Advanced Mechanics II. Phys 303

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Contents

LECTURE 1
Review of Phys 302. Newtonian formulation.

1.1. Introduction

- Syllabus. Exams. Homeworks. Grades. Office hours etc.
- Structure (tentative) of the course.
- Questions, interruptions etc.
- Homeworks, cheating, study groups.
- Homework sessions, Office hour.
- Student evaluations for the last semester.
  - No solutions for the homeworks.
  - How to prepare for the exams — banks of problems.
  - Study materials: my lecture notes, books, ask questions.
  - Homework grading and comments.
  - Non-optimal use of the black board.
- What is my role?
- Life is good and physics is great!

1.2. Newtonian formulation

- Inertial frame of references.
- A particle position is described by a time dependent position vector $\vec{r}(t)$. The vectors velocity and acceleration are the time derivatives of the position vector
$$\vec{v} = \frac{d\vec{r}}{dt}, \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2}.$$
- The equation of motion is
$$\vec{F} = m\vec{a}$$
- The forces are the result of INTERaction.
$$\vec{F}_{12} = -\vec{F}_{21}$$
- Write (draw) down all the forces that act on a body. Remember, that the force is always a result of INTERaction.
- Chose a Cartesian system of coordinates in the inertial frame of reference.
- Write down the components of the forces in the chosen system of coordinates.
- For each component write down the equation of motion
$$F_i = ma_i, \qquad \dot{\vec{p}} = \vec{F}.$$
- Solve the resulting system of generally nonlinear second order differential equations.
- Use the initial conditions in order to find the motion $\vec{r}(t)$.

Pros:

- Very straight forward and intuitive.
- Very general – the nature of the forces does not matter, as long as you know them.

Cons:

- The symmetries and corresponding conservation laws are hidden.
- Difficult to use in anything but the inertial frame and Cartesian coordinates. (fictitious forces etc)
- Very quickly becomes cumbersome. Easy to make mistakes.

Examples. Wedge. Wedge with friction. Oscillator. Pendulum.

1.3. Conservation laws.

Momentum conservation law.

- Center of mass motion.
- Inelastic collision.
- Rocket motion,

$$Mv = (M + dM)(v + dv) + (V_0 - v)dM, \qquad \frac{dv}{V_0} = -\frac{dM}{M}, \qquad v_f - v_i = -V_0 \log \frac{M_f}{M_i}$$

Angular momentum conservation law.

- Angular momentum
$$\vec{J} = \vec{r} \times \vec{p}$$
- Torque
$$\vec{\tau} = \vec{r} \times \vec{F}$$
- Central internal forces.
- $$\dot{\vec{J}} = \vec{\tau}_{ex}.$$

Examples:

- A bullet in a disc with the fixed axis.
- A bullet in a disc-like wheel with no friction. What height should the bullet strike for the wheel to roll without slipping?
- At what point should the stick strike so that the striking hand feel good?

LECTURE 2
Review of Phys 302. Energy conservation.

Example: At what point should the stick strike so that the striking hand feels good?

- Energy conservation law.
  - Work
  $$\mathcal{A} = \int_A^B \vec{F} \cdot d\vec{r}$$
  It depends on the path from A to B.
  - Kinetic energy.
  $$\mathcal{A} = \int \vec{F} \cdot d\vec{r} = \int m\frac{d\vec{v}}{dt} \cdot \vec{v}dt = \int m\vec{v} \cdot d\vec{v} = \Delta\frac{m\vec{v}^2}{2}.$$
  - Conservative forces.
  $$\vec{F} = -\frac{\partial U}{\partial \vec{r}}$$
  - Potential energy U. If forces depend on coordinates only it does not mean that the force is conservative and the function U exists.
  - On a closed contour the work of a conservative force equals zero.
  - For a conservative force the work does not depend on the path.
  - Total energy
  $$E = \frac{m\vec{v}^2}{2} + U.$$
  is conserved
  $$\frac{dE}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{\partial U}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} = \left(m\vec{a} - \vec{F}\right) \cdot \vec{v} = 0$$

Examples: A wall with rope and a cart. Elastic 1D collision.

Examples of both Energy and momentum conservation: Elastic collision in 2D (case of equal masses.)

Example of 1D motion under conservative force.

$$\int_{x_0}^{x_f} \frac{dx}{\sqrt{E - U(x)}} = \pm\sqrt{\frac{2}{m}}(t - t_0), \qquad x(t = t_0) = x_0, \qquad E = \frac{mv_0^2}{2} + U(x_0).$$

Example: $1D$, the graph $U(x)$.

LECTURE 3
Review of Phys 302. Lagrangian and Hamiltonian formulations.

3.1. Lagrangian formulation.

- Action
$$S = \int_A^B L dt$$
- Hamilton principle. Minimum of action. Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t)$. Euler-Lagrange equation.
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$
This is a condition for the functional S to be at extremum.
- Classical mechanics states that if the Lagrangian is
$$L = K - U$$
Then the solution of the Euler-Lagrange equation gives the trajectory.
- The Euler-Lagrange equation is a second order (nonlinear) differential equation.
- Generalized momentum (canonical)
$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$
- Conservation of generalized momentum (ignorable coordinates).
- Conservation of energy – no explicit time dependence in the Lagrangian.
$$E = \sum_i p_i \dot{q}_i - L$$

Cons:

- Only conservative forces.

Pros:

- General coordinates.
- Only one scalar function L needs to be constructed. Easier.
- Symmetries are more transparent.

Examples:

- Pendulum in the accelerating car.

The technique of minimizing a functional is not used in mechanics exclusively. There are a lot of problems where such techniques are useful. The conservation lows will also be applicable there, but will, in general, have different meaning.

3.2. Hamiltonian formulation

- Phase space $(\{q_i\}, \{p_i\})$.
- Poisson brackets $\{p_i, q_j\}$.
  - Antisymmetric.
  - Bilinear.
  - For a constant c, $\{f, c\} = 0$.
  - $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$.
  - Jacobi's identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.
- Hamiltonian $H(\{q_i\}, \{p_i\})$.
- Hamiltonian equation of motion: for any function f on phase space $\dot{f} = \{H, f\}$, in particular

$$\dot{p}_i = \{H, p_i\}, \qquad \dot{q}_i = \{H, q_i\}$$

- In canonical coordinates and momenta

$$\{p_i, q_j\} = \delta_{i,k}, \qquad \{f, g\} = \sum_i \left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right).$$

-

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

- If we know the Lagrangian, then the momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ are canonical and the Hamiltonian is given by:

$$H(p_i, q_i) = \sum_i p_i \dot{q}_i - L, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Imoprtant:

- Lagrangian is a function of generalized coordinates and velocities.
- Hamiltonian is a function of generalized coordinates and momenta.

3.3. Motion in 2D in a central field

- Motion in 2D in a central field.

$$L = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} - U(r)$$

$$\frac{d}{dt}m\dot{r} = mr\dot{\phi}^2 - \frac{\partial U}{\partial r}, \qquad \frac{d}{dt}mr^2\dot{\phi} = 0$$

So we the angular momentum

$$L_\phi = mr^2\dot{\phi}$$

is conserved. We can use then $\dot{\phi} = \frac{L_\phi}{mr^2}$ and write

$$\frac{d}{dt}m\dot{r} = \frac{L_\phi^2}{mr^3} - \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{L_\phi^2}{2mr^2} + U\right)$$

This is a motion in $1D$ in the effective central potential

$$U_{eff}(r) = U(r) + \frac{L_\phi^2}{2mr^2}.$$

We then know the solution. As $\frac{m\dot{r}^2}{2} + U_{eff}(r) = E$, we have

$$\frac{dr}{\sqrt{E - U_{eff}(r)}} = \pm\sqrt{\frac{2}{m}}dt, \qquad dt = \frac{mr^2}{L_\phi}d\phi$$

or

$$\pm d\phi = \frac{L_\phi}{\sqrt{2m}}\frac{1}{r^2}\frac{dr}{\sqrt{E - U_{eff}(r)}}$$

- Kepler orbits. Let's use the gravitational potential energy

$$U(r) = -\frac{GMm}{r},$$

then we have

$$U_{eff} = -\frac{GMm}{r} + \frac{L_\phi^2}{2mr^2}$$

and

$$\phi - \phi_0 = \frac{L_\phi}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr'}{r'^2}\frac{1}{\sqrt{E + \frac{GMm}{r'} - \frac{L_\phi^2}{2mr'^2}}}$$

- For $E < 0$ this expression will give $r(\phi)$ for Kepler orbits.
- For $E > 0$ it will give the unbounded orbits. In particular we can use it to compute the angle between the incoming and outgoing velocities at infinity — the scattering angle.

LECTURE 4
Probability density. Disintegration of a particle.

4.1. Math preliminaries.

4.1.1. Probability density.

- Probability density.
  - Probability.
  - Probability density.
  - Positive definite.
  - Normalization.
  - Averaging: $\langle v^2 \rangle = \int v^2 \rho(v) dv$, or more generally $\langle f(v) \rangle = \int f(v)\rho(v) dv$.
- Examples:
  - Uniform distribution.
    - The probability for x is uniformly distributed in the interval $[a, b]$.
    - The probability density is a constant $\rho(x) = A$.
    - The constant must be found from the normalization condition:

$$\int_a^b \rho(x) dx = 1, \quad \text{so} \quad A = \frac{1}{b-a}.$$

    - Average x

$$\langle x \rangle = \int_a^b x\rho(x) dx = \frac{1}{b-a} \int_a^b x dx = \frac{a+b}{2}.$$

      as expected.
    - We also can compute average of x^2:

$$\langle x^2 \rangle = \int_a^b x^2 \rho(x) dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{b^2 + ab + a^3}{3}.$$

      Notice, $\langle x^2 \rangle \neq \langle x \rangle^2$.
  - $1D$ quantum harmonic oscillator:
    - Potential energy $U(x) = \frac{m\omega^2 x^2}{2}$: the wave function of the ground state is $\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$.

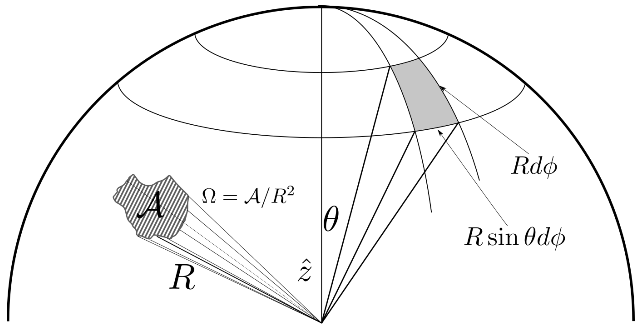


Figure 1. Solid Angle.

- The probability density for the coordinate is
$$\rho(x) = |\psi(x)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{\hbar}x^2}, \qquad \int_{-\infty}^{\infty}\rho(x)dx = 1.$$
These are the results of Quantum mechanics. We here take them for granted and only use the resulting probability density.
- Average position in the ground state is given by
$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)dx = 0.$$
- However,
$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\rho(x)dx = \frac{1}{2}\frac{\hbar}{m\omega}.$$
- In particular average potential energy
$$\langle U(x)\rangle = \int_{-\infty}^{\infty} U(x)\rho(x)dx = \frac{m\omega^2}{2}\langle x^2\rangle = \frac{1}{4}\hbar\omega.$$
- We also can ask what is probability density for the potential energy U? We see, that $x = \sqrt{2U/m\omega^2}$, so $dx = \frac{dU}{\sqrt{2Um\omega^2}}$.
$$\rho(x)dx = \rho\left(\sqrt{2U/m\omega^2}\right)\frac{dU}{\sqrt{2Um\omega^2}} = \frac{1}{\sqrt{2\pi U\hbar\omega}}e^{-\frac{U}{\hbar\omega}}dU.$$
so
$$\rho_U(U) = \frac{1}{\sqrt{2\pi U\hbar\omega}}e^{-\frac{U}{\hbar\omega}}, \qquad \int_0^{\infty}\rho_U(U)dU = 1.$$

- Change of variables.
  Let's say we have a probability density $\rho(v)$ to find a speed between v and $v+dv$. We want to find the probability density to find a kinetic energy between K and $K+dK$. The probability is
$$dp = \rho(v)dv$$
The kinetic energy is $K = \frac{mv^2}{2}$, or $v = \sqrt{2K/m}$, so $dv = \frac{dK}{m\sqrt{2K/m}}$, and
$$dp = \rho(\sqrt{2K/m})\frac{1}{m\sqrt{2K/m}}dK, \qquad \rho_K(K) = \rho(\sqrt{2K/m})\frac{1}{m\sqrt{2K/m}}.$$
- Notice the change of differential!

