} = Ku_{jj}du_{kk} + 2\mu\tilde{u}_{ik}(du_{ik} - \frac{1}{3}\delta_{ik}du_{jj}) \\ =& Ku_{jj}du_{kk} + 2\mu\tilde{u}_{ik}du_{ik} = Ku_{jj}\delta_{ik}du_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}\right)du_{ik} \\ =& \left[Ku_{jj}\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}\right)\right]du_{ik}, \end{aligned} \tag{25.10}$$

so comparison with (25.4) gives

$$\sigma_{ik} = Ku_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}). \tag{25.11}$$

- This equation gives σ_{ij} if we know u_{ij}. It only works for uniform isotropic media.

Note that $\sigma_{jj} = 3Ku_{jj}$, so that

$$u_{jj} = \frac{\sigma_{jj}}{3K}.$$

We now use this equation for u_{jj} in (25.11), and then solve it for u_{ik}:

$$u_{ik} = \delta_{ik}\frac{\sigma_{jj}}{9K} + \frac{1}{2\mu}\left(\sigma_{ik} - \frac{1}{3}\sigma_{jj}\delta_{ik}\right). \tag{25.12}$$

- This equation gives u_{ij} if we know σ_{ij}. It only works for uniform isotropic media.

In the above the first term has a finite trace and the second term has zero trace.

This is the desired result:

- knowing the stress σ_{ij} we can find the strain u_{ij} by (25.12).
- knowing the strain u_{ij} we can find the stress σ_{ij} by (25.11).

All *isotropic* materials in the linear approximation are described by just two constants.

LECTURE 26
Elastic Moduli.

- Students evaluation 04-19-2023 until 05-03-2023.

26.1. Results of last lecture

- The elastic properties of an isotropic material in the linear approximation are characterized by two constants/parameters K and μ. They are called elastic moduli. Both of them are positive: $\mu > 0$ and $K > 0$.
- μ is called shear modulus or modulus of rigidity, K is inverse (isothermal) comressibility.
- These moduli can be expressed through Lamé coefficients λ and μ: $K = \lambda + \frac{2}{3}\mu$, and μ is the same.
- The elastic moduli allow one to connect stress and strain tensors.

$$\sigma_{ij} = K u_{kk}\delta_{ij} + 2\mu(u_{ij} - \frac{1}{3}\delta_{ij}u_{kk}). \tag{26.1}$$

$$u_{ij} = \delta_{ij}\frac{\sigma_{kk}}{9K} + \frac{1}{2\mu}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right), \tag{26.2}$$

  where $K > 0$ and $\mu > 0$.
- Taking the trace of either equation we get

$$u_{kk} = \frac{\sigma_{kk}}{3K}. \tag{26.3}$$

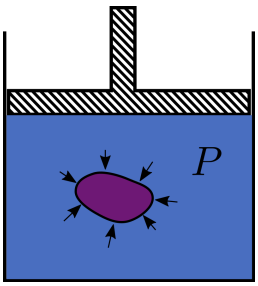
This lecture is about the physical meaning of the elastic moduli.

The two elastic moduli K and μ can be expressed through Young's moduli: E and σ. E is called Young's modulus, or the modulus of extension. σ is called Poisson's ratio.

The equation of equilibrium is $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$. Generally, we need to find a solution of this equation (it is really three equations, as there is a free index i) which satisfies the boundary conditions. It is a complicated problem in general. However, in some simple cases we can use our intuition to guess the solution, and then check it.

26.2. Bulk Modulus and Young's Modulus

In this part we will guess σ_{ij}, check the guess, and find the corresponding deformation.



26.2.1. Hydrostatic compression.

We consider an object submerge into incompressible liquid. There is no gravity. The liquid is under a hydrostatic pressure P.

- For hydrostatic compression the force on one side of any small tile in the liquid is always perpendicular to that tile and "into" this tile.
- So the vector of force and the vector area of the tile's side have the exactly opposite directions.
- It means that the stress tensor σ_{ij} inside the object is $\sigma_{ij} = -P\delta_{ij}$ on the *boundary* of the object.
- However, as this $\sigma_{ij} = -P\delta_{ij}$ is constant, it also satisfies the equation $\frac{\partial \sigma_{ij}}{\partial x_j}$ *inside* the object.
- The tensor $\sigma_{ij} = -P\delta_{ij}$ is symmetric. So this tensor satisfies the boundary condition, the equation for equilibrium, and is symmetric. It is then the stress tensor of the object under hydrostatic compression.
- Using (26.2) we find the strain tensor u_{ij}

$$u_{ij} = -\frac{P}{3K}\delta_{ij}.$$

- Taking the trace of the strain tensor u_{kk} we find (see also 26.3)

$$u_{kk} = -\frac{P}{K}. \qquad \text{(hydrostatic compression)} \tag{26.4}$$

So we found the full description of the deformation of an isotropic, uniform object under hydrostatic compression.

26.2.1.1. *Physical meaning of constant K.* Consider an object of volume V under the pressure P. If we change the pressure by dP keeping the temperature constant, the volume will change by dV, where $\frac{dV}{V} = du_{kk} = -\frac{dP}{K}$, so

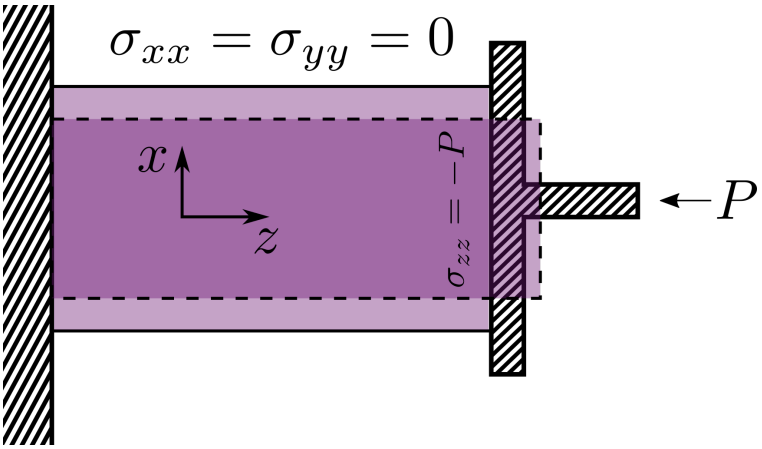
$$\frac{1}{K} = -\frac{du_{kk}}{dP} = -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_T. \tag{26.5}$$

So K is inverse isothermal compressibility $\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ as defined in thermodynamics.

26.2.2. Uni-direction compression.

A stick of length L and cross-section A is compressed from right with force $F = PA$. We want to find how the stick is deformed.

- The stick is along z direction, left end of a stick meets an unmovable wall. The force is applied to the right end of the stick in the negative z direction.



- On the right boundary of the stick the force per unit area is P. The direction of this force is opposite to the direction of the area vector.
- So on the right boundary the stress tensor has value $\sigma_{zz} = -P$, and all other components are zero.
- On the sides of the stick, there are no forces. So on the side boundary of the stick all components σ_{ix} and σ_{iy} are zero, including $\sigma_{xx} = \sigma_{yy} = 0$.
- Consider a stress tensor of the following form

$$\hat{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P \end{pmatrix}.$$

  - This tensor satisfies the boundary conditions.
  - As it is constant, it is obviously satisfies the equilibrium condition $\frac{\partial \sigma_{ij}}{\partial x_j}$ inside the stick.
  - It is symmetric.

  So this tensor is the correct solution for the stress tensor inside the stick!
- For this tensor we have $\sigma_{kk} = -P$
- Using (26.2) we see that $u_{ij} = 0$ for $i \neq j$, and

$$u_{xx} = u_{yy} = \frac{P}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right), \tag{26.6}$$

$$u_{zz} = -\frac{P}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right) = -\frac{P}{E}, \qquad E \equiv \frac{9K\mu}{3K+\mu}. \tag{26.7}$$

- So the strain tensor is diagonal matrix with u_{xx}, u_{yy}, and u_{zz} on the diagonal.
- Notice, that for positive pressure (compression) u_{zz} is always negative, as both $K > 0$ and $\mu > 0$, and hence $E > 0$.

So we found the full description of the deformation of an isotropic, uniform stick under uni-direction force.

The coefficient in front of P in the equation for u_{zz} is called the *coefficient of extension.* Its inverse E is called *Young's modulus,* or the *modulus of extension.*

In particular we can find the spring constant of the stick. As the wall on the left is stationary, in the equation $u_i(\vec{r}) = \int_{\vec{r}_0,\Gamma}^{\vec{r}} u_{ij} dx_j$ we take the point $\vec{r}_0$ to be on the left end of the stick, the path Γ to be a line parallel to the z axis, and the final point $\vec{r}$ to be on the

right end of the stick. Then we find, that the right end is shifted along the z direction (Only u_z is non-zero) by

$$u_z = \Delta z = \int_0^L u_{zz} dz = u_{zz} L = -\frac{PL}{E} = -\frac{L}{AE} F = -\frac{F}{k}, \qquad k = \frac{AE}{L},$$

where A is the area where pressure is applied and L is the length of the sample ($\Delta z \ll L$). So we see, that the object behaves as a spring with the spring constant k.

26.2.2.1. *Poisson's ratio.* For the previous experiment we can define *Poisson's ratio* σ via

$$u_{xx} = -\sigma u_{zz}. \tag{26.8}$$

- A word of caution. The Poisson ratio is defined by (26.8) **ONLY IN THE EXPERIMENT CONSIDERED ABOVE**.
- It is not a universal rule that $u_{xx} = -\sigma u_{zz}$ by any means. It is so **ONLY** for the above experiment.
- In particular, if in the above experiment the purple stic is placed within a hard rigid tube (see next section), then u_{xx} will be zero, but u_{zz} will not be zero.

Then we find that

$$\sigma = -\frac{u_{xx}}{u_{zz}} = \frac{(\frac{1}{2\mu} - \frac{1}{3K})}{(\frac{1}{3K} + \frac{1}{\mu})} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}. \tag{26.9}$$

Since K and μ are positive, the maximum value for σ is $\frac{1}{2}$ and the minimum value is -1. All materials in Nature (except some!!!) have $\sigma > 0$.

$$-1 < \sigma < 1/2.$$

Notice, that the volume is changing by $\frac{\delta dV}{dV} = u_{ii} = u_{zz}(1 - 2\sigma)$, so if $\sigma = 1/2$ the volume does not change – incompressible liquid. The requirements that when we compress the volume cannot increase is the requirement that $\sigma < 1/2$.

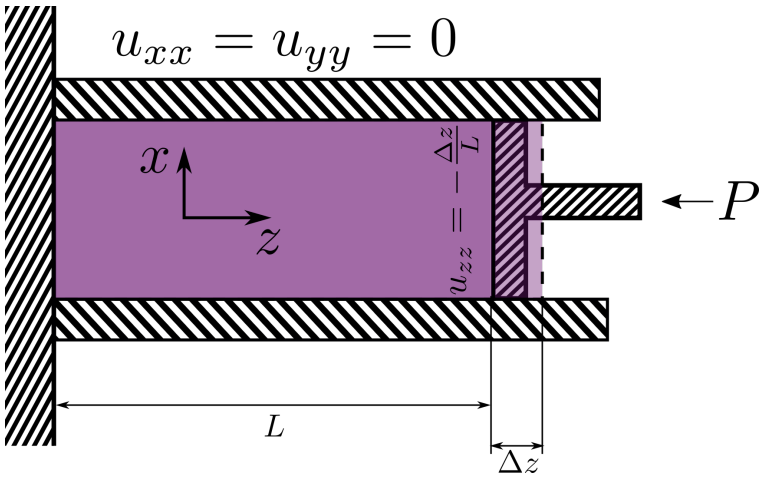
Often one uses E and σ instead of K and μ. We leave it to the reader to show that

$$\begin{aligned} \lambda &= \frac{E\sigma}{(1-2\sigma)(1+\sigma)}, \\ \mu &= \frac{E}{2(1+\sigma)}, \\ K &= \frac{E}{3(1-2\sigma)}. \end{aligned} \tag{26.10}$$

- In the two examples we considered we guessed σ_{ij} and then found u_{ij}.
- In the next two examples we will guess u_{ij}, find the corresponding σ_{ij}, check that it satisfies the equilibrium condititon, and then find what boundary conditions are required.

26.2.3. Uni-direction compression in a hard tube.

Let's now consider a uni-direction compression, but now the purple material is placed inside a hard rigid tube. There is no friction between the purple material and the tube. We apply the pressure P and we want to know Δz.



Notice, that this case is very different from the one we considered above. In this case we cannot say that $\sigma_{xx} = \sigma_{yy} = 0$, as the hard tube applies pressure to the sides. So we need to find a different approach.

This approach is reverse to what we used before. We first guess the strain tensor u_{ij}, then compute σ_{ij}, then check that σ_{ij} is symmetric and satisfies the equation $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$, and then knowing σ_{ij} we find the pressure P. This way we connect the pressure P and Δz.

Looking at the picture, and keeping in mind, that the purple material is uniform, it is natural to guess

$$u_{xx} = u_{yy} = 0, \qquad u_{zz} = -\frac{\Delta z}{L}, \qquad u_{i\neq j} = 0.$$

So

$$\hat{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Delta z/L \end{pmatrix}.$$

in this case $u_{kk} = u_{zz} = -\Delta z/L$, and using (26.1) we find

$$\sigma_{xx} = \sigma_{yy} = -\frac{\Delta z}{L}\left(K - \frac{2}{3}\mu\right), \qquad \sigma_{zz} = -\frac{\Delta z}{L}\left(K + \frac{4}{3}\mu\right), \qquad \sigma_{i\neq j} = 0$$

The first thing we notice the σ_{ij} is diagonal and is thus symmetric. It is also constant in space, so the equation $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$ is satisfied! So it is indeed a correct solution and our guess was right!

Now using (26.10) we express the above results through E and σ.

$$\sigma_{xx} = \sigma_{yy} = -\frac{\Delta z}{L}\frac{E\sigma}{(1-2\sigma)(1+\sigma)}, \qquad \sigma_{zz} = -\frac{\Delta z}{L}\frac{E(1-\sigma)}{(1-2\sigma)(1+\sigma)}$$

(in case of $\sigma < 0$ the purple material must be glued to the tube in order for $u_{xx} = u_{yy} = 0$ to be true)

In particular, we see, that

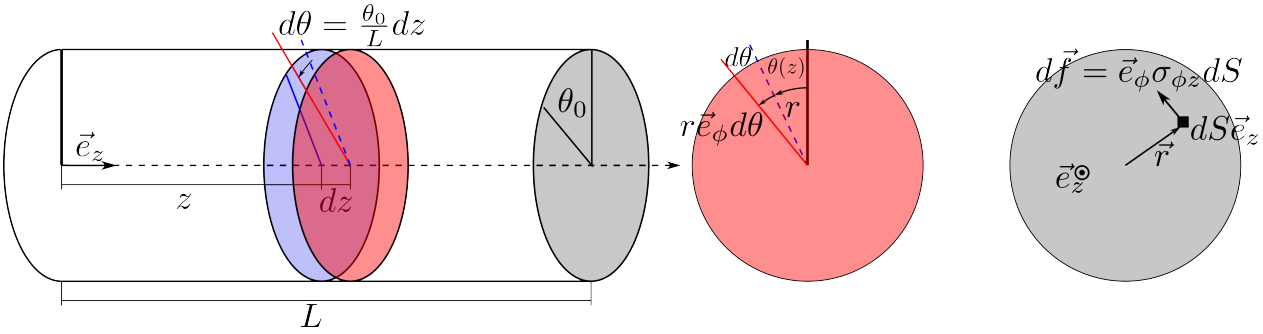
$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma}{1-\sigma}\sigma_{zz}.$$

Also the force F is

$$F = A\sigma_{zz} = -\frac{A}{L}\frac{E(1-\sigma)}{(1-2\sigma)(1+\sigma)}\Delta z = -k\Delta z, \qquad k = \frac{A}{L}\frac{E(1-\sigma)}{(1-2\sigma)(1+\sigma)}$$

Notice, that although the purple material is the same in both experiments, the "spring constants" are different.

26.2.4. Twisted rod.



Let's take a circular rod of radius a and length L and twist its right end (keeping the left end as it was) by a small angle θ_0. We want to calculate the torque needed to be applied to the right end to the this.

- We first guess the right solution.