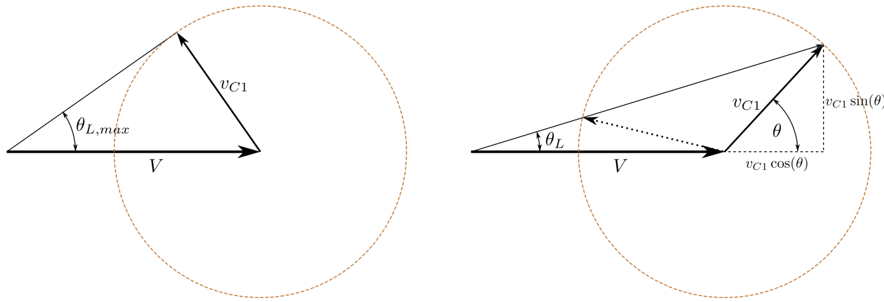


Figure 2. Illustrations to the equations (4.1) and (4.2).

- Notice, that if the distribution is uniform in one variable it may not be uniform in the other!

4.1.2. Solid angle.

- An angle in radians is the ratio of the arc's length to the radius.
- Analogously, the solid angle is the ration of the area of the patch on the surface of a sphere to the square of the radius of the sphere.
- A small solid angle $d\Omega$ is given by $d\Omega \sin\theta d\theta d\phi$, see figure 1.

4.2. Disintegration of a particle.

We want to consider the following problem: A particle has an internal energy ϵ. At some point in time this energy is released and the particle disintegrates into two particles of masses m_1 and m_2. There is a detector in the laboratory. The detector can detect the direction and magnitude of the velocity of the particle 1. The initial particle has velocity V in the laboratory frame of references. We want to know what the detector will show if we have of a beam of disintegrating particles.

- Disintegration of a single particle.
  - In the center of mass system of reference the initial momentum is zero and the initial energy is ϵ, so the conservation laws give

$$m_1 v_{C1} + m_2 v_{C2} = 0, \qquad \frac{m_1 v_{C1}^2}{2} + \frac{m_2 v_{C2}^2}{2} = \epsilon,$$

$$\frac{m_1 v_{C1}^2}{2} + \frac{m_1^2 v_{C1}^2}{2m_2} = \frac{m_1 v_{C1}^2}{2}\left(1 + \frac{m_1}{m_2}\right) = \epsilon$$

    The direction of the velocity V_{C1} is arbitrary with the uniform distribution – this means that there is no control over the orientation of the initial particle.
  - We are observing the process in the laboratory frame of references. In this frame the center of mass has a velocity V. In the laboratory system of reference

$$\vec{v}_{L1} = \vec{V} + \vec{v}_{C1}.$$

  - Kinematics show (see figure)

$$\sin(\theta_{L,max}) = \frac{v_{C1}}{V}, \quad \text{if} \quad V > v_{C1} \tag{4.1}$$

and

$$\tan\theta_L = \frac{v_{C1}\sin(\theta)}{v_{C1}\cos(\theta)+V}, \tag{4.2}$$

or

$$\cos\theta = -\frac{V}{v_{C1}}\sin^2(\theta_L) \pm \cos(\theta_L)\sqrt{1-\frac{V^2}{v_{C1}^2}\sin^2(\theta_L)}.$$

For $v_{C1} > V$ the result is one-to-one, we must take the + sign, so that $\theta(\theta_L = 0) = 0$, for $v_{C1} < V$ the result is not one-to-one: for a single θ_L in laboratory frame, there are two θs in the center of mass frame.

LECTURE 5
Scattering cross-section.

5.1. Disintegration of many particles.

- All detectors are on a sphere of a large radius R_d.
- We are watching only particle number 1.
- In each disintegration we know the speed v_{C1} if we know the internal energy of the initial particle ϵ, or vice verse

$$\frac{m_1 v_{C1}^2}{2}\left(1+\frac{m_1}{m_2}\right)=\epsilon.$$

  The speed in the center of mass reference frame is the same for all particles number 1 which are the products of disintegrating initial particles.
- The direction of the vector $\vec{v}_{C1}$ is arbitrary. We assume that there is no preferential direction (the way the initial particle was set up) and any direction of $\vec{v}_{C1}$ is equally probable.
- In the center of mass ref. frame the probability to find the particle 1 in the solid angle $d\Omega$ is uniform and is given by

$$dp=\frac{R^2 d\Omega}{4\pi R^2}=\frac{\sin(\theta)d\theta d\phi}{4\pi}$$

  The angle θ is the angle between the vectors of the velocity of the center of mass $\vec{V}$ and velocity $\vec{v}_{C1}$ of the particle 1 *in the center of mass reference frame!!!!!*
- The probability density to find the velocity $\vec{v}_{C1}$ direction between the angle θ and $\theta+d\theta$ is (we do not care about the angle ϕ.)

$$dp=d\theta\int_0^{2\pi}\frac{\sin(\theta)d\phi}{4\pi}=\frac{1}{2}\sin(\theta)d\theta.$$

  (One should check that the total is 1.)
- Now we have everything (see previous lecture) to find the distribution of particles over θ_L – the angle measured in laboratory frame of references.
- Instead of doing that we will find the distribution of the kinetic energy of the particles 1 in the laboratory frame of ref. (**Important note:** in the center of mass frame of ref. the distribution of the kinetic energies is trivial, as all particles 1 have the same

speed v_{C1}, so they have the same kinetic energy. However, in the lab. frame of ref. it is not so.)

- In the laboratory reference frame $\vec{v}_{L1} = \vec{V} + \vec{v}_{C1}$, so
$$v_{L1}^2 = V^2 + v_{C1}^2 + 2Vv_{C1}\cos(\theta).$$
where θ is the angle in the center of mass ref. frame.
- The kinetic energy of the particle 1 in the laboratory ref. frame is
$$K = \frac{m_1 v_{L1}^2}{2} = \frac{m_1 V^2}{2} + \frac{m_1 v_{C1}^2}{2} + m_1 V v_{C1}\cos(\theta),$$
$$dK = -m_1 V v_{C1}\sin(\theta)d\theta.$$
or
$$\sin(\theta)d\theta = -\frac{1}{m_1 V v_{C1}}dK$$
- We can ignore the minus sign in the above (it only tells us that the particles with larger kinetic energy have smaller angle)
$$dp = \frac{1}{2}\sin\theta d\theta = \frac{1}{2m_1 V v_{C1}}dK.$$
The uniform distribution: $\rho_K = \frac{1}{2m_1 V v_{C1}}$
- The maximum kinetic energy is $K_{max} = \frac{m_1(V+v_{C1})^2}{2}$. The minimum is $K_{min} = \frac{m_1(V-v_{C1})^2}{2}$.
- One can check that
$$\int_{K_{min}}^{K_{max}} \rho_K dK = 1$$
- So if one knows V and m_1, one can measure $\rho(K)$. If it is uniform, then on knows that what happens is particle disintegration. Measuring the width of the distribution, one finds v_{C1}. Then one can find ϵ — learn something about the initial particles!

5.2. Scattering.

- Set up of a scattering problem. Experiment, detector, etc.
- Energy. Impact parameter. The scattering angle. Impact parameter as a function of the scattering angle $\rho(\theta)$.
- Flux of particle. Same energy, different impact parameters, different scattering angles.
- The scattering problem, n — the flux, number of particles per unit area per unit time. dN the number of particles scattered between the angles θ and $\theta + d\theta$ per unit time. A suitable quantity to describe the scattering
$$d\sigma = \frac{dN}{n}.$$
- **It has the units of area and is called differential cross-section.**
- If we know the function $\rho(\theta)$, then only the particles which are in between $\rho(\theta)$ and $\rho(\theta + d\theta)$ are scattered at the angle between θ and $\theta + d\theta$. So $dN = n2\pi\rho d\rho$, or
$$d\sigma = 2\pi\rho d\rho = 2\pi\rho\left|\frac{d\rho}{d\theta}\right|d\theta$$

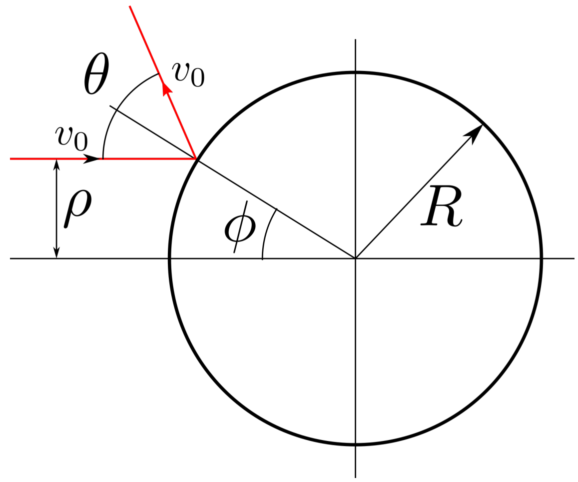


Figure 1. The scattering processes from: left, rigid sphere; right, spherical square potential $-U_0$.

(The absolute value is needed because the derivative is usually negative.)

- Often $d\sigma$ refers not to the scattering between θ and $\theta + d\theta$, but to the scattering to the solid angle $d\omega = 2\pi \sin\theta d\theta$. Then

$$d\sigma = \frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| d\omega$$

5.2.1. Example: Scattering from a perfectly rigid sphere

- Cross-section for scattering of particles from a perfectly rigid sphere of radius R.
  - The scattering angle $\theta = 2\phi$.
  - $R\sin\phi = \rho$, so $\rho = R\sin(\theta/2)$.
  - $$d\sigma = \frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| d\omega = \frac{1}{4}R^2 d\omega$$
  - Independent of the incoming energy. The scattering does not probe what is inside.
  - The total cross-section area is
$$\sigma = \int d\sigma = \frac{1}{4}R^2 2\pi \int_0^\pi \sin\theta d\theta = \pi R^2$$

LECTURE 6
Rutherford's formula.

6.1. Example of scattering.

- We defined the scattering differential cross-section

$$d\sigma = \frac{dN}{n},$$

where dN number of particles per unit time captured by a given detector and n is the flux of the particles in the beam.

- We found that

$$d\sigma = \frac{\rho(\theta,\phi)}{\sin\theta}\left|\frac{d\rho(\theta,\phi)}{d\theta}\right| d\omega$$

where ρ is the impact parameter, θ and ϕ are the coordinates of the detector and also the scattering angle, and $d\omega = \sin\theta d\theta d\phi$ is the solid angle of the detector.

- In a axially symmetric potential noting depends on ϕ. One then can use a detector which is a ring and integrate over ϕ. Then the result is

$$d\sigma = 2\pi\rho(\theta)\left|\frac{d\rho(\theta)}{d\theta}\right| d\theta.$$

6.1.1. Example: Scattering from a spherical potential well.

- Cross-section for scattering of particles from a spherical potential well of depth U_0 and radius R.
  - Energy conservation

$$\frac{mv_0^2}{2} = \frac{mv^2}{2} - U_0, \qquad v = v_0\sqrt{1+\frac{2U_0}{mv_0^2}} = v_0\sqrt{1+U_0/E}$$

  - Angular momentum conservation

$$v_0\sin\alpha = v\sin\beta, \qquad \sin\alpha = n(E)\sin\beta, \qquad n(E) = \sqrt{1+U_0/E}$$

  - Scattering angle (see figure)

$$\theta = 2(\alpha-\beta)$$

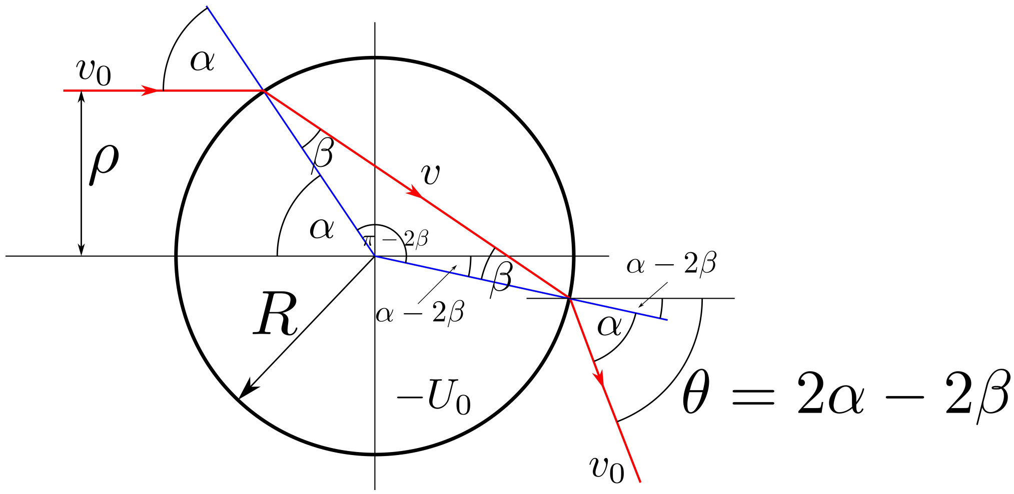


Figure 1. The scattering processes from: left, rigid sphere; right, spherical square potential $-U_0$.

– Impact parameter

$$\rho = R\sin\alpha$$

– So we have

$$\frac{\rho}{R} = \sin(\alpha) = n\sin(\beta) = n\sin(\alpha-\theta/2) = n\sin\alpha\cos(\theta/2) - n\cos\alpha\sin(\theta/2)$$
$$= n\frac{\rho}{R}\cos(\theta/2) - n\sqrt{1-\rho^2/R^2}\sin(\theta/2)$$

$$\rho^2 = R^2\frac{n^2\sin^2(\theta/2)}{1+n^2-2n\cos(\theta/2)}.$$

– The differential cross-section is

$$d\sigma = \frac{R^2n^2}{4\cos(\theta/2)}\frac{(n\cos(\theta/2)-1)(n-\cos(\theta/2))}{(1+n^2-2n\cos(\theta/2))^2}d\omega$$

– Differential cross-section depends on E/U_0, where E is the energy of incoming particles. By measuring this dependence we can find U_0 from the scattering.

– The scattering angle changes from 0 ($\rho=0$) to θ_{max}, where $\cos(\theta_{max}/2) = 1/n$ (for $\rho = R$). The total cross-section is the integral

$$\sigma = \int_0^{\theta_{max}} d\sigma = \pi R^2.$$

It does not depend on energy or U_0.

- Negative U_0.
  – Consider a negative U_0 – this is not a well, but a bump.
  – Then for $E > U_0$

$$n = \sqrt{1-|U_0|/E} < 1$$

and it is imaginary for $E < U_0$. This is a rigid sphere — the particle cannot get into the potential.

  – For $E > U_0$, or $n < 1$, there is no solution of equation $\sin(\alpha) = n\sin(\beta)$ for $\alpha > \alpha_{cr} = \sin^{-1}(n)$.

- So for $R\sin(\alpha_{cr}) = nR < \rho < R$ the particle does not penetrate inside the potential.
- In this range of impact parameters we will have a reflection from a "rigid" sphere.

6.2. Rutherford experiment.

- What is the question?
- Experiment set up.
- Expected result from Thomson model.
- Obtained result.

LECTURE 7
Rutherford's formula. Intro in oscillations.

7.1. Rutherford formula.

7.1.1. Arbitrary central potential

Consider the scattering of a particle of initial velocity v_∞ from the central force given by the potential energy $U(r)$.

- The energy is
$$E = \frac{mv_\infty^2}{2}.$$
- The angular momentum is given by
$$L_\phi = mv_\infty \rho,$$
where ρ is the impact parameter.
- The trajectory is given by
$$\pm(\phi - \phi_0) = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{r} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}, \qquad U_{eff}(r) = U(r) + \frac{L_\phi^2}{2mr^2}$$

where r_0 and ϕ_0 are some distance and angle on the trajectory.

At some point the particle is at the closest distance r_0 to the center. The angle at this point is ϕ_0 (the angle at the initial infinity is zero.) Let's find the distance r_0. As the

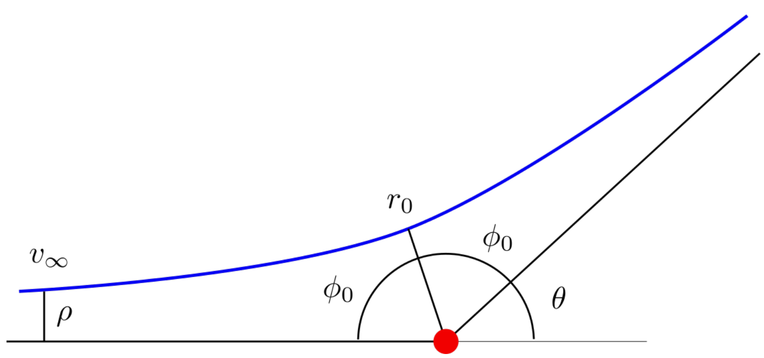


Figure 1. Rutherford experiment.

energy and the angular momentum are conserved and at the closest point the velocity is perpendicular to the radius we have

$$E = \frac{mv_0^2}{2} + U(r_0), \qquad L_\phi = mr_0v_0.$$

so we find that the equation for r_0 is

$$U_{eff}(r_0) = E.$$

This is, of course, obvious from the picture of motion in the central field as a one dimensional motion in the effective potential $U_{eff}(r)$.

The angle ϕ_0 is then given by

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}. \tag{7.1}$$

From geometry the scattering angle θ is given by the relation

$$\theta + 2\phi_0 = \pi. \tag{7.2}$$

So we see, that for a fixed v_0 the energy E is given, but the angular momentum L_ϕ depends on the impact parameter ρ. The equation (7.1) then gives the dependence of ϕ_0 on ρ. Then the equation (7.2) gives the dependence of the scattering angle θ on the impact parameter ρ. If we know that dependence, we can calculate the scattering cross-section.

$$d\sigma = \frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| d\omega$$

7.1.2. Coulomb potential.

Let's say that we have a repulsive Coulomb interaction

$$U = \frac{\alpha}{r}, \qquad \alpha > 0$$

In this case the geometry gives

$$\theta = \pi - 2\phi_0.$$

Let's calculate ϕ_0

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}}$$

where r_0 is the value of r, where the expression under the square root is zero.

Let's take the integral

$$\int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E - \alpha x - x^2\frac{L_\phi^2}{2m}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E + \frac{\alpha^2 m}{2L_\phi^2} - \frac{L_\phi^2}{2m}(x + \frac{\alpha m}{L_\phi^2})^2}}$$

$$= \sqrt{\frac{2m}{L_\phi^2}} \int_0^{1/r_0} \frac{dx}{\sqrt{\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4} - (x + \frac{\alpha m}{L_\phi^2})^2}}$$

changing $\sqrt{\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4}} \sin\psi = x + \frac{\alpha m}{L_\phi^2}$ we find that the integral is

$$\sqrt{\frac{2m}{L_\phi^2}} \int_{\psi_1}^{\pi/2} d\psi,$$

where $\sin(\psi_1) = \frac{\alpha m}{L_\phi^2}\left(\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4}\right)^{-1/2}$ So we find that

$$\phi_0 = \pi/2 - \psi_1$$

or

$$\cos\phi_0 = \sin\psi_1 = \frac{\alpha m}{L_\phi^2}\left(\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4}\right)^{-1/2}.$$

Using $L_\phi = \rho\sqrt{2mE}$ this gives

$$\sin\frac{\theta}{2} = \frac{\alpha}{2E}\left(\rho^2 + \frac{\alpha^2}{4E^2}\right)^{-1/2}$$

or

$$\frac{\alpha^2}{4E^2}\cot^2\frac{\theta}{2} = \rho^2$$

The differential cross-section then is

$$d\sigma = \frac{d\rho^2}{d\theta}\frac{1}{2\sin\theta}d\omega = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}d\omega$$

- Notice, that the total cross-section diverges at small scattering angles.

7.2. Rutherford formula analysis.

Rutherford's formula.

$$d\sigma = \frac{d\rho^2}{d\theta}\frac{1}{2\sin\theta}d\omega = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}d\omega$$

Divergence of the forward scattering in the ideal case – the cross-section of the beam.

- The beam. How do you characterize it?
- What is measured?
- The statistics. How much data we need to collect to get certainty of our results?
- The beam again. Interactions.
- Final state interactions.
- The forward scattering diverges.
- The cut off of the divergence is given by the size of the atom.
- Back scattering. Almost no dependence on θ.
- Energy dependence $1/E^2$.
- Plot $d\sigma$ as a function of $1/(4E)^2$, expect a straight line at large $1/(4E)^2$.
- The slope of the line gives α^2.
- What is the behavior at very large E? What is the crossing point?
- The crossing point tells us the size of the nucleus $d\sigma = \frac{R^2}{4}d\omega$.

7.3. Small oscillations.

Problem with one degree of freedom: $U(x)$. The Lagrangian is

$$L = \frac{m\dot{x}^2}{2} - U(x).$$

The equation of motion is

$$m\ddot{x} = -\frac{\partial U}{\partial x}$$

If the function $U(x)$ has an extremum at $x = x_0$, then $\left.\frac{\partial U}{\partial x}\right|_{x=x_0} = 0$. Then $x = x_0$ is a (time independent) solution of the equation of motion.

Consider a small deviation from the solution $x = x_0 + \delta x(t)$. Assuming that δx stays small during the motion we have

$$U(x) = U(x_0 + \delta x) \approx U(x_0) + U'(x_0)\delta x + \frac{1}{2}U''(x_0)\delta x^2 = U(x_0) + \frac{1}{2}U''(x_0)\delta x^2$$

The equation of motion becomes

$$m\ddot{\delta x} = -U''(x_0)\delta x$$

- If $U''(x_0) > 0$, then we have small oscillations with the frequency
$$\omega^2 = \frac{U''(x_0)}{m}$$
This is a stable equilibrium.
- If $U''(x_0) < 0$, then the solution grows exponentially, and at some point our approximation becomes invalid. The equilibrium is unstable.

Look at what it means graphically.

7.3.1. Noise and dissipation.

Generality: consider a system with infinitesimally small dissipation and external perturbations. The perturbations will kick it out of any unstable equilibrium. The dissipation will bring it down to a stable equilibrium. It may take a very long time.

After that the response of the system to small enough perturbations will be defined by the small oscillations around the equilibrium

LECTURE 8
Oscillations. Many degrees of freedom.

8.1. Examples in $1D$.

- $U(x) = \frac{kx^2}{2} + \frac{\gamma x^4}{4}$, where $k > 0$ and $\gamma > 0$.
  - First, we find the equilibrium positions. The equation is
  $$0 = \frac{\partial U}{\partial x} = x(k + \gamma x^2).$$
  for $\gamma > 0$, $k > 0$ there is only one real solution: $x = 0$.
  - We need to compute
  $$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=0} = k > 0.$$
  - The frequency of the small oscillations is
  $$\omega^2 = \frac{1}{m}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=0} = k/m.$$
- $U(x) = -\frac{kx^2}{2} + \frac{\gamma x^4}{4}$, where $k > 0$ and $\gamma > 0$.
  - First, we find the equilibrium positions. The equation is
  $$0 = \frac{\partial U}{\partial x} = x(-k + \gamma x^2)$$
  for $\gamma > 0$, $k > 0$ there are three real solution: $x = 0$, $x_0 = \pm\sqrt{k/\gamma}$.

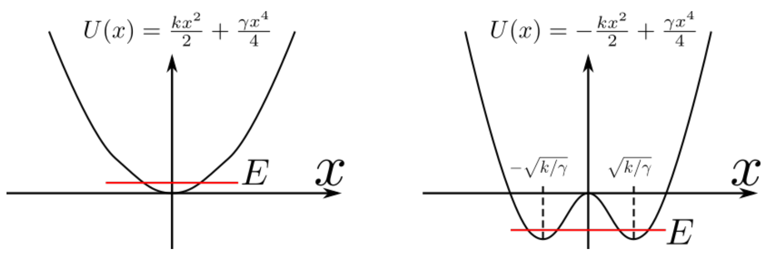


Figure 1. Examples of $1D$ potentials.

– We need to compute

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=0} = -k < 0, \qquad \left.\frac{\partial^2 U}{\partial x^2}\right|_{x=\pm\sqrt{k/\gamma}} = 2k > 0.$$

So the equilibrium point $x = 0$ is unstable. The two equilibrium points $x = \pm\sqrt{k/\gamma}$ are stable. The small oscillations are only possible around the stable points. So we have

$$\omega^2 = \frac{1}{m}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=\pm\sqrt{k/\gamma}} = 2k/m.$$

- Look at what it means graphically.

8.2. Full solution in $1D$.

The equation

$$\ddot{x} = -\omega^2 x, \qquad \omega^2 = U''(x_0)/m > 0, \qquad U'(x_0) = 0$$

General solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t + \phi) = \Re\tilde{C}e^{i\omega t}, \qquad \tilde{C} = Ce^{i\phi}.$$

The two arbitrary constants A and B, or C and ϕ, or one complex $\tilde{C}$ must be found from the initial conditions.

$$x(0) = x_0, \qquad \dot{x}(0) = v_0.$$

8.3. Special $\omega = 0$ case.

- In the case $\omega = 0$, the equation becomes

  $$\ddot{x} = 0, \qquad x(0) = x_0, \quad \dot{x}(0) = v_0$$

  with an obvious solution

  $$x(t) = x_0 + v_0 t.$$

- Let's obtain the same result from the general solution

  $$x(t) = A\cos(\omega t) + B\sin(\omega t).$$

- If one naively plugs $\omega = 0$ in it, one gets $x(t) = A$, which is obviously incorrect. It is physically incorrect, but also mathematically, as a solution of a second order differential equation must depend on 2 arbitrary constant, while here we have only one.
- To do it correctly and to see where the problem is, one has to first satisfy the initial conditions and write

  $$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t).$$

- Now we see, that we cannot simply plug $\omega = 0$, as we cannot divide by zero. Instead we must take a limit $\omega \to 0$ at fixed time t.
- Then the argument of the sin function is small. Using $\sin(\omega t) \sim \omega t$, we restore the right answer.

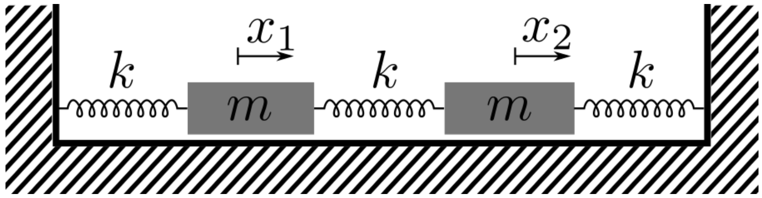


Figure 2. Two blocks.