If z is the coordinate along the axis of the rod, then the cross-section at distance z from the left end (we are twisting the right end) is rotated by the angle $\theta(z) = \frac{z}{L}\theta_0$

Two cross-sections a distance dz from each other are twisted by the angle $d\theta = \frac{\theta_0}{L} dz$ with respect to each other. So a point at distance r from the center on the cross-section at $z + dz$ is shifted by the vector $d\vec{u} = r d\theta \vec{e}_\phi = r\frac{\theta_0}{L} dz \vec{e}_\phi$ in comparison to that point in the cross-section at z. So the ϕ component of the vector $d\vec{u}$ is given by $du_\phi = r\frac{\theta_0}{L} dz$ and all other components are zero $u_r = u_z = 0$. We thus see that the strain tensor is

$$u_{z\phi} = u_{\phi z} = \frac{1}{2}\left(\frac{du_\phi}{dz} + \frac{du_z}{d\phi}\right) = \frac{1}{2} r \frac{\theta_0}{L}$$

and all other elements are zero.

The relation between u_{ij} and σ_{ij} is local, so we can write them in any local system of coordinates. So as the strain tensor is trace-less

$$\sigma_{z\phi} = \sigma_{\phi z} = \mu r \frac{\theta_0}{L}.$$

and all other elements are zeros.

- Notice, that this stress tensor is symmetric and $\frac{\partial \sigma_{z\phi}}{\partial z} = \frac{\partial \sigma_{z\phi}}{\partial \phi} = 0$, so the condition of equilibrium is satisfied and our guess was right.

Now we calculate the torque we need to apply to the right end. Consider a small area dS of the right end of the rod at a point at distance r from the center. The vector of area is $d\vec{S} = dS\vec{e}_z$, it has only z component. As we know, $df_i = \sigma_{ij} dS_j$, so in this case $df_i = \sigma_{iz} dS$. But the only nonzero element of σ_{iz} is $\sigma_{\phi z}$. So the force will have only ϕ component. So the force we need to apply to the element dS is

$$d\vec{f} = \sigma_{\phi z} dS \vec{e}_\phi.$$

The torque of this force with respect to the center is along z direction and is given by

$$d\vec{\tau} = \vec{r} \times d\vec{f} = r\sigma_{\phi z} dS \vec{e}_r \times \vec{e}_\phi = r\sigma_{\phi z} dS \vec{e}_z.$$

As all $d\vec{\tau}$ are in the same direction z, we can simply add them up. So the total torque is

$$\tau = \int r\sigma_{\phi z} dS = \int_0^a r\mu r \frac{\theta_0}{L} r dr d\phi = \mu \frac{\theta_0}{L} \int_0^a r^3 dr d\phi = \frac{\pi}{2} \frac{\mu}{L} a^4 \theta_0.$$

So we found how torque depends on θ_0 and how it depends on the dimensions of the rod. Notice, that the dependence on the rod's radius a is a^4, this power of 4 is typical for elastic problems. Also notice, that the relation between the torque and the angle involves only the sheer modulus μ and does not depend on K. So twisting rods is a good way to measure μ!

We can measure μ in this experiment by the following way

(a) Prepare rods of different radii and lengths from the same material.
(b) For each rod measure torque τ as a function of the twist angle θ.
(c) For each rod plot τ as a function of θ. Verify, that for small enough angle τ/θ does not depend on θ and is just a constant. This constant is a slope of each graph at small θ.
(d) Plot this constant (the slope) as a function of $\frac{\pi a^4}{2L}$. Verify, that the points are on a straight line for small $\frac{\pi a^4}{2L}$. The slope of this line at small $\frac{\pi a^4}{2L}$ is the sheer modulus μ.
(e) A better way to measure $\frac{\pi a^4}{2L}$ is by measuring the frequency of oscillations of a disk on known moment of inertia hanged on a thread/rod.

A word of caution:

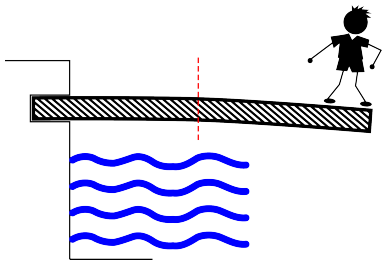
- All the examples I presented were specifically crafted in such a way, that we did not have to solve the partial differential equations $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$.
- We simply guessed the correct solutions.
- Generally, this is not the case. One has to set up the boundary conditions and solve the equations $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$.
- There are many techniques to do that, but they are beyond this course.

LECTURE 27
Small deformation of a beam.

27.1. Statement of the problem.

Let's consider a narrow rigid beam of length L with rectangular cross-section of area A. The beam is held in place almost horizontally by some supports. One can think that the beam is held/clamped by its left end. We are interested in the beam deformation under gravity.

For example, let's consider a diving board shown on the picture. It is in equilibrium. All of it is.



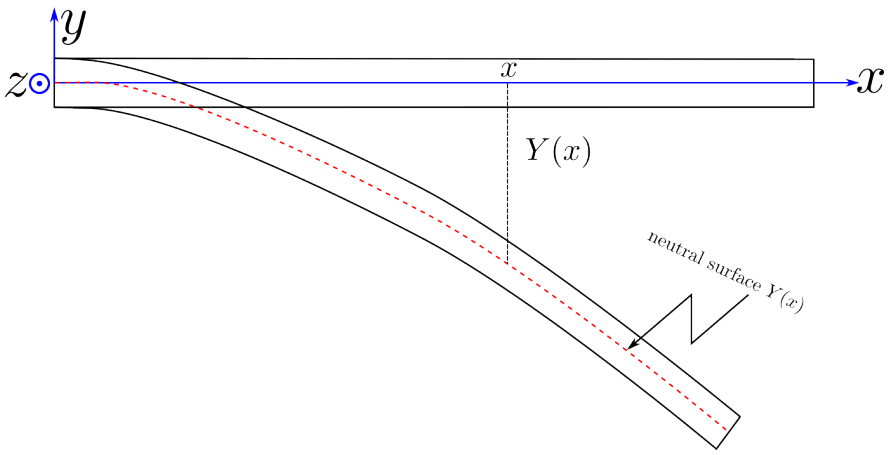
- Question to the students: Look at the part of the diving board to the right of the red dashed line. What keeps this part stationary?

In order for the board to be stationary, it has to deform, bend. When it is bent there are forces and torques acting on each cross-section. We need to find these forces and torques for a given deformation.

We also want to describe the deformation of the beam. However, we do not want to find the full deformation of the beam – this is a difficult problem, and we do not need it. We want to describe an overall "shape" of the beam only. So we want to disregard the changes in cross sections, but we cannot disregard the forces that act in each cross-section.

As the beam is narrow and rigid we expect that the deformation of the shape will lead to only negligible deformations in cross-sections. We also expect the beam to be almost horizontal.

- x coordinate is along undeformed beam, y is perpendicular to it, pointing up.
- Nothing depends on z. $\hat{z}$ points towards us.
- The beam is made of a material with known Young's modulus E.
- We will describe the "shape" of the beam by the shape of *neutral surface*.