8.4. Many degrees of freedom.

Consider two equal masses in $1D$ connected by springs of constant k to each other and to the walls.

There are two coordinates: x_1 and x_2.

There are two modes $x_1 - x_2$ and $x_1 + x_2$.

The potential energy of the system is

$$U(x_1, x_2) = \frac{kx_1^2}{2} + \frac{k(x_1 - x_2)^2}{2} + \frac{kx_2^2}{2}$$

The Lagrangian

$$L = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{kx_1^2}{2} - \frac{k(x_1 - x_2)^2}{2} - \frac{kx_2^2}{2}$$

The equations of motion are

$$m\ddot{x}_1 = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -2kx_2 + kx_1$$

These are two second order differential equations. Total they must have four solutions. Let's look for the solutions in the form

$$x_1 = A_1 e^{i\omega t}, \qquad x_2 = A_2 e^{i\omega t}$$

then

$$-\omega^2 m A_1 = -2kA_1 + kA_2$$
$$-\omega^2 m A_2 = -2kA_2 + kA_1$$

or

$$(2k - m\omega^2)A_1 - kA_2 = 0$$
$$(2k - m\omega^2)A_2 - kA_1 = 0$$

or

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

In order for this set of equations to have a non trivial solution we must have

$$\det \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} = 0, \qquad (2k - m\omega^2)^2 - k^2 = 0, \qquad (k - m\omega^2)(3k - m\omega^2) = 0$$

There are two modes with the frequencies

$$\omega_a^2 = k/m, \qquad \omega_b^2 = 3k/m$$

and corresponding eigen vectors

$$\begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix} = A^a \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix} = A^b \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The general solution then is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_a t + \phi_a) + A^b \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_b t + \phi_b)$$

- There are four arbitrary constants A^a, A^b, ϕ_a, ϕ_b which must be obtained from the initial conditions.
- Picture of the eigen modes.
- Symmetry.

What will happen if the masses and springs constants are different?

Repeat the previous calculation for arbitrary m_1, m_2, k_1, k_2, k_3.

The Lagrangian

$$L = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} - \frac{k_1 x_1^2}{2} - \frac{k_2 (x_1 - x_2)^2}{2} - \frac{k_3 x_2^2}{2}.$$

The equations of motion

$$\begin{aligned} m_1 \ddot{x}_1 &= -(k_1 + k_2) x_1 + k_2 x_2 \\ m_2 \ddot{x}_2 &= k_2 x_1 - (k_3 + k_2) x_2. \end{aligned}$$

We look for the solution in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{i\omega t} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

The equations become

$$\begin{pmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

There is a non-trivial solution iff

$$\det \begin{pmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 \end{pmatrix} = 0.$$

It will have two solutions for ω^2 (it may be degenerate). Correspondingly it will have two eigen vectors.

$$\omega^2 = \omega_a^2 \longrightarrow \begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix}, \qquad \omega^2 = \omega_b^2 \longrightarrow \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix}.$$

The general solution then is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix} \cos(\omega_a t + \phi_a) + b \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix} \cos(\omega_b t + \phi_b).$$

The four constants a, ϕ_a, b, and ϕ_b must be found from the initial conditions.

General scheme.

LECTURE 9
Oscillations. Many degrees of freedom.

9.1. Example of $\omega = 0$.

A rail and two objects with masses m and M connected by a spring k.

The Lagrangian is

$$L = \frac{m\dot{x}_1^2}{2} + \frac{M\dot{x}_2^2}{2} - \frac{k(x_1 - x_2)^2}{2}.$$

Notice, that the Lagrangian depends only on $x_1 - x_2$ and velocities.

The equations of motion are

$$m\ddot{x}_1 = -k(x_1 - x_2)$$
$$M\ddot{x}_2 = -k(x_2 - x_1)$$

Looking for a solution in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t},$$

then

$$\begin{pmatrix} -\omega^2 m + k & -k \\ -k & -\omega^2 M + k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

So the determinant equal zero requirement gives

$$\omega^4 mM - k(m+M)\omega^2 = 0.$$

Two solutions

$$\omega^2 = 0, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \omega^2 = k\frac{m+M}{mM}, \quad \begin{pmatrix} M \\ m \end{pmatrix}.$$

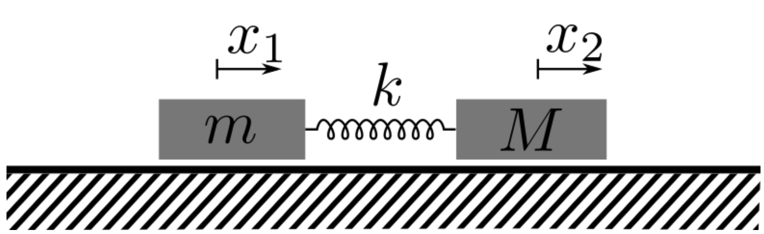


Figure 1. Two blocks, zero mode.

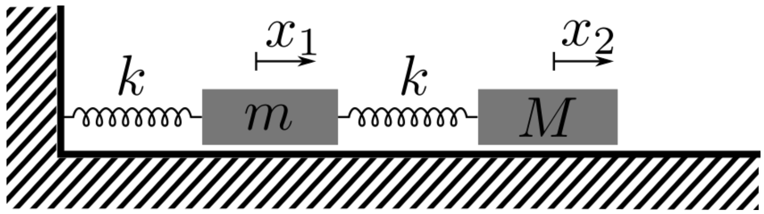


Figure 2. Two blocks, mode disappearing.

The full solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (v_0 t + x_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} M \\ m \end{pmatrix} \cos(\omega t + \phi)$$

- Role of symmetry for zero mode.
  - As the Lagrangian depends only on $x_1 - x_2$ (and the velocities), the Lagrangian will not change if we add the same constant ϵ to both x_1 and x_2.
  - This is the symmetry of the problem.
  - It means that if we find a solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ which minimizes the Action, then the solution $\begin{pmatrix} x_1(t)+\epsilon \\ x_2(t)+\epsilon \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will also minimize the Action (but it may not satisfy the initial conditions)
  - Notice, that ϵ multiplies the mode that corresponds to $\omega = 0$.
  - This is a very general situation.

9.2. Example of mode disappearing.

A rail a wall and to objects with masses m and M connected by a spring k with each other and by the spring k through mass m with the wall. We want to consider two cases $m = 0$ and $m \to 0$.

The Lagrangian is

$$L = \frac{m\dot{x}_1^2}{2} + \frac{M\dot{x}_2^2}{2} - \frac{kx_1^2}{2} - \frac{k(x_1 - x_2)^2}{2}$$

Notice, that now the Lagrangian depends not only on the combination $x_1 - x_2$, but also on x_1. This breaks the symmetry discussed before. It means that we do not expect $\omega = 0$ mode.

The equations of motion are

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$
$$M\ddot{x}_2 = -k(x_2 - x_1)$$

Looking for a solution in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t},$$

then

$$\begin{pmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 M + k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

So the determinant equal zero requirement gives

$$\omega^4 mM - k(m+2M)\omega^2 + k^2 = 0.$$

- Notice, that if we put $m = 0$ directly in the equation we will get a single solution $\omega^2 = k/2M$. However, for any, no matter how small $m \neq 0$ we must have two modes! What happens to the second mode as $m \to 0$?

The two solutions are

$$\omega^2 = k\frac{2M + m \pm \sqrt{4M^2 + m^2}}{2mM}$$

For $m \ll M$, we need to keep only linear in m terms in the numerator

$$\omega^2 = k\frac{m + 2M \pm 2M}{2mM} = \begin{cases} \frac{2k}{m}, & \text{for the “+” sign} \\ \frac{k}{2M}, & \text{for the “−” sign} \end{cases},$$

or two normal frequencies

$$\omega_-^2 = \frac{k}{2M}, \qquad \omega_+^2 = \frac{2k}{m} \to \infty.$$

Physical picture:

- Mode ω_-: when m is very small, we have two springs in series (spring constant $k/2$) acting on mass M.
- Mode ω_+: when m is very small, M almost does not move we have two springs in parallel (spring constant $2k$) acting on mass m.
- When $m \to 0$ acceleration goes to infinity, so does the corresponding frequency.

9.3. General situation.

Let's consider a general situation in detail. We start from an arbitrary Lagrangian

$$L = K(\{\dot{q}_i\}, \{q_i\}) - U(\{q_i\})$$

Very generally the kinetic energy is zero if all velocities are zero. It will also increase if any of the velocities increase.

- The harmonic oscillator equations are linear.
- We expect harmonic oscillations around a stable equilibrium position.
- In order to have linear equations we need the Lagrangian only up to the second order in both velocities and coordinate shifts from the equilibrium position.

It is assumed that the potential energy has a minimum at some values of the coordinates $q_i = q_{i0}$. Let's first change the definition of the coordinates $x_i = q_i - q_{0i}$. We rewrite the Lagrangian in these new coordinates.

$$L = K(\{\dot{x}_i\}, \{x_i\}) - U(\{x_i\})$$

We can take the potential energy to be zero at $x_i = 0$, also as $x_i = 0$ is a minimum we must have $\partial U/\partial x_i|_{\{x_i\}=0} = 0$.

Let's now assume, that the motion has very small amplitude. We then can use Taylor expansion in both $\{\dot{x}_i\}$ and $\{x_i\}$ up to the second order.

The time reversal invariance demands that only even powers of velocities can be present in the expansion. Also as the kinetic energy is zero if all velocities are zero, we have $K(0, \{x_i\}) =$

0, so we have

$$K(\{\dot{x}_i\},\{x_i\}) \approx \frac{1}{2}\sum_{i,j}\left.\frac{\partial K}{\partial \dot{x}_i \partial \dot{x}_j}\right|_{\dot{x}=0,x=0} \dot{x}_i\dot{x}_j = \frac{1}{2}k_{ij}\dot{x}_i\dot{x}_j,$$

where the constant matrix k_{ij} is symmetric and positive definite. (I use the small letter k_{ij} to denote it to remind that it originates from the kinetic energy. Physically, it is more as a mass matrix.)

For the potential energy we have

$$U(\{x_i\}) \approx \frac{1}{2}\sum_{i,j}\left.\frac{\partial U}{\partial x_i \partial x_j}\right|_{x=0} x_i x_j = \frac{1}{2}u_{ij}x_i x_j,$$

where the constant matrix u_{ij} is symmetric. If $x = 0$ is indeed a minimum, then the matrix u_{ij} is also symmetric and positive definite. (I use the small letter u_{ij} to denote it to remind that it originates from the potential energy. Physically, it is more as a spring constant matrix.)

The Lagrangian is then

$$L = \frac{1}{2}k_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}u_{ij}x_i x_j$$

where k_{ij} and u_{ij} are just constant matrices and summation over the repeated indexes i and j is assumed. The Lagrange equations are

$$k_{ij}\ddot{x}_j = -u_{ij}x_j,$$

where the summation over the repeated index j is assumed.

We are looking for the solutions in the form

$$x_j^a = A_j^a e^{i\omega_a t}.$$

The letter a denotes the eigen/normal modes: $a = 1, \ldots, N$, where N is the number of degrees of freedom.

Substituting the solution in this form in the equations of motion we get:

$$\left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a = 0, \tag{9.1}$$

where the summation over the index j is assumed.

In order for this linear equation to have a nontrivial solution we must have

$$\det\left(\omega_a^2 k_{ij} - u_{ij}\right) = 0$$

After solving this equation we can find all N of eigen/normal frequencies ω_a and the eigen/normal modes of the small oscillations A_i^a.

We can prove, that all ω_a^2 are positive (if U is at minimum.) Let's substitute the solutions ω_a and A_j^a into equation (9.1), multiply it by $(A_i^a)^*$ and sum over the index i.

$$\sum_{ij}\left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a \left(A_i^a\right)^* = 0.$$

From here we see

$$\omega_a^2 = \frac{\sum_{ij} u_{ij} A_j^a \left(A_i^a\right)^*}{\sum_{ij} k_{ij} A_j^a \left(A_i^a\right)^*}$$

As both matrices k_{ij} and u_{ij} are symmetric and positive definite, we have the ratio of two positive real numbers in the RHS. So ω_a^2 must be positive and real.

LECTURE 10
Examples of coupled oscillations. Oscillators with time-dependent parameters.

10.1. Examples.

10.1.1. Two masses, splitting of symmetric and anitsymmetric modes.

This is the problem of two identical masses m connected with each other with the spring k and to the walls with the springs K.

The Lagrangian is almost the same as we have derived before

$$L = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{Kx_1^2}{2} - \frac{k(x_1 - x_2)^2}{2} - \frac{Kx_2^2}{2}$$

The kinetic and potential energies have the forms

$$K = \frac{1}{2}(\dot{x}_1, \dot{x}_2)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \qquad U = \frac{1}{2}(x_1, x_2)\begin{pmatrix} K+k & -k \\ -k & K+k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

so the matrices $\hat{k}$ and $\hat{u}$ are

$$\hat{k} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad \hat{u} = \begin{pmatrix} K+k & -k \\ -k & K+k \end{pmatrix},$$

and the matrix $\omega^2\hat{k} - \hat{u}$ is

$$\begin{pmatrix} \omega^2 m - K - k & k \\ k & \omega^2 m - K - k \end{pmatrix}.$$

To find the eigen/normal modes we need to find such ω^2 at which the determinant of this matrix is zero.

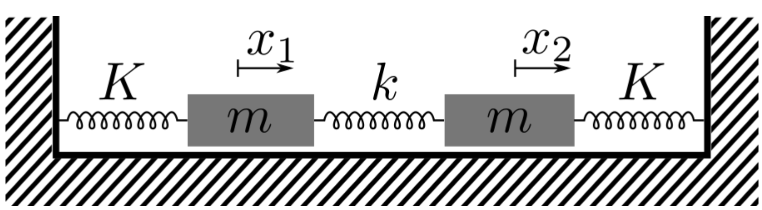


Figure 1. Two blocks.

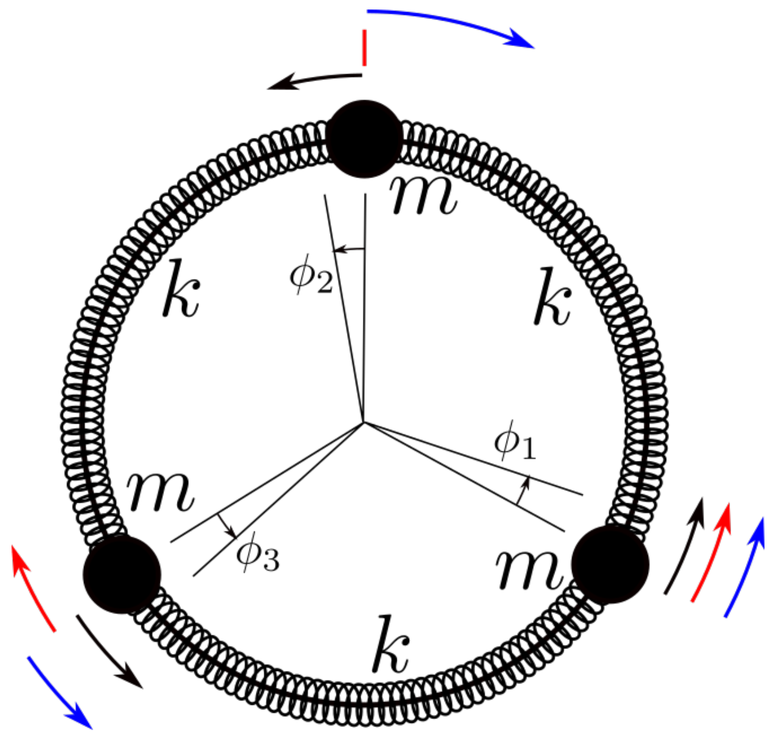


Figure 2. Three masses on a ring.

Notice that

- If the masses were not interacting $k = 0$, then the matrix would be already diagonal and we would have to degenerate eigen frequencies $\omega^2 = K/m$.
- The interaction between the masses (degrees of freedom) appears as the off-diagonal matrix element k.
- This off-diagonal matrix element will split the degeneracy of the two modes.

The equation is

$$\left(\omega^2 m - K - k\right)^2 - k^2 = 0,$$

with the solutions

$$\omega_+^2 = \frac{K + 2k}{m}, \quad A_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \omega_-^2 = \frac{K}{m}, \quad A_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These are symmetric and antisymmetric modes. It is a very general situation for a symmetric system, that the interaction splits the degeneracy and the new modes are symmetric/antisymmetric pair.

10.1.2. Three masses on a ring. Symmetries. Zero mode.

The Lagrangian

$$L = \frac{mR^2\dot{\phi}_1^2}{2} + \frac{mR^2\dot{\phi}_2^2}{2} + \frac{mR^2\dot{\phi}_3^2}{2} - \frac{kR^2(\phi_1 - \phi_2)^2}{2} - \frac{kR^2(\phi_2 - \phi_3)^2}{2} - \frac{kR^2(\phi_3 - \phi_1)^2}{2}$$

Using the vector

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

The corresponding matrices are

$$\hat{k} = \begin{pmatrix} mR^2 & 0 & 0 \\ 0 & mR^2 & 0 \\ 0 & 0 & mR^2 \end{pmatrix}, \qquad \hat{u} = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix}.$$

So we need to solve the equation

$$\det \begin{pmatrix} \omega^2 mR^2 - 2kR^2 & kR^2 & kR^2 \\ kR^2 & \omega^2 mR^2 - 2kR^2 & kR^2 \\ kR^2 & kR^2 & \omega^2 mR^2 - 2kR^2 \end{pmatrix} = 0,$$

which gives

$$\omega^2 m \left(\omega^2 m - 3k\right)^2 = 0.$$

We see, that there are three modes: one is the zero mode $\omega = 0$, the other two are degenerate $\omega^2 = 3k/m$. The corresponding eigen modes are

$$\omega = 0 \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \omega^2 = 3k/m \longrightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

(Any linear combination of the degenerate modes is also a normal mode.)

- The zero mode exists because of the continuous symmetry. It would be present even if all the masses and all the springs were different.
- The degeneracy is the consequence of the discrete symmetry, the fact that all the springs are the same and all masses are the same. This degeneracy would be lifted if we take, say, one of the masses to be different form the other two.

The modes are shown on the figures with different colors.

10.2. Oscillations with time-dependent parameters.

- Oscillations with parameters depending on time.

$$L = \frac{1}{2} m(t) \dot{x}^2 - \frac{1}{2} k(t) x^2.$$

The Lagrange equation

$$\frac{d}{dt} m(t) \frac{d}{dt} x = -k(t) x.$$

We change the definition of time

$$m(t) \frac{d}{dt} = \frac{d}{d\tau}, \qquad \frac{d\tau}{dt} = \frac{1}{m(t)}$$

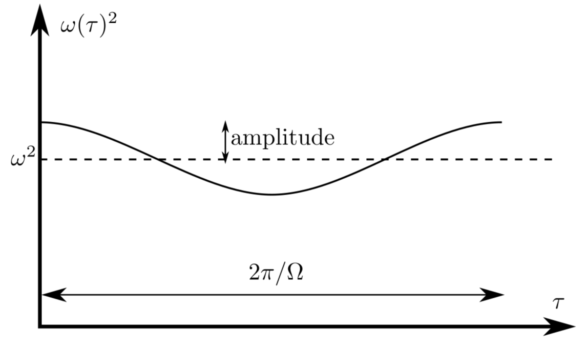
then the equation of motion is

$$\frac{d^2 x}{d\tau^2} = -mkx.$$

So without loss of generality we can consider an equation

$$\ddot{x} = -\omega^2(\tau)x$$

- The most interesting is the situation when $\omega(\tau)$ is by itself a periodic function with frequency Ω. In this case the system returns back to where it was after time $2\pi/\Omega$.
- We call Ω the frequency of *change* of ω.
- Different time scales. Three different cases: $\Omega \gg \omega$, $\Omega \ll \omega$, and $\Omega \sim \omega$.

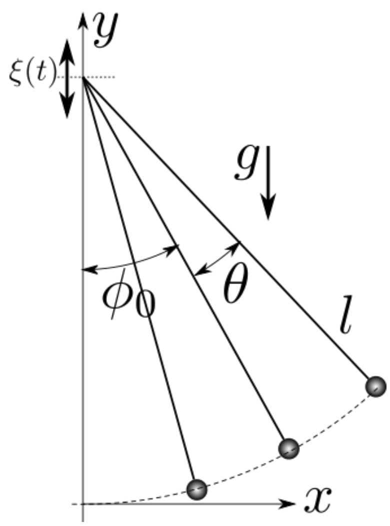


  - $\Omega \gg \omega$ — Kapitza pendulum. (demo) Criteria: $\overline{\left(\dot{\xi}\right)^2} > gl$.
    - Importance of the time scale separation.
    - Averaging out fast processes – a natural thing to do.
    - Importance of non-linearity.
    - Universal mechanism – averaging over fast degrees of freedom leads to the change of the dynamics of the slow degree of freedom through non-linearity.
  - $\Omega \ll \omega$ — Foucault pendulum as an example of slow change of the parameter $\Delta\phi$ =solid angle of the path. (quantum: Berry phase 1984; classical: Hannay angle 1985.)
    - Topological in nature.
    - Universal.
  - $\Omega \sim \omega$ — parametric resonance ($\omega_p = \frac{2}{n}\omega_0$)

$$\ddot{x} = -\omega^2(t)x, \qquad \omega^2(t) = \omega_0^2(1 + h\cos(\omega_p t)), \quad h \ll 1$$

  Different from the usual resonance:
    - If the initial conditions $x(t=0) = 0$, $\dot{x}(t=0) = 0$, then $x(t) = 0$.
    - Frequency ω_p is a fraction of ω_0.
    - At finite dissipation one must have a finite amplitude h in order to get to the resonance regime.

LECTURE 11
Kapitza pendulum $\Omega \gg \omega$.



11.1. Set up of the problem.

- We have a standard rigid pendulum of length l in the gravitation field g. Somebody/something is shaking the pivot point of the pendulum vertically or horizontally very fast, but with a small amplitude. So the position of the pivot point with respect to its average is given by $\xi(t)$, where $|\xi(t)| \ll l$.
- Time scales difference. The shaking is very fast. So the frequency of the function $\xi(t)$, which we call Ω is much larger than the natural frequency of the pendulum $\omega = \sqrt{g/l}$, or $\Omega \gg \omega$.
- Expected results. We assume that the function $\xi(t)$ is so fast, that we do not actually see the motion of the pivot point. The pendulum then will perform some very complicated motion, but this motion can be split into a very fast one, which we do not see, and a slow "average" one. We are interested in that slow motion only. We will be able to clearly define these two motion because there is large separation of the time scales. We then expect, that the fast motion will change — "renormalize" — the slow motion in comparison with the motion of a normal pendulum of length l.

We will consider two cases — vertical and horizontal shaking — separately. One can also consider a general case, when both are present.

11.2. Vertical displacement.

11.2.1. Exact Lagrangian and exact equation of motion.

The coordinates

$$\begin{array}{ll} x = l\sin\phi & \dot{x} = l\dot{\phi}\cos\phi \\ y = l(1-\cos\phi)+\xi \quad , & \dot{y} = l\dot{\phi}\sin\phi + \dot{\xi} \end{array}$$

The Lagrangian is

$$L = \frac{ml^2}{2}\dot{\phi}^2 + ml\dot{\phi}\dot{\xi}\sin\phi + mgl\cos\phi,$$

where I dropped the terms that are full derivatives over time. The equation of motion is

$$\ddot{\phi} + \frac{\ddot{\xi}}{l}\sin\phi = -\omega^2\sin\phi, \qquad \omega^2 = g/l.$$

This is a non-linear second order differential equation with time dependent coefficients. We cannot find the exact solution of it, but we do not want to. The exact solution will have both slow and fast motions. What we want is to "average" over the fast motion and to find an effective equation for the slow motion only.

11.2.2. Averaging over fast motion.

Look for the solution

$$\phi = \phi_0 + \theta, \qquad \bar{\theta} = 0$$

- What does averaging mean? Separation of the time scales $2\pi/\Omega \ll 2\pi/\omega$. Consider the time T such that $2\pi/\Omega \ll T \ll 2\pi/\omega$. During this time the fast motion goes over many cycles, while the slow motion is almost does not happen.
- The averaging then means $\bar{\theta} = \frac{1}{T}\int_0^T \theta(t)dt$. Notice, that this is not full averaging. The full averaging would require taking a limit $T \to \infty$. We do not do that.
- Notice, that with this averaging procedure we write
$$\overline{\theta(t)} = \frac{1}{T}\int_0^T \theta(t)dt = 0, \quad \text{while} \quad \overline{\phi_0(t)} = \frac{1}{T}\int_0^T \phi_0(t)dt = \phi_0(t),$$
as $\phi_0(t)$ is almost a constant during time T.
- Such procedure only makes sense if the result does not depend on T as long as $2\pi/\Omega \ll T \ll 2\pi/\omega$. The validity of separating the scales must also be checked afterwards.

We expect θ to be small, but $\dot{\theta}$ and $\ddot{\theta}$ are **NOT** small. Using the smallness of θ and the Taylor expansion we have

$$\ddot{\phi}_0 + \ddot{\theta} + \frac{\ddot{\xi}}{l}\sin\phi_0 + \frac{\ddot{\xi}}{l}\theta\cos\phi_0 = -\omega^2\sin\phi_0 - \omega^2\theta\cos\phi_0 \tag{11.1}$$

The frequency of the function ϕ_0 is small, so the fast oscillating functions (the last three terms in the left had side and the last term in the right hand side) must cancel each other. So

$$\ddot{\theta} + \frac{\ddot{\xi}}{l}\sin\phi_0 + \frac{\ddot{\xi}}{l}\theta\cos\phi_0 = -\omega^2\theta\cos\phi_0.$$

Neglecting the term proportional to small θ we get $\ddot{\theta} + \frac{\ddot{\xi}}{l}\sin\phi_0 = 0$, or

$$\theta = -\frac{\xi}{l}\sin\phi_0.$$

As $\bar{\xi} = 0$, the requirement $\bar{\theta} = 0$ fixes the other terms coming from the integration.