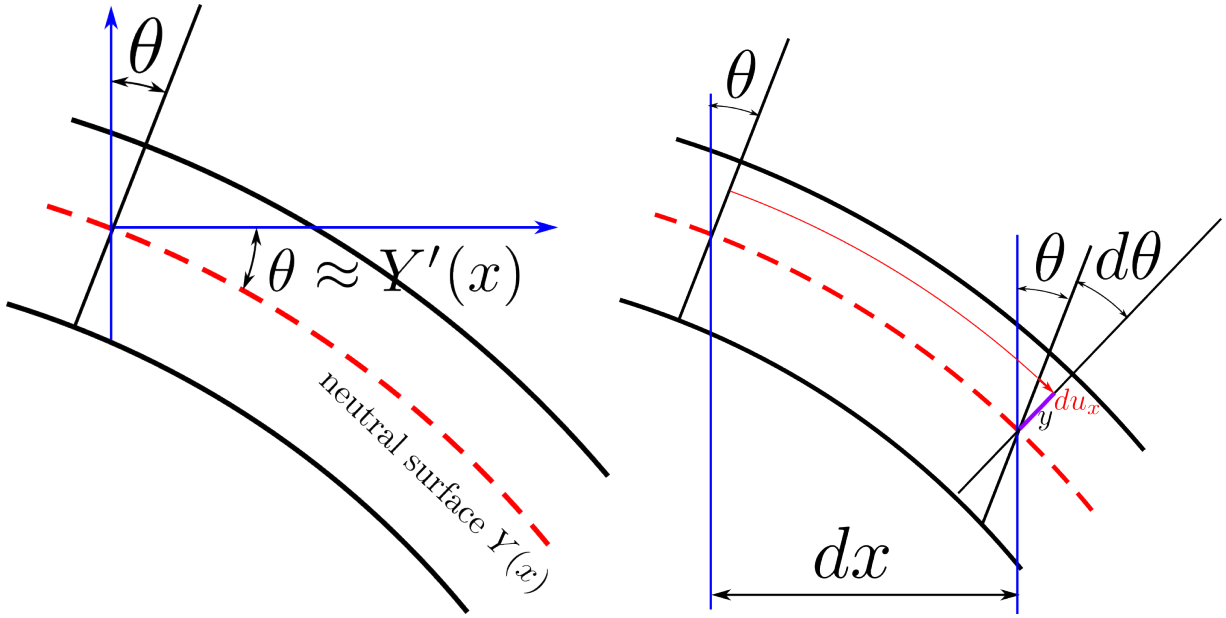
- *Neutral surface:* In any cross-section part of it (lower) is squeezed and part of it (upper) is stretched. So there is a point in the beam's cross-section at given x which is neither squeezed, nor stretched. This point for the cross-section at x has a y coordinate Y. So we have a neutral surface $Y(x)$. This function is what we will use to describe the shape of the beam. It is this function $Y(x)$ that we want to find.
- Deformation is small, $|Y'(x)| \ll 1$.

Under these conditions the (small) angle between a cross-section at x in the deformed beam makes with the same cross-section in un-deformed beam is given by

$$\theta(x) \approx Y'(x).$$

So the change of the angle $\theta(x)$ between two near corss-sections at points x and $x + dx$ is

$$d\theta = Y''(x)dx.$$



27.2. Torques and Forces in equilibrium.

27.2.1. Torques for a given shape $Y(x)$.

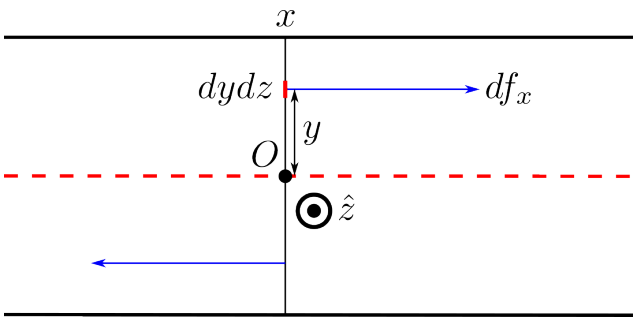
The neutral surface is neither stretched, nor compressed. The line which is a distance y from this surface is stretched (compressed) in x direction by $du_x = yd\theta = yY''dx$, so we have

$$u_{xx} = \frac{\partial u_x}{\partial x} = y\frac{\partial^2 Y(x)}{\partial x^2}.$$

- The stretching (compression) proportional to the second derivative, as the first derivative describes the uniform rotation of the beam.

There is no confining in the y or z directions, so we find that

$$\sigma_{xx} = -Eu_{xx} = -Ey\frac{\partial^2 Y(x)}{\partial x^2}.$$



Consider a cross-section of the beam at point x. The force in the x direction of the $dydz$ element of the beam is $df_x = \sigma_{xx}dzdy$. The torque with respect to a point O on the neutral surface which acts from the **right** part on the **left**

$$\vec{\tau}(x) = -\hat{e}_z \int y\sigma_{xx}dydz = \hat{e}_z E\frac{\partial^2 Y(x)}{\partial x^2}\int y^2 dzdy = \hat{e}_z IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad I = \frac{\int y^2 dydz}{\int dydz}.$$

(A is the cross-section area.) This torque is in the negative (see picture) $\hat{z}$ direction as Y'' is negative. Its z component is:

$$\tau(x) = IAE\frac{\partial^2 Y(x)}{\partial x^2},$$

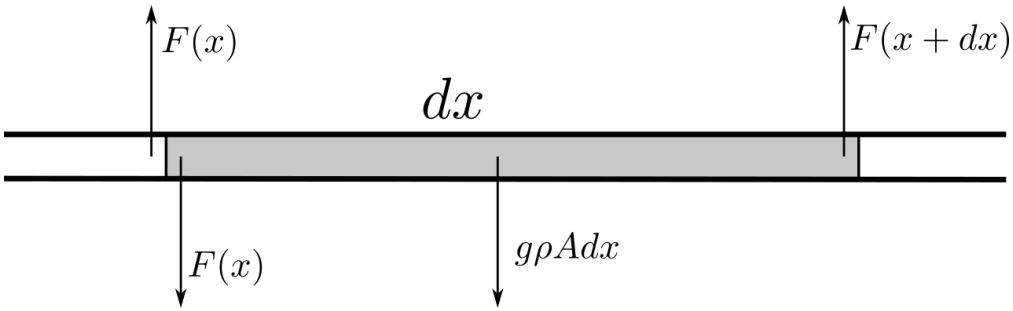
This is our first equation. It gives the $\hat{z}$ component of the torque acting **from right on left** of the cross-section of the beam at x.

- This equation simply states how torque in a cross-section depends on the shape $Y(x)$.
- Notice, that the cross-section area A and the quantity I are just properties of an unstretched beam. They are just some constants that characterize the beam and they do not depend on its bending.

So far we have not used the fact that the beam is at equilibrium.

27.2.2. Equilibrium.

The beam is at equilibrium. So if we take a small portion of it, between x and $x + dx$, the total force and torque on it must be zero. Let's consider these two conditions one by one:

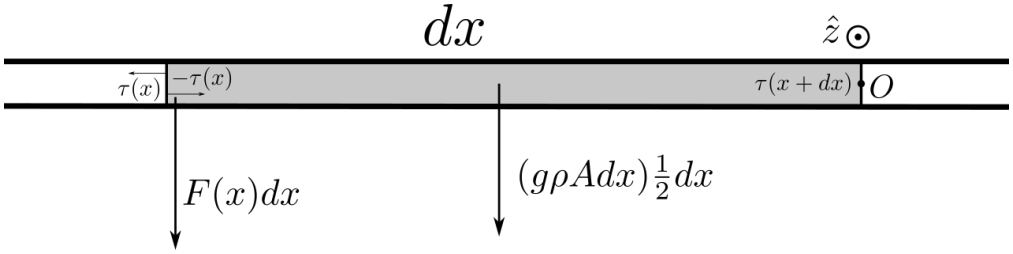


Force: Let's say that the y component of the force on the cross-section at x with which the **right** side is acting on the **left** is $F(x)$ (positive is $\hat{e}_y$ direction). The **left** side of a cross-section at x applies to the **right** side the force $-F(x)$ So demanding that the total force acting on the small piece of length dx (the shaded region on the figure) be zero we get

$$F(x+dx) - F(x) - \rho g A dx = 0, \qquad \frac{\partial F}{\partial x} = \rho g A.$$

(positive direction is up, along $\hat{y}$.) **This is our second equation. This is our first equation.** It gives the $\hat{y}$ component of the force acting **from right on left** of the cross-section of the beam at x.

- This equation encodes the first equilibrium condition that the forces acting on a piece dx sum up to zero.



Torque: The total torque (with respect to the point $x + dx$ — point O in the figure) acting on this portion is

$$\tau(x+dx) - \tau(x) + F(x)dx + \frac{1}{2} m \rho g A (dx)^2 = 0, \qquad \frac{\partial \tau}{\partial x} = -F(x).$$

This is our third equation.

- This equation encodes the second equilibrium condition that the torques acting on a piece dx sum up to zero.

27.3. Shape $Y(x)$

Now we collect all three equations:

$$\tau(x) = IAE \frac{\partial^2 Y(x)}{\partial x^2}, \qquad \frac{\partial F}{\partial x} = \rho g A, \qquad \frac{\partial \tau}{\partial x} = -F(x).$$

From the last two equations we find

$$\frac{\partial^2 \tau}{\partial x^2} = -\frac{\partial F}{\partial x} = -\rho g A$$

Using the first equation we get

$$IAE\frac{\partial^4 Y(x)}{\partial x^4} = -\rho g A.$$

This is the fourth order differential equation for the function $Y(x)$!