Now we take the equation (11.1) and average it over the time T.

$$\ddot{\phi}_0 + \frac{\overline{\theta\ddot{\xi}}}{l}\cos\phi_0 = -\omega^2\sin\phi_0 \tag{11.2}$$

We now have for the second term in the left hand side

$$\overline{\theta\ddot{\xi}} = -\overline{\xi\ddot{\xi}}\frac{1}{l}\sin\phi_0.$$

To simplify it a bit we use

$$\overline{\xi\ddot{\xi}} = \frac{1}{T}\int_0^T \xi\ddot{\xi}dt = -\frac{1}{T}\int_0^T (\dot{\xi})^2 dt = -\overline{(\dot{\xi})^2}.$$

(One important technical note. The expression $\overline{(\dot{\xi})^2}$ means that you first take the derivative, then square, then average. The order of operations is important. Any other order of the same operations will give you zero.)

Our averaged equation (11.2) then becomes

$$\ddot{\phi}_0 = -\left(\omega^2\sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0\right) = -\frac{\partial}{\partial\phi_0}\left(-\omega^2\cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0\right)$$

11.2.3. Effective potential.

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2\cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0.$$

Notice:

- the first term in the effective potential energy is simply the standard gravitational term;
- the second term, however, comes from the "averaging" procedure;
- the second term is **NOT** small in comparison to the first, as $\dot{\xi}$ is not small.

Now we analyze this effective potential energy the usual way.

11.2.3.1. *The equilibrium.* The equilibrium positions are given by

$$\frac{\partial U}{\partial\phi_0} = \omega^2\sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0 = 0, \qquad \sin\phi_0\left(\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_0\right) = 0$$

We see, that if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$, a pair of new solutions appears.

11.2.3.2. *The stability of equilibrium points.* The stability is defined by the sign of

$$\frac{\partial^2 U}{\partial \phi_0^2} = \omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{l^2} \cos 2\phi_0$$

One see, that

- $\phi_0 = 0$ is always a stable solution.
- $\phi_0 = \pi$ is unstable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} > 1$, but becomes stable if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$ are unstable.

11.2.3.3. *The oscillations around the equilibrium.* For ϕ_0 close to π we can introduce $\phi_0 = \pi + \tilde{\phi}$, where $\tilde{\phi}$ is small.

$$\ddot{\tilde{\phi}} = -\omega^2 \left(\frac{\overline{(\dot{\xi})^2}}{l^2 \omega^2} - 1 \right) \tilde{\phi}$$

We see, that for $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$, the oscillations in the upper point have the frequency

$$\tilde{\omega}^2 = \omega^2 \left(\frac{\overline{(\dot{\xi})^2}}{l^2 \omega^2} - 1 \right)$$

Remember, that above calculation is correct if Ω of the ξ is much larger than ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot{\xi})^2} \approx \Omega^2 \xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime ($\overline{(\dot{\xi})^2}/l^2\omega^2 \sim 1$) is at

$$\Omega^2 \sim \omega^2 \frac{l^2}{\xi_0^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

11.3. Horizontal displacement.

The derivation is analogous to the vertical case (see also the next lecture).

If ξ is horizontal, then it is convenient to redefine the angle $\phi_0 \to \pi/2 + \phi_0$, then the shake contribution changes sign and we get

$$U_{eff} = -\omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{4l^2} \cos 2\phi_0$$

The equilibrium position is found by

$$\frac{\partial U_{eff}}{\partial \phi_0} = \sin\phi_0 \left(\omega^2 - \frac{\overline{(\dot{\xi})^2}}{l^2} \cos\phi_0 \right).$$

Let's write U_{eff} for small angles, then (dropping the constant.)

$$U_{eff} \approx \frac{\omega^2}{2} \left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \right) \phi_0^2 + \frac{\omega^2}{24} \left(4\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} - 1 \right) \phi_0^4$$

If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$, then

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{8}\phi_0^4.$$

- Spontaneous symmetry braking.

LECTURE 12

SKIP. Oscillations with parameters depending on time. Kapitza pendulum. Horizontal case.

Let's consider a shaken pendulum without the gravitation force acting on it. The fast shaking is given by a fast time dependent vector $\vec{\xi}(t)$. This vector defines a direction in space. I will call this direction $\hat{z}$, so $\vec{\xi}(t) = \hat{z}\xi(t)$.

The amplitude ξ is small $\xi \ll l$, where l is the length of the pendulum, but the shaking is very fast $\Omega \gg \omega$, the frequency of the pendulum motion (without gravity it is not well defined, but we will keep in mind that we are going to include gravity later.)

Let's now use a non inertial frame of reference connected to the point of attachment of the pendulum. In this frame of reference there is a artificial force which acts on the pendulum. This force is

$$\vec{f} = -\ddot{\xi} m \hat{z}.$$

If the pendulum makes an angle ϕ with respect to the axis $\hat{z}$, then the torque of the force $\vec{f}$ is $\vec{\tau} = \vec{r} \times \vec{f}$, its magnitude $\tau = lf \sin\phi$ — the positive direction is defined by the positive direction of the angle. So the equation of motion

$$ml^2\ddot{\phi} = lm\ddot{\xi}\sin\phi, \qquad \ddot{\phi} = \frac{\ddot{\xi}}{l}\sin\phi$$

Now we split the angle onto slow motion described by ϕ_0 – a slow function of time, and fast motion $\theta(t)$ a fast oscillating function of time such that $\bar{\theta} = 0$.

We then have

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0 + \theta)$$

Notice the non linearity of the RHS.

As $\theta \ll \phi_0$, we can use the Taylor expansion

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0) + \frac{\ddot{\xi}\theta}{l}\cos(\phi_0) \tag{12.1}$$

Double derivatives of θ and ξ are very large, so in the zeroth order we can write

$$\ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0), \qquad \theta = \frac{\xi}{l}\sin(\phi_0).$$

Now averaging the equation (12.1) in the way described in the previous lecture we get

$$\ddot{\phi}_0 = \frac{\overline{\ddot{\xi}\theta}}{l}\cos(\phi_0) = \frac{\overline{\ddot{\xi}\xi}}{l^2}\sin(\phi_0)\cos(\phi_0)$$

or

$$\ddot{\phi}_0 = \frac{\overline{\ddot{\xi}\theta}}{l}\cos(\phi_0) = -\frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0)$$

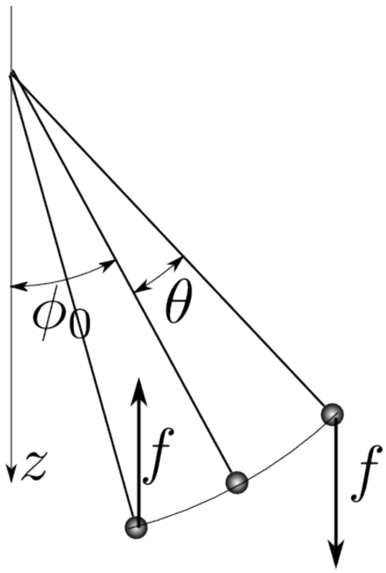


Figure 1. The Kapitza pendulum.

What is happening is illustrated on the figure. If ξ is positive, then $\ddot{\xi}$ is negative, so the torque is negative and is larger, because the angle $\phi = \phi_0 + \theta$ is larger. So the net torque is negative!

12.0.1. Vertical.

Now we can get the result from the previous lecture. We just need to add the gravitational term $-\omega^2 \sin\phi_0$.

$$\ddot{\phi}_0 = -\omega^2 \sin\phi_0 - \frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0).$$

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2 \cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0$$

The equilibrium positions are given by

$$\frac{\partial U}{\partial \phi_0} = \omega^2 \sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0 = 0, \qquad \sin\phi_0\left(\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_0\right) = 0$$

We see, that if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$, a pair of new solutions appears.

The stability is defined by the sign of

$$\frac{\partial^2 U}{\partial \phi_0^2} = \omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos 2\phi_0$$

One see, that

- $\phi_0 = 0$ is always a stable solution.
- $\phi_0 = \pi$ is unstable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} > 1$, but becomes stable if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$ are unstable.

For ϕ_0 close to π we can introduce $\phi_0 = \pi + \tilde{\phi}$

$$\ddot{\tilde{\phi}} = -\omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)\tilde{\phi}$$

We see, that for $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$ the frequency of the oscillations in the upper point have the frequency

$$\tilde{\omega}^2 = \omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)$$

Remember, that above calculation is correct if Ω of the ξ is much larger then ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot{\xi})^2} \approx \Omega^2 \xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime is at

$$\Omega^2 > \omega^2 \frac{l^2}{\xi^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

12.0.2. Horizontal.

If ξ is horizontal, the it is convenient to redefine the angle $\phi_0 \to \pi/2 + \phi_0$, then the shake contribution changes sign and we get

$$U_{eff} = -\omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{4l^2} \cos 2\phi_0$$

The equilibrium position is found by

$$\frac{\partial U_{eff}}{\partial \phi_0} = \sin\phi_0 \left(\omega^2 - \frac{\overline{(\dot{\xi})^2}}{l^2} \cos\phi_0\right).$$

Let's write U_{eff} for small angles, then (dropping the constant.)

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{24}\left(4\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} - 1\right)\phi_0^4$$

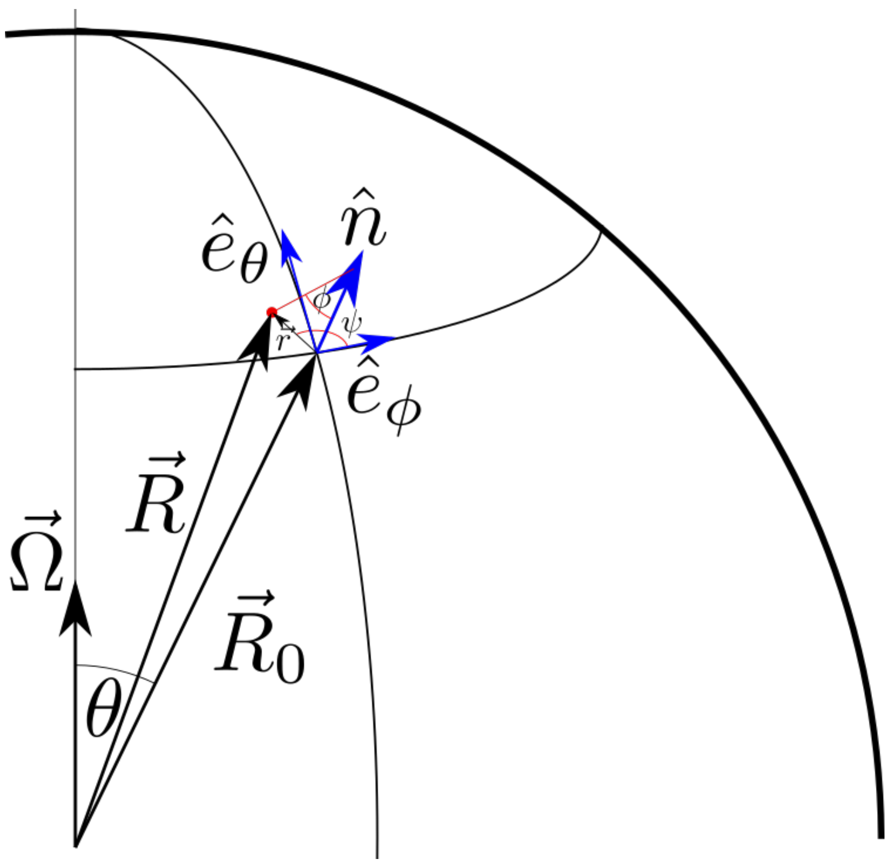
If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$, then

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{8}\phi_0^4.$$

- Spontaneous symmetry braking.

LECTURE 13

SKIP. Oscillations with parameters depending on time. Foucault pendulum.



The opposite situation, when the change of parameters is very slow – adiabatic approximation.

In rotation, for any vector $\vec{l}$ which is just rotates with angular velocity $\vec{\Omega}$ we have

$$\dot{\vec{l}} = \vec{\Omega} \times \vec{l}.$$

In our local system of coordinate (not inertial) a radius-vector is

$$\vec{r} = x\vec{e}_\phi + y\vec{e}_\theta.$$

So the position vector $\vec{R}$ in the inertial system of coordinates is

$$\vec{R} = \vec{R}_0 + \vec{r} = \vec{R}_0 + x\vec{e}_\phi + y\vec{e}_\theta.$$

So the velocity $\vec{v} = \dot{\vec{R}}$ is

$$\dot{\vec{R}} = \Omega \times \vec{R}_0 + \dot{x}\vec{e}_\phi + \dot{y}\vec{e}_\theta + x\vec{\Omega} \times \vec{e}_\phi + y\vec{\Omega} \times \vec{e}_\theta$$

where we used

$$\dot{\vec{R}}_0 = \vec{\Omega} \times \vec{R}_0, \qquad \dot{\hat{e}}_\phi = \vec{\Omega} \times \hat{e}_\phi, \qquad \dot{\hat{e}}_\theta = \vec{\Omega} \times \hat{e}_\theta.$$

We chose the system of coordinate such that $e_\phi \perp \vec{\Omega}$. Then

$$\vec{v}^2 = \Omega^2 R_0^2 \sin^2\theta + \dot{x}^2 + \dot{y}^2 + x^2\Omega^2 + y^2\Omega^2\cos^2\theta + 2\dot{x}\Omega R_0 \sin\theta - y\Omega^2 R_0 \sin(2\theta) + \Omega(\dot{y}x - \dot{x}y)\cos\theta + 2xy\Omega^2$$

where we used

$$(\Omega \times \vec{R}_0)^2 = \Omega^2 R_0^2 \sin^2\theta, \quad (\vec{\Omega} \times \vec{e}_\phi)^2 = \Omega^2, \quad (\vec{\Omega} \times \vec{e}_\theta)^2 = \Omega^2\cos^2\theta,$$
$$[\Omega \times \vec{R}_0] \cdot \vec{e}_\phi = R_0\Omega\sin\theta, \quad [\Omega \times \vec{R}_0] \cdot \vec{e}_\theta = 0, \quad [\Omega \times \vec{R}_0] \cdot [\vec{\Omega} \times \vec{e}_\phi] = 0,$$
$$[\Omega \times \vec{R}_0] \cdot [\vec{\Omega} \times \vec{e}_\theta] = -\Omega^2 R_0 \sin\theta\cos\theta, \quad \vec{e}_\phi \cdot [\vec{\Omega} \times \vec{e}_\theta] = -\Omega\cos\theta, \quad [\vec{\Omega} \times \vec{e}_\phi] \cdot [\vec{\Omega} \times \vec{e}_\theta] = \Omega^2$$

Let's consider the $\vec{v}^2$ term by term having in mind, that $m\vec{v}^2/2$ is the first term in the Lagrangian. Then we see

- The first term $\Omega^2 R_0^2 \sin^2\theta$ can be dropped, as it is a constant.
- The term $2\dot{x}\Omega R_0 \sin\theta$ can be dropped, as it is a full derivative.
- The term $-y\Omega^2 R_0 \sin(2\theta)$ comes from the centripetal force (it is proportional $\Omega^2 R_0$) It simply changes the potential energy a little. As it is definitely a lot less than gravity we can also drop it.

We then have

$$\vec{v}^2 \to \dot{x}^2 + \dot{y}^2 + x^2\Omega^2 + y^2\Omega^2\cos^2\theta + \Omega(\dot{y}x - \dot{x}y)\cos\theta + 2xy\Omega^2$$

Now we remember, that Ω is small. So we also drop all the terms of the order of Ω^2 (we keep the term linear in Ω, as we will need to do the experiment during one day, so $\Omega T \sim 1$.) So we have

$$\vec{v}^2 \to \dot{x}^2 + \dot{y}^2 + \Omega(\dot{y}x - \dot{x}y)\cos\theta$$

For a pendulum we have

$$x = l\phi\cos\psi, \quad y = l\phi\sin\psi$$

so

$$\dot{x}^2 + \dot{y}^2 = l^2\dot{\phi}^2 + l^2\phi^2\dot{\psi}^2$$
$$x\dot{y} - y\dot{x} = l^2\phi^2\dot{\psi}$$

and

$$v^2 \to l^2\dot{\phi}^2 + l^2\phi^2\dot{\psi}^2 + 2\Omega l^2\phi^2\dot{\psi}\cos\theta$$

The Lagrangian then is (the gravitation potential energy is $mgl(1-\cos\phi) \approx \frac{1}{2}mgl\phi^2 = \frac{1}{2}ml^2\omega^2\phi^2$, where $\omega^2 = g/l$)

$$L = \frac{mv^2}{2} + mgl\cos\phi = \frac{mv^2}{2} - \frac{1}{2}ml^2\omega^2\phi^2$$

and the Lagrangian equations ($\omega^2 = g/l$)

$$\ddot{\phi} = -\omega^2\phi + \phi\dot{\psi}^2 + 2\Omega\phi\dot{\psi}\cos\theta$$
$$2\phi\dot{\phi}\dot{\psi} + \phi^2\ddot{\psi} + 2\phi\dot{\phi}\Omega\cos\theta = 0$$

We will see, that $\dot{\psi} \sim \Omega$. As we have already neglected all terms of the order of Ω^2 we also must neglect all terms of the order of $\Omega\dot{\psi}$, $\dot{\psi}^2$, and $\ddot{\psi}$:

$$\ddot{\phi} = -\omega^2\phi$$
$$\dot{\psi} = -\Omega\cos\theta$$

The total change of the angle ψ for the period is

$$\Delta\psi = -\Omega T\cos\theta = -2\pi\cos\theta.$$

- Geometrical meaning. As ψ is an angle, we can always add 2π to it. We then have $\Delta\psi = 2\pi(1-\cos\theta)$ — the solid angle defined by the path of the pendulum!!
- Parallel transport on a sphere:
  - What is parallel transport?
  - What is straight line?
  - Triangle is three points connected by the straight lines.
  - We can draw triangle with all angles $\pi/2$ degrees.
  - Parallel transport a vector along such a triangle.
  - Upon returning it will turn $\pi/2$.
  - The area of such a triangle is 1/8th of the area of the sphere.
  - The solid angle is $4\pi/8 = \pi/2$.

LECTURE 14

SKIP. Oscillations with parameters depending on time. Foucault pendulum. General case.

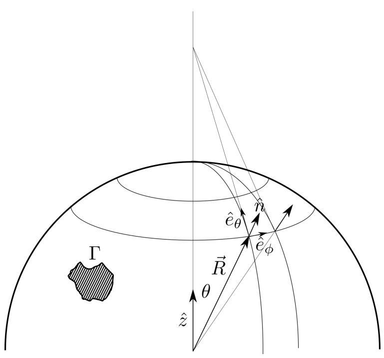


Figure 1

We want to move a pendulum around the world along some closed trajectory. The question is what angle the plane of oscillations will turn after we return back to the original point?

We assume that the earth is not rotating.

We assume that we are moving the pendulum slowly.

First of all we need to decide on the system of coordinates. For our the simple case we can do it in the following way.

(a) We choose a global unit vector $\hat{z}$. The only requirement is that the z line does not intersect our trajectory.

(b) After that we can introduce the angles θ and ϕ in the usual way. (strictly speaking in order to introduce ϕ we also need to introduce a global vector $\hat{x}$, thus introducing a full global system of coordinates.)

(c) In each point on the sphere we introduce it's own system/vectors of coordinates $\hat{e}_\phi$, $\hat{e}_\theta$, and $\hat{n}$, where $\hat{n}$ is along the radius-vector $\vec{R}$, $\hat{e}_\phi$ is orthogonal to both $\hat{n}$ and $\hat{z}$, and $\hat{e}_\theta = \hat{n} \times \hat{e}_\phi$.

We then have

$$\hat{e}_\theta^2 = \hat{e}_\phi^2 = \hat{n}^2 = 1, \qquad \hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\theta \cdot \hat{n} = \hat{e}_\phi \cdot \hat{n} = 0.$$

Let's look how the coordinate vectors change when we change a point where we siting. So let as change our position by a small vector $d\vec{r}$. The coordinate vectors then change by $\hat{e}_\theta \to \hat{e}_\theta + d\hat{e}_\theta$, etc. Where $d\hat{e}_\theta$, $d\hat{e}_\phi$, and $d\hat{n}$ will be proportional to $d\vec{r}$. We then see that

$$\hat{e}_\theta \cdot d\hat{e}_\theta = \hat{e}_\phi \cdot d\hat{e}_\phi = \hat{n} \cdot d\hat{n} = 0, \qquad \hat{e}_\theta \cdot d\hat{e}_\phi + d\hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\theta \cdot d\hat{n} + d\hat{e}_\theta \cdot \hat{n} = \hat{e}_\phi \cdot d\hat{n} + d\hat{e}_\phi \cdot \hat{n} = 0.$$

or

$$\begin{aligned} d\hat{e}_\theta &= a\hat{e}_\phi + b\hat{n} \\ d\hat{e}_\phi &= -a\hat{e}_\theta + c\hat{n} \\ d\hat{n} &= -b\hat{e}_\theta - c\hat{e}_\phi \end{aligned} \tag{14.1}$$

Where coefficients a, b, and c are linear in $d\vec{r}$.

Let's now assume, that our $d\vec{r}$ is along the vector $\hat{e}_\phi$. Then it is clear, that $d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi$, and $d\hat{e}_\theta = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi$.

If $d\vec{r}$ is along the vector $\hat{e}_\theta$, then $d\hat{e}_\phi = 0$, and $d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta$.

Collecting it all together we have

$$\begin{aligned}
d\hat{e}_\theta &= -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi - \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{n} \\
d\hat{e}_\phi &= \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\theta - \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{n} \\
d\hat{n} &= \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta + \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi
\end{aligned}$$

Notice, that these are purely geometrical formulas.

There is more algebraic way to find the coefficients a, b, and c. Consider a vector $\vec{r} = R\hat{n}$. We have $d\vec{r} = Rd\hat{n} + \hat{n}dR$. Using the last of the equations (14.1) we get $d\vec{r} = -Rb\hat{e}_\theta - Rc\hat{e}_\phi + \hat{n}dR$. Multiplying this equation by $\hat{e}_\theta$ and by $\hat{e}_\phi$ we will find the previous results for b and c. Now we notice, that our definition of $\hat{e}_\phi$ is such, that $\hat{z}\cdot\hat{e}_\phi = 0$, so $\hat{z}\cdot d\hat{e}_\phi = 0$. Using the second of the equations (14.1) we find $-a\hat{z}\cdot\hat{e}_\theta + c\hat{z}\cdot\hat{n} = 0$. As $\hat{z}\cdot\hat{n} = \cos\theta$, and $\hat{z}\cdot\hat{e}_\theta = \sin\theta$, we find $c = a/\tan\theta$.

Now let's consider a pendulum. In our local system of coordinates it's radius vector is

$$\vec{\xi} = x\hat{e}_\theta + y\hat{e}_\phi = \xi\cos\psi\hat{e}_\theta + \xi\sin\psi\hat{e}_\phi.$$

The velocity is then

$$\dot{\vec{\xi}} = \dot{\xi}(\cos\psi\hat{e}_\theta + \sin\psi\hat{e}_\phi) + \xi\dot{\psi}(-\sin\psi\hat{e}_\theta + \cos\psi\hat{e}_\phi) + \xi(\cos\psi\frac{\partial\hat{e}_\theta}{\partial\vec{r}} + \sin\psi\frac{\partial\hat{e}_\phi}{d\vec{r}})\frac{d\vec{r}}{dt}.$$

When we calculate $\dot{\vec{\xi}}^2$ we only keep terms no more than first order in $d\vec{r}/dt$

$$\dot{\vec{\xi}}^2 \approx \dot{\xi}^2 + \xi^2\dot{\psi}^2 + 2\xi^2\dot{\psi}\frac{\hat{e}_\phi\cdot\partial\hat{e}_\theta}{\partial\vec{r}}\frac{d\vec{r}}{dt} = \dot{\xi}^2 + \xi^2\dot{\psi}^2 + 2\xi^2\dot{\psi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt}$$

The potential energy does not depend on ψ, so the Lagrange equation for ψ is simply $\frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}} = 0$. Moreover, as ξ is fast when we take the derivative $\frac{d}{dt}$ we differentiate only ξ. Then

$$4\xi\dot{\xi}\dot{\psi} + 4\xi\dot{\xi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt} = 0$$

so

$$\dot{\psi} = -\frac{1}{R\tan\theta}\frac{R\sin\theta d\phi}{dt} = -\cos\theta\frac{d\phi}{dt}$$

Finally,

$$d\psi = -\cos\theta d\phi.$$

LECTURE 15
Oscillations with parameters depending on time. Parametric resonance.

15.1. Parametric resonance. Generalities.

Now we consider a situation when the parameters of the oscillator depend on time and the frequency of this dependence is comparable to the frequency of the oscillator. We start from the equation

$$\ddot{x} = -\omega^2(t)x,$$

where $\omega(t)$ is a periodic function of time. The interesting case is when $\omega(t)$ is almost a constant ω_0 with a small correction which is periodic in time with period T. Then the case which we are interested in is when $2\pi/T$ is of the same order as ω_0. We are going to find the resonance conditions. Such resonance is called "parametric resonance".

First we notice, that if the initial conditions are such that $x(t=0)=0$, and $\dot{x}(t=0)=0$, then $x(t)=0$ is the solution and no resonance happens. This is very different from the case of the usual resonance.

Let's assume, that we found two linearly independent solutions $x_1(t)$ and $x_2(t)$ of the equation. All the solutions are just linear combinations of $x_1(t)$ and $x_2(t)$.