The general solution of this equation is simply

$$Y(x) = -\frac{\rho g}{24IE}x^4 + \frac{C_3}{6}x^3 + \frac{C_2}{2}x^2 + C_1 x + C_0.$$

$$\tau(x) = IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad \text{along } \hat{z} \text{ direction}$$

$$F(x) = -IAE\frac{\partial^3 Y(x)}{\partial x^3}, \qquad \text{along } \hat{y} \text{ direction} \tag{27.1}$$

Both the force and the torque is from the **right** on the **left** side of a cross-section at x.

- The constants C_0, C_1, C_2, and C_3 must be found from the boundary conditions.
- As there are four unknown constants we must have four boundary conditions. Or two conditions for each end of the beam.

27.4. A beam with free end. A diving board.

We need to determine four unknown constants. C_0, C_1, C_2, and C_3.

We take $Y = 0$ at $x = 0$ — fixing the position of one end — which gives $C_0 = 0$. Another condition is that at $x = 0$ the board is horizontal – **the end is clamped** ,

$$Y'(x = 0) = 0$$

This determines $C_1 = 0$.

As there is nothing to the right of the very last cross-section, at the other end ($x = L$ — the length of the board) both the force and the torque are zero — **it is a free end condition**. So we get the conditions

$$F(x = L) \sim \left.\frac{\partial^3 Y(x)}{\partial x^3}\right|_{x=L} = 0, \qquad \tau(x = L) \sim \left.\frac{\partial^2 Y(x)}{\partial x^2}\right|_{x=L} = 0.$$

These two conditions determine $C_3 = \frac{\rho g}{IE}L$ and $C_2 = -\frac{\rho g}{2IE}L^2$.

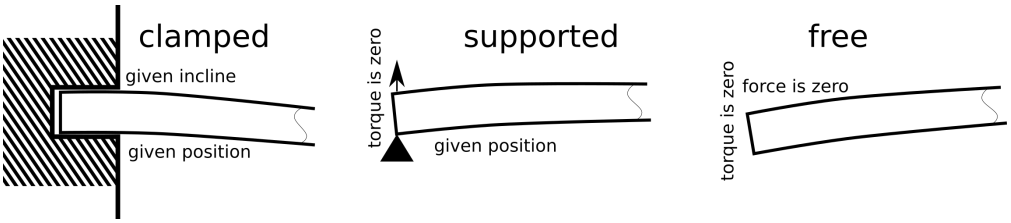
$$Y(x) = -\frac{\rho g}{24IE}x^2\left(x^2 - 4xL + 6L^2\right).$$

In particular,

$$Y(x = L) = -\frac{\rho g}{8IE}L^4.$$

- Notice the proportionality to the fourth power.
- If ρg is larger, then the free end of the beam hangs lower.
- If E is smaller — the beam is less rigid — the free end hangs lower.

27.5. Different models for the boundary conditions.



- Clamped. The position Y, as well as inclination Y' are given on that end. These are two conditions.
- Supported. There is a support at the end. So the position Y at this end is given, but the inclination is not. There is a force at the end – the force that supports the end at the given position, but the torque is zero. Again, it is two conditions at this end: The position Y, and the torque.
- Free. There is nothing at the end. So both force and torque at this end are zero. Again, it is two conditions.

As any beam has two ends, and we have two conditions at each end, we have total four conditions — exactly the number of conditions we need to find the values of four constants C_0, C_1, C_2, and C_3.

27.6. Phenomenological approach.

The equation for the shape of the beam is very simple. There is, then, a question: can we derive it simpler?

Let's start with trying to understand, how the potential energy $E[Y(x)]$ of the beam depends on arbitrary shape $Y(x)$. If we find this potential energy, we will be able to find the equation for the shape, by demanding, that for the true shape the potential energy is at minimum.

We will still consider only very small deformations (it is possible to relax this condition, but math becomes more complicated).

The potential energy of the beam consists of two terms: the gravitational potential energy E_g, and the elastic potential energy E_e. As the function $Y(x)$ is the height of the board at point x, a small piece of the board of length dx at point x has potential energy $Y(x)\rho g A dx$. The total gravitational potential energy is simply the sum of the potential energies of all pieces

$$E_g = \int_0^L Y(x)\rho g A dx.$$

Now we need to write the elastic energy. Let's see what it can depend on. The elastic energy cannot depend on the function $Y(x)$ itself, as if we add a constant to $Y(x)$ the shape of the board will not change, so elastic energy must not change.

The elastic energy also cannot depend on the derivative $Y'(x)$ of the function $Y(x)$. The argument is the following. Let's assume, that $Y'(x)$ is a constant, then the beam is a straight line with some inclination. But this means that the shape of the beam did not change! So the elastic energy can only be non zero if $Y'(x)$ is not constant. In the lowest order then the elastic energy can only depend on $Y''(x)$.

If we consider the board with $Y''(x) = a$ and compare it with the board with $Y''(x) = -a$, then in both cases the elastic energy must be the same! So in the lowest order the elastic

energy must be proportional to $(Y''(x))^2$. The coefficient of the proportionality is some phenomenological parameter. So we can write

$$E_e = \int_0^L \frac{IAE}{2} \left(\frac{\partial^2 Y}{\partial x^2} \right)^2 dx.$$

(the form of the phenomenological coefficient $\frac{IAE}{2}$ does not follow from anywhere, it is just a phenomenological parameter. One can make an argument, that it must be proportional to the cross-section area A and the Young modulus E, the rest is simply written in this form because I know the answer.)

Now the total potential energy is

$$E[Y(x)] = \int_0^L \left(Y(x)\rho g A + \frac{IAE}{2} \left(\frac{\partial^2 Y}{\partial x^2} \right)^2 \right) dx.$$

We need to find the function $Y(x)$ which provides the minimum of this functional. Such function then is given by Euler equation. In comparison to our usual Lagrangian equation in this case we have functional which depends on the second derivative, so the derivation of the Euler equation is a bit different, but goes the same way as before. I leave it to the reader to derive it. The result is

$$IAE \frac{\partial^4 Y}{\partial x^4} + \rho g A = 0.$$

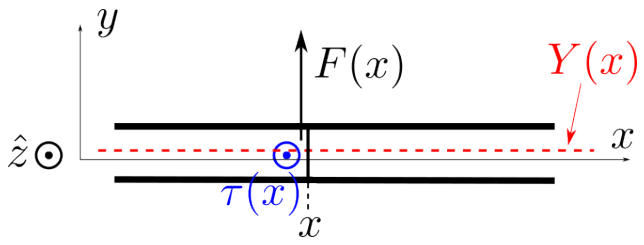
Which is the same equation as before.

One can also find the force at the end. In order to do that, we only need to find the energy $E(Y_L)$ as the function of the position Y_L of the end, then the force is $F = -\frac{\partial E}{\partial Y_L}$. In order to find the function $E(Y_L)$ one needs to solve the above equation with boundary conditions Y_0, Y_0', Y_L, Y_L' (fourth order equations requires four conditions, two on each end) find the function $Y(x|Y_0, Y_0', Y_L, Y_L')$ which satisfies the equation with the given boundary conditions. Then one substitutes this function to the expression for energy and computes the integral. The result is the energy $E(Y_0, Y_0', Y_L, Y_L')$, which will depend on the boundary conditions. Then the force is computed as the derivative of the potential energy.

LECTURE 28
A rigid beam on three supports.

- Students evaluation ends May 1, 2024.
- Final exam, Tuesday, May 7, 8am — 10am.

28.1. Results of the previous lecture.



For small deformations, the shape of the beam is described by the shape of the neutral surface. As nothing depends on z coordinate, the neutral surface is described by a function $Y(x)$. This function is given by

$$
\begin{aligned}
Y(x) &= -\frac{\rho g}{24IE}x^4 + \frac{C_3}{6}x^3 + \frac{C_2}{2}x^2 + C_1 x + C_0. \\
\tau(x) &= IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad \text{along } \hat{z} \text{ direction} \\
F(x) &= -IAE\frac{\partial^3 Y(x)}{\partial x^3}, \qquad \text{along } \hat{y} \text{ direction}
\end{aligned}
\tag{28.1}
$$

Both the force and the torque is from the **right** on the **left** side of a cross-section at x. The positive direction of the force is up. The positive direction of the torque is toward us.

- The constants C_0, C_1, C_2, and C_3 must be found from the boundary conditions.
- On each of the two ends of a beam we have two conditions (free, supported, or clamped). It is total 4 conditions — exactly as many as we need to determine 4 constants.

28.2. The force on the middle.

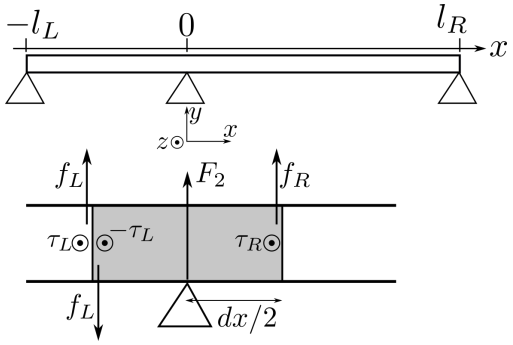
28.2.1. Why?

Consider an absolutely rigid $E = \infty$ horizontal beam with its ends fixed. Let's see how the force on the central support changes as a function of height h of this support. For $h < 0$ the force is zero. For $h > 0$ the force is infinite and $h \to 0_-$ and $h \to 0_+$ are very different. So the situation is unphysical. It means that the order of limits first $E \to \infty$ and then $h \to 0$ is wrong. We need to take the limits in the opposite order: first take $h = 0$ and then $E \to \infty$. In this order the limits are well defined. So we need to solve the static horizontal beam on three supports for large, but finite E and then take the limit $E \to \infty$ at the very end, when we already know the solution. Luckily we know how to solve this problem for large E!

28.2.2. Two beams.

The beam is of length L. The central support has a coordinate $x = 0$ and is at the distance l_L from the left end and at the distance l_R from the right end ($l_R + l_L = L$).

In the previous lecture we saw, that we need four boundary conditions to define the shape of the beam. Each end of the beam gives two conditions. However, the central support will give another set of conditions. It looks like the problem will be over-determined.



It is not so. There is one place, an infenitesimal piece around the central support, where the previous lecture's calculation fails, as in that calculation we only considered elastic forces and the gravity. Here there is another force — the force from the support — which must be included in the equilibrium condition for that one infinitesimal piece. Outside of this one infinitesimal piece we have only elastic forces and the gravity, so outside of this one piece the calculations of the previous lecture are valid.

The central support exerts a force F_2 on the beam. This force is at a single point.

- It means that there is a jump/discontinuity in the internal elastic forces at $x = 0$.
- However, for everything outside of the infinitesimal piece around the central support we can use the results of the previous lecture. So everything is piece-wise continuous on the left and on the right of $x = 0$. We then use two functions $Y_L(x)$ and $Y_R(x)$ to describe the shape to the left and to the right of the central support.

We then have the shape which is given by

$$Y_L(-l_L < x < 0) \qquad \text{and} \qquad Y_R(0 < x < l_R)$$

As all supports are at the same height we must have $Y_L(x = 0) = Y_L(x = -l_L) = Y_R(x = 0) = Y_R(x = l_R) = 0$, so

$$Y_L = -\tfrac{\rho g}{24IE} x(x + l_L)\left(x^2 + C_1^L x + C_0^L\right) \qquad \text{for } -l_L < x < 0$$
$$Y_R = -\tfrac{\rho g}{24IE} x(x - l_R)\left(x^2 + C_1^R x + C_0^R\right) \qquad \text{for } 0 < x < l_R$$

The form of these functions may look different from what we have used in the previous lecture. However, the statement of equilibrium which we derived in the previous lecture

simply demands, that the function $Y(x)$ is a polynomial of the fourth order with the coefficient in front of x^4 be $-\frac{\rho g}{24IE}$. Both functions above are exactly of this kind with the additional requirement, that all supports are at the same height.

We thus have 4 unknown constants. We need four boundary conditions. Two boundary conditions are given by the fact, that there is no torque on the left and on the right sides of the beam: $\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=-l_L} = 0$ and $\left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=l_R} = 0$.

28.2.2.1. *The central support.* The other two boundary conditions must come from the central support. First, it is clear, that the beam must be smooth at $x = 0$, so $\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0}$.

To find the last boundary condition we compute the total torque on the infinitesimal element of length dx centered at $x = 0$. The total torque must be zero, so we have (Positive torque is towards us, τ_L and τ_R, are torques on a given cross-section from the right part to the left part. For f_R and f_L positive is up and are applied on a given gross-section from the right part to the left part. See figure.)

$$f_R dx/2 + f_L dx/2 + \tau_R - \tau_L = 0, \quad \text{in } dx \to 0 \text{ limit}, \quad \tau_R(0) = \tau_L(0).$$

As torque is the second derivative of Y (the left and right parts are made of the same material), this condition means that the second derivative from the left and from the right must be the same. So the boundary conditions are

- The torque at $x = 0$ is continuous: $\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=0}$.
- The beam is smooth at $x = 0$: $\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0}$.
- The torques on both ends are zero, $\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=-l_L} = 0$ and $\left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=l_R} = 0$.

We thus have four conditions and four unknowns.

In order to find the force *from the middle support on the beam* F_2, lets again consider a small (length dx) element right on top of the middle support. The sum of all forces must be zero, so we get (see figure)

$$F_2 + f_R - f_L - \rho A g dx = 0.$$

Taking the limit $dx \to 0$ we find

$$F_2 = f_L - f_R = IAE\left(\left.\frac{d^3 Y_R}{dx^3}\right|_{x=0} - \left.\frac{d^3 Y_L}{dx^3}\right|_{x=0}\right) = -\frac{\rho g A}{4}\left(C_1^R - C_1^L - l_R - l_L\right).$$

(Check the units.) We see, that if we know $C_1^R - C_1^L$, then we will know the force we are interested in.

Let's see what the boundary conditions give one by one:

- $$C_0^L + l_L C_1^L = C_0^R - l_R C_1^R.$$
- $$l_L C_0^L = -l_R C_0^R.$$
- $$3l_R^2 + 2C_1^R l_R + C_0^R = 0, \qquad 3l_L^2 - 2C_1^L l_L + C_0^L = 0.$$

These are four linear equation for four unknowns. We only need the combination $C_1^R - C_1^L$ from them. Solving the equations we find

$$C_1^R - C_1^L = -\frac{1}{2}(l_R + l_L)\frac{l_R^2 + l_R l_L + l_L^2}{l_R l_L}.$$

and hence the force is

$$F_2 = \frac{\rho g A}{8}(l_R + l_L)\left(1 + \frac{(l_R + l_L)^2}{l_R l_L}\right) = \frac{Mg}{8}\left(1 + \frac{L^2}{l_L(L - l_L)}\right).$$

After this we find that

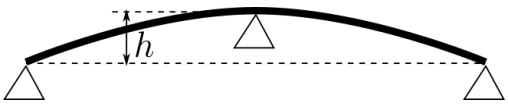
$$F_L = \frac{Mg}{8}\left(3 + \frac{l_L}{L} - \frac{L}{l_L}\right), \qquad F_R = \frac{Mg}{8}\left(3 + \frac{L - l_L}{L} - \frac{L}{L - l_L}\right).$$

In particular

- The answer does not depend on E! So the limit $E \to \infty$ is well defined!
- If $l_L = L/2$, we have $F_2 = \frac{5}{8}Mg$, $F_L = F_R = \frac{3}{16}Mg$. The guy at the center carries more than half of the total weight!
- If $l_L \to 0$ ($l_L \to L$), then F_2 and F_L (F_R) diverges. Why?

28.3. The force as a function of h.

Now let's finish this problem and compute how the force F_2 depends on the height h of the middle support. We simplify the problem by considering the middle support to be in the center: $l_R = l_L \equiv l = L/2$.



We expect that the result for the force on the center support will be linear in h as for a spring.

- This is different from the situation of two unstretched springs. The difference is the torques that appears at bending.

So the result should have the form $F(h) = -\frac{5}{8}Mg - kh$. The spring constant k will depend on the Young modulus E. It is also clear, that if we fix the position of the ends (this is what we do for the solution) the spring constant will not depend on g, as it will be the same even without gravity. The force is always proportional to the combination EIA.

- The dimensional analysis then gives $kh \sim \frac{EIA}{l^2}\frac{h}{l}$. ($E$ has units of pressure, A is cross-section area, I has units of length square.)

The prefactor should be just a number.

Again we have two functions $Y_L(x)$ and $Y_R(x)$ and the following boundary conditions

- At both ends we must have $Y = 0$, so

$$Y_L(x = -l) = 0, \qquad Y_R(x = l) = 0.$$

- At $x = 0$ we must have $Y = h$ for both parts, so

$$Y_L(x = 0) = h, \qquad Y_R(x = 0) = h.$$

- The torque is continuous at the center

$$\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=0}.$$

- The beam is smooth at the center

$$\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0},$$

- The torques at the ends are zero.

$$\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=-l} = 0, \qquad \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=l} = 0,$$

The force *on the support from the beam* is given by (different sign then before)

$$F = -IAE\left(\left.\frac{d^3 Y_R}{dx^3}\right|_{x=0} - \left.\frac{d^3 Y_L}{dx^3}\right|_{x=0}\right).$$

The first two conditions are satisfied by the functions of the form

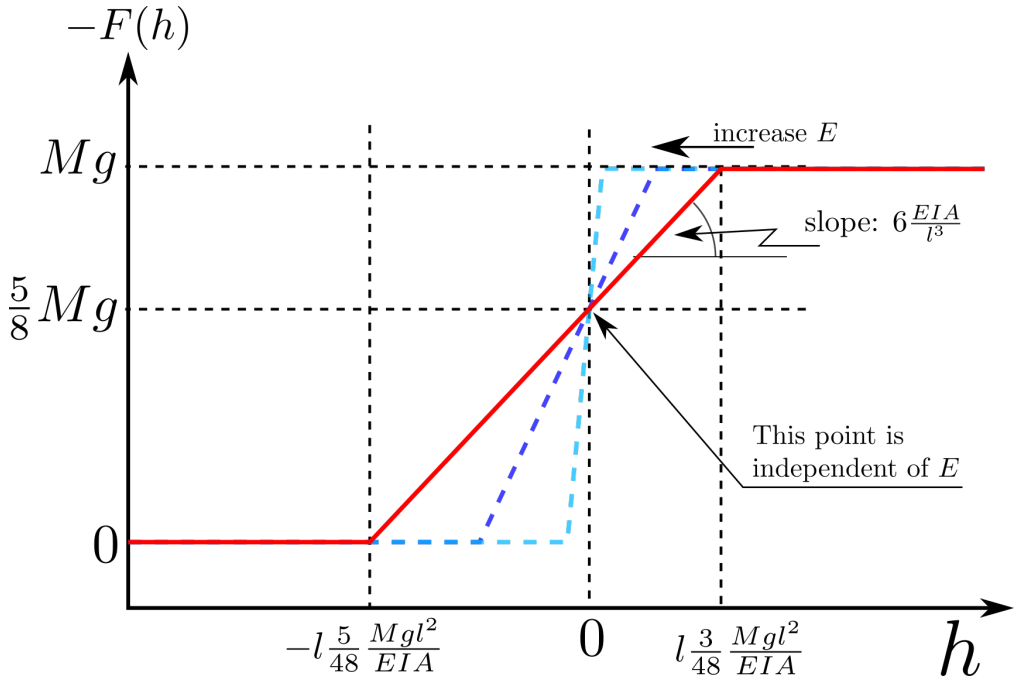
$$Y_L(x) = -\frac{\rho g}{24IE}(x+l)\left(x^3 + C_2^L x^2 + C_1^L x - \frac{24IE}{\rho g l}h\right)$$
$$Y_R(x) = -\frac{\rho g}{24IE}(x-l)\left(x^3 + C_2^R x^2 + C_1^R x + \frac{24IE}{\rho g l}h\right)$$

The rest four conditions are enough to determine four unknown constants. As the result we have for the force *on the support*

$$F(h) = -\frac{5}{8}Mg - 6\frac{EIA}{l^2}\frac{h}{l}.$$

It has the expected form. One can see, that

$$F(h=0) = -\frac{5}{8}Mg$$
$$F = 0, \quad \text{for } h = -l\tfrac{5}{48}\tfrac{Mgl^2}{EIA}$$
$$F = -Mg, \quad \text{for } h = l\tfrac{3}{48}\tfrac{Mgl^2}{EIA}.$$



28.4. N supports.

Let's see if this scheme will work if we have N supports: two at the ends of a beam and $N-2$ somewhere in between.

- We will have $N-1$ functions Y, each requires 4 constants, so we need $4(N-1)$ conditions.
- We have two conditions on both ends — 4 conditions.
- On each of the $N-2$ supports in the middle we have:
  - Two conditions that the functions on the left and on the right have the same fixed (the height of the support) value — 2 conditions.
  - The smoothness — match of the first derivatives on the left to the that of on the right — 1 condition.
  - The torque condition — match of the second derivatives on the left to the that of on the right — 1 condition.
  - Total 4 conditions on each of the middle support.
- So total for all middle supports we have $4(N-2)$ conditions.
- Adding the conditions on the ends we have $4(N-2)+4 = 4(N-1)$ conditions.
- The total number of conditions equals to the total number of the unknowns!

So the scheme which we derived will work for any number of supports.

LECTURE 29
Hydrodynamics of Ideal Fluid: Mass conservation and Euler equation.

- Students evaluation ends May 1, 2024.

29.1. Hydrostatics.

For the statics of liquid we can use the elastic theory. The main difference between the solid body and the liquid is that the liquid has zero sheer coefficient. In this case the equation

$$\sigma_{ik} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}).$$

tells us that the stress tensor is diagonal and we can use $\sigma_{ij} = -P\delta_{ij}$. The constant P is called pressure. We then have

$$\frac{\delta V}{V} = u_{ii} = \frac{\sigma_{ii}}{3K} = -\frac{P}{K}.$$

The constant K is then given by the equation of state for the liquid.

The static equilibrium condition is that the sum of all forces on a volume dV is zero. The force from stress tensor on the volume dV is $\frac{\partial \sigma_{ij}}{\partial x_j}$, the force from gravity is $g_i \rho dV$, where ρ is the liquid's density. So the static equilibrium condition reads:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\rho g_i,$$

gives

$$\frac{\partial P}{\partial x_i} = \rho g_i, \qquad \vec{\nabla} P = \rho \vec{g}$$

So the pressure at the point of depth h is $P = \rho g h$.

Consider a small volume dV. The force which acts on it is the weight $\rho \vec{g} dV$ and the force of the hydrostatic pressure. We see, that the force of the hydrostatic pressure is

$$d\vec{f} = -dV\vec{\nabla} P$$

.

29.2. Hydrodynamics of ideal liquid.

- Separation of scales.
- Separation of time scales.
- Universality.

Ideal fluid means that there is no viscosity.

29.2.1. Mass conservation.

The liquid is now moving. Mass current: amount of mass dM through an area dS during time dt is $dM = Idt$, I is proportional to dS and depends on the orientation, so $I = \vec{j} \cdot d\vec{S}$. $\vec{j}$ is the mass current density and is

$$\vec{j} = \rho \vec{v}.$$

Mass conservation, consider an arbitrary volume V

- during time dt the amount of which crosses the boundary of the volume V going in is $\delta m = -dt \oint_{\partial V} \vec{j} \cdot d\vec{S}$, where $d\vec{S}$ points out. Using Gauss theorem we can rewrite this as $\delta m = -dt \int_V \vec{\nabla} \cdot \vec{j} dV$.
- During the same time dt the change of mass inside the volume V is $dt \int_V \dot{\rho} dV$.
- As the ONLY WAY (this is our mass conservation statement) for the mass inside to change is for it to cross the boundary we must have

$$-dt \int_V \vec{\nabla} \cdot \vec{j} dV = dt \int_V \dot{\rho} dV$$

- As it is correct for ANY volume V we have

$$\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0.$$

This is called the continuity equation. It represents the fact that the mass cannot appear or disappear. It will also be correct for any conserved quantity with the correct definition of "current density".

29.2.2. Another Euler equation.

We can describe the flow of liquid in two different ways:

- Describe the position and the velocity of the "liquid particles" as the function of time.
- Introduce the fields $\rho(\vec{r}, t)$, $P(\vec{r}, t)$, and $\vec{v}(\vec{r}, t)$ of density, pressure, and velocity and describe the dynamics of these fields.

Describe the two point of views.

- the field $\vec{v}(\vec{r}, t)$ describes the velocity at the point $\vec{r}$ at time t. It is NOT a velocity of an object! it's time derivative $\frac{\partial \vec{v}}{\partial t}$ is NOT an acceleration of an object. We cannot use the Newton's laws for it.

Instead we must consider a small volume dV at point $\vec{r}$ at time t. This volume has mass ρdV and is an object to which we can apply the Newton's law $\vec{F} = m\vec{a}$.

- The force which acts on this volume is $-dV \nabla P$.
- Considering the vector field $\vec{v}(\vec{r}, t)$ as given at each point and at each time(!):
  - At time t our volume dV has the velocity $\vec{v}(\vec{r}, t)$.
  - At time $t + dt$ this volume/object will shift to the position $\vec{r}' = \vec{r} + \vec{v}(\vec{r}, t)dt$.

  - So its velocity at time $t + dt$ is

$$\vec{v}(\vec{r}', t + dt) = \vec{v}(\vec{r} + \vec{v}(\vec{r}, t)dt, t + dt) \approx \vec{v}(\vec{r}, t) + (\nabla_i \vec{v})v_i dt + \frac{\partial \vec{v}}{\partial t} dt.$$

  - So during time dt the velocity of the volume/object dV has changed by $dv = dt(\vec{v} \cdot \nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t} dt$.
  - Its acceleration then is $\vec{a} = \frac{dv}{dt} = (\vec{v} \cdot \nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t}$.
- Now we can write $\vec{F} = m\vec{a}$:

$$-dV \nabla P = \rho dV \left((\vec{v} \cdot \nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t} \right)$$

The equation of the vector field $\vec{v}(\vec{r}, t)$ time evolution (Euler equation) is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho} \vec{\nabla} P$$

In case there is gravity there is extra force $\rho dV \vec{g}$ on this volume/object, so the equation is modified

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho} \vec{\nabla} P + \vec{g}.$$

This equation together with the continuity equation *and the equation of state* are the full set of equations which must be supplied with the boundary conditions.

- The equation of state – how ρ depends on P (and may be temperature) – is what distinguishes one liquid from another.
- All together we have three equations (five in components) for three (again five in components) fields $\vec{v}$, P, and ρ.
- Student evaluations.

LECTURE 30
Hydrodynamics of Ideal Fluid: Incompressible fluid, potential flow.

- Students evaluation 04-19-2023 until 05-03-2023.

30.1. Incompressible liquid.

In case of incompressible liquid the equation of state is particularly simple: the density is constant. So we have

$$\vec{\nabla}\cdot\vec{v}=0, \qquad \frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\vec{\nabla}\left(\frac{P}{\rho}+\Phi_g\right).$$

In this case we can use the following trick (more formal justification can be found in the cutout): we will be looking for the solution in the form

$$\vec{v}=\nabla\phi.$$

Notice, that this implies that $\mathrm{curl}\vec{v}=0$.

The continuity equation then gives

$$\Delta\phi=0.$$

This is so called potential flow. We need to supplement this equation with the boundary conditions. The simplest one is that on each boundary the component of the fluid velocity perpendicular to the boundary equals to the component of the boundary velocity perpendicular to the boundary.

Now substituting $\vec{v}=\vec{\nabla}\phi$ into the Euler equation and using $(\vec{v}\cdot\vec{\nabla})\vec{v}=\partial^i\phi\partial^i\partial^j\phi=\partial^i\phi\partial^j\partial^i\phi=\frac{1}{2}\partial^j(\partial^i\phi)^2=\nabla\frac{1}{2}v^2$ we find

$$\vec{\nabla}\left(\frac{\partial\phi}{\partial t}+\frac{P}{\rho}+\Phi_g+\frac{1}{2}v^2\right)=0,$$

and finally (notice, that the function f does not depend on the coordinate and thus must be given by the boundary conditions) we get the equation for P.

$$\frac{\partial\phi}{\partial t}+\frac{P}{\rho}+\Phi_g+\frac{1}{2}v^2=f(t)$$

Using the formula

$$\vec{v}\times\mathrm{curl}\vec{v}=\frac{1}{2}\vec{\nabla}v^2-(\vec{v}\cdot\vec{\nabla})\vec{v}$$

we can rewrite the Euler equation as

$$\frac{\partial\vec{v}}{\partial t}-\vec{v}\times\mathrm{curl}\vec{v}=-\vec{\nabla}\left(\frac{P}{\rho}+\Phi_g+\frac{1}{2}v^2\right).$$

If we now take curl of both sides we'll get

$$\frac{\partial}{\partial t}\mathrm{curl}\vec{v}-\mathrm{curl}(\vec{v}\times\mathrm{curl}\vec{v})=0$$

Notice, that this equation is identically satisfied if $\mathrm{curl}\vec{v}=0$. Which in turn identically satisfied by $\vec{v}=\vec{\nabla}\phi$ for some function ϕ.

30.2. Potential flow around a moving sphere

Consider a sphere of radius R moving with the velocity $\vec{u}$ in the ideal incompressible fluid. The flow of the fluid around the sphere is potential, so we need to solve the equation

$$\Delta\phi = 0, \qquad \vec{n}\cdot\vec{v}|_{\text{on sphere}} = \vec{n}\cdot\vec{u}, \qquad \vec{v}|_{r\to\infty} \to 0.$$

where the boundary conditions demand that the normal component of the fluid on the sphere equals the normal component of the element of the sphere.

The function ϕ is the scalar. It must linearly depend on the velocity $\vec{u}$ as both the Laplace equation and the boundary conditions are linear. This is analogous to the dipole field in the electrostatics, so the solution must be of the form

$$\phi = a\vec{u}\cdot\vec{\nabla}\frac{1}{r},$$

where a is an arbitrary constant which must be found from the boundary conditions. This is the field produced by the dipole $\vec{d} = a\vec{u}$, so the velocity (electric field) is

$$\vec{v} = \frac{a}{r^3}\left[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}\right].$$

So on the sphere surface we have

$$\vec{v}\cdot\vec{n}|_{r=R} = \frac{2a}{R^3}(\vec{u}\cdot\vec{n})$$

and we see, that $a = \frac{R^3}{2}$ and

$$\phi = -\frac{R^3}{2r^2}\vec{u}\cdot\vec{n}, \qquad \vec{v} = \frac{R^3}{2r^3}\left[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}\right].$$

In order to calculate the pressure use $\frac{\partial\phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}v^2 = \frac{P_0}{\rho}$. We then need to calculate $\frac{\partial\phi}{\partial t}$. In order to do we must remember, that the sphere is moving, so we need to think about the potential ϕ as a function of the position of the center of the sphere $\vec{r}_0$ and its velocity $\vec{u}$: $\phi(\vec{r} - \vec{r}_0, u)$, then we have

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\vec{u}}\cdot\dot{\vec{u}} - \vec{u}\cdot\nabla\phi.$$

We then find

$$P = P_0 + \frac{1}{8}\rho u^2(9\cos^2\theta - 5) + \frac{1}{2}\rho R\vec{n}\cdot\frac{d\vec{u}}{dt}.$$

We can calculate the total force acting on the sphere

$$\vec{F} = \oint P d\vec{S}.$$

The integration of the first two terms in P gives zero. For the last term we find

$$F_i = \frac{1}{2}\rho R\frac{du_j}{dt}4\pi R^2\overline{n_j n_i} = \frac{1}{2}\frac{4\pi}{3}\rho R^3\frac{du_i}{dt}.$$

(it is clear that $\overline{n_j n_i}$ must be diagonal. Also $\overline{n_x n_x} = \overline{n_y n_y} = \overline{n_z n_z}$ and $\overline{n_i n_i} = 1$) So we find that

$$\vec{F} = \frac{1}{2}\rho R\frac{du_j}{dt}4\pi R^2\overline{n_j n_i} = \frac{2\pi}{3}\rho R^3\frac{d\vec{u}}{dt}.$$

Notice:

- Without the viscosity the force is zero if the velocity of the sphere does not change.

- The liquid just effectively changes the mass of the sphere by the value

$$\frac{1}{2}\frac{4}{3}\pi R^3\rho.$$

Half of the expelled mass.