If a function $x_1(t)$ is a solution, then function $x_1(t+T)$ must also be a solution, as T is a period of $\omega(t)$. It means, that the function $x_1(t+T)$ is a linear combination of functions $x_1(t)$ and $x_2(t)$. The same is true for the function $x_2(t+T)$. So we have

$$\begin{pmatrix} x_1(t+T) \\ x_2(t+T) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

We can always choose such $x_1(t)$ and $x_2(t)$ that the matrix is diagonal. In this case

$$x_1(t+T) = \mu_1 x_1(t), \qquad x_2(t+T) = \mu_2 x_2(t)$$

so the functions are multiplied by constants under the translation on one period. The most general functions that have this property are

$$x_1(t) = \mu_1^{t/T} X_1(t), \qquad x_2(t) = \mu_2^{t/T} X_2(t),$$

where $X_1(t)$, and $X_2(t)$ are periodic functions of time.

The numbers μ_1 and μ_2 cannot be arbitrary. The functions x_1 and x_2 satisfy the Wronskian equation

$$W(t) = \dot{x}_1 x_2 - \dot{x}_2 x_1 = \text{const}$$

So on one hand $W(t+T) = \mu_1\mu_2 W(t)$, on the other hand $W(t)$ must be constant. So

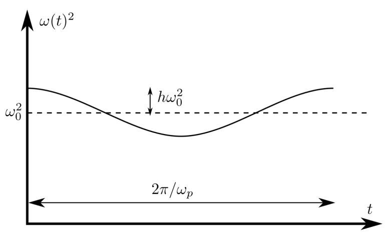
$$\mu_1\mu_2 = 1.$$

Now, if x_1 is a solution so must be x_1^*. It means that either both μ_1 and μ_2 are real, or $\mu_1^* = \mu_2$. In the later case we have $|\mu_1| = |\mu_2| = 1$ and no resonance happens. In the former case we have $\mu_2 = 1/\mu_1$ (either both are positive, or both are negative). Then we have

$$x_1(t) = \mu^{t/T} X_1(t), \qquad x_2(t) = \mu^{-t/T} X_2(t).$$

We see, that one of the solutions is unstable, it increases exponentially with time. This means, that a small initial deviation from the equilibrium will exponentially grow with time. This is the parametric resonance.

15.2. Resonance.



Let's now consider the following dependence of ω on time

$$\omega^2 = \omega_0^2(1 + h\cos\omega_p t)$$

where $h \ll 1$.

- The most interesting case is when $\omega_p \sim 2\omega_0$.
- Explanation:
  - We write the equation in the following form

    $$\ddot{x} = -\omega_0^2 x - h\omega_0^2 x\cos(\omega_p t).$$

  - Now we notice, that as $h \ll 1$ the second term in the RHS is small.
  - We then try to construct the perturbation theory and see at which ω_p it fails.
  - In order to see this, we notice that in the zeroth order in h we have an equation

    $$\ddot{x}_0 = -\omega_0^2 x_0$$

    which has a solution

    $$x_0 = \cos(\omega_0 t)$$

  - We then look for the solution in the form

    $$x = \cos(\omega_0 t) + h x_1(t) + h^2 x_2(t) + \ldots.$$

  - We substitute this solution into our equation and keep only the terms linear in h. We get

    $$\ddot{x}_1 + \omega_0^2 x_1 = \omega_0^2 \cos(\omega_0 t)\cos(\omega_p t)$$

– This is the usual oscillator with a periodic force in the RHS. We see, that this periodic force has the form

$$\cos(\omega_0 t)\cos(\omega_p t) = \frac{1}{2}\cos(\omega_p - \omega_0)t + \frac{1}{2}\cos(\omega_p + \omega_0)t.$$

– Notice, that if $\omega_p = 2\omega_0$, then the first term in the force is $\cos(\omega_0 t)$ — it has exactly the same frequency, as the oscillator $\ddot{x}_1 + \omega_0^2 x_1$. This is a resonance condition! So the solution $x_1(t)$ will grow exponentially and eventually will become larger than x_0 no matter how small h is. This means that the perturbation series brakes down.
– So the situation when $\omega_p \approx 2\omega_0$ has to be treated differently, and we should expect a resonance there.

So I will take $\omega_p = 2\omega_0 + \epsilon$, where $\epsilon \ll \omega_0$. The equation of motion is

$$\ddot{x} + \omega_0^2 x[1 + h\cos(2\omega_0 + \epsilon)t] = 0$$

(Mathieu's equation)

We seek the approximate solution in the form

$$x = a(t)\cos(\omega_0 + \epsilon/2)t + b(t)\sin(\omega_0 + \epsilon/2)t$$

We need to substitute the solution in this form into our equation. We notice

- The identity

$$\cos(\omega_0 + \epsilon/2)t\cos(2\omega_0 + \epsilon)t = \frac{1}{2}\cos 3(\omega_0 + \epsilon/2)t + \frac{1}{2}\cos(\omega_0 + \epsilon/2)t$$

  shows that in the RHS we will have the terms with the frequency $3(\omega_0 + \epsilon/2)$, and with the frequency $\omega_0 + \epsilon/2$. The terms with frequency $3(\omega_0 + \epsilon/2)$ are far away from the resonance, so we just drop them and retain only the "most dangerous" terms of frequency $\omega_0 + \epsilon/2$.
- We expect the de-tuning ϵ at which the resonance condition still holds to be of the order of h.
- We assume that the functions $a(t)$ and $b(t)$ are slow functions of time and assume (and later check) that $\dot{a} \sim \epsilon a$ and $\dot{b} \sim \epsilon b$.
- Finally, we retain only the terms linear in h and ϵ. In particular, this means dropping $\ddot{a}$ and $\ddot{b}$ terms, as they are of the order of ϵ^2, but keeping $\dot{a}$ and $\dot{b}$ as they are of the order of ϵ.

The result is

$$-\omega_0\left(2\dot{a} + b\epsilon + \frac{1}{2}h\omega_0 b\right)\sin(\omega + \epsilon/2)t + \omega_0\left(2\dot{b} - a\epsilon + \frac{1}{2}h\omega_0 a\right)\cos(\omega + \epsilon/2)t = 0$$

In order for this equation to be satisfied at all times the coefficients in front of $\sin(\omega + \epsilon/2)t$ and $\cos(\omega + \epsilon/2)t$ must both be zero.

So we have a pair of equations

$$2\dot{a} + b\epsilon + \frac{1}{2}h\omega_0 b = 0$$
$$2\dot{b} - a\epsilon + \frac{1}{2}h\omega_0 a = 0$$

These are linear first order differential equations with the time independent coefficients, so we look for the solution in the form $a, b \sim a_0, b_0 e^{st}$, then

$$2sa_0 + (\epsilon + h\omega_0/2)b_0 = 0,$$
$$(-\epsilon + h\omega_0/2)a_0 + 2sb_0 = 0.$$

This is a pair of homogeneous linear equations. The compatibility condition (determinant is zero) gives

$$s^2 = \frac{1}{4}\left[(h\omega_0/2)^2 - \epsilon^2\right].$$

- Notice, that e^s is what we called μ before.
- Depending on the value of ϵ we will have s either purely imaginary or purely real.
- If $\epsilon > h\omega_0/2$, then s is imaginary and $|\mu| = |e^s| = 1$ – no resonance.
- The condition for the resonance is that s is real,

. It means that the resonance happens for

$$-\frac{1}{2}h\omega_0 < \epsilon < \frac{1}{2}h\omega_0$$

- The range of frequencies for the resonance depends on the amplitude h. On a swing the more you move the less precise you have to be!
- The amplification $s = \frac{1}{2}\sqrt{(h\omega_0/2)^2 - \epsilon^2}$, also depends on the amplitude h and on ϵ.
- In case of dissipation the solution acquires a decaying factor $e^{-\lambda t}$, so s should be substituted by $s - \lambda$. Then in order for the instability to occur we must have $s > \lambda$ so the range of instability is given by $\frac{1}{4}\left[(h\omega_0/2)^2 - \epsilon^2\right] > \lambda^2$:

$$-\sqrt{(h\omega_0/2)^2 - 4\lambda^2} < \epsilon < \sqrt{(h\omega_0/2)^2 - 4\lambda^2}$$

- At finite dissipation the parametric resonance requires finite amplitude $h = 4\lambda/\omega_0$. On a rusty swing you cannot get a resonance if your h is too small.
- Other resonances occur $\omega_0/\omega_p = n/2$.

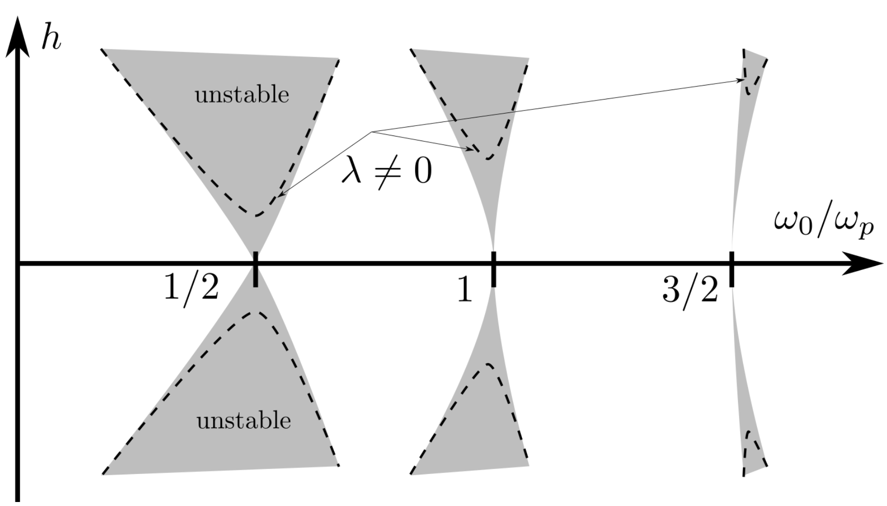


Figure 1

LECTURE 16
Oscillations of an infinite series of springs. Oscillations of a rope. Phonons.

16.1. Series of springs.

Consider one dimension string of N masses m connected with identical springs of spring constants k. The first and the last masses are connected by the same springs to walls. The question is what are the normal modes of such system?

- The difference between the infinite number of masses and finite, but large — zero mode.

This system has N degrees of freedom, so we must find N modes. We call x_i the displacement of the ith mass from its equilibrium position. The Lagrangian is:

$$L = \sum_{i=1}^{N} \frac{m\dot{x}_i^2}{2} - \frac{k}{2}\sum_{i=0}^{N}(x_i - x_{i+1})^2, \qquad x_0 = x_{N+1} = 0.$$

16.1.1. First solution

The matrix $-\omega^2 k_{ij} + u_{ij}$ is

$$-\omega^2 k_{ij} + u_{ij} = \begin{pmatrix} -m\omega^2 + 2k & -k & 0 & \dots & \dots \\ -k & -m\omega^2 + 2k & -k & 0 & \dots \\ 0 & -k & -m\omega^2 + 2k & -k & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

This is $N \times N$ matrix. Let's call its determinant D_N. We then see

$$D_N = (-m\omega^2 + 2k)D_{N-1} - k^2 D_{N-2}, \qquad D_1 = -m\omega^2 + 2k, \qquad D_2 = (-m\omega^2 + 2k)^2 - k^2$$

This is a linear difference equation with constant coefficients. The solution should be of the form $D_N = a^N$. Then we have

$$a^2 = (-m\omega^2 + 2k)a - k^2, \qquad a = \frac{-m\omega^2 + 2k \pm i\sqrt{m\omega^2(4k - m\omega^2)}}{2}.$$

Notice, that $|a|^2 = k^2$.

The general solution is a linear combination of the two solutions (with a and $\bar{a}$) which we found with arbitrary coefficients. However, D_N must be real, so the coefficients must also be complex conjugated. Thus the general solution and the initial conditions are

$$D_N = Aa^{N-1} + \bar{A}\bar{a}^{N-1}, \qquad A + \bar{A} = -m\omega^2 + 2k, \qquad Aa + \bar{A}\bar{a} = (-m\omega^2 + 2k)^2 - k^2.$$

The solution is

$$A = \frac{a^2}{a - \bar{a}}, \qquad \bar{A} = -\frac{\bar{a}^2}{a - \bar{a}}.$$

(I used here $-m\omega^2 + 2k = a + \bar{a}$ and $(-m\omega^2 + 2k)^2 - k^2 = a^2 + a\bar{a} + \bar{a}^2$.) Now in order to find the normal frequencies we need to solve the following equation for ω.

$$D_N = \frac{a^2}{a - \bar{a}} a^{N-1} - \frac{\bar{a}^2}{a - \bar{a}} \bar{a}^{N-1} = 0, \qquad \text{or} \qquad \left(\frac{a}{\bar{a}}\right)^{N+1} = 1.$$

As $|a|^2 = k^2$, we present a as

$$a = ke^{i\phi}, \qquad \cos\phi = \frac{-m\omega^2 + 2k}{2k},$$

then our equation reads

$$e^{2i\phi(N+1)} = 1, \qquad 2\phi(N+1) = 2\pi n, \qquad \text{where} \quad n = 1 \dots N.$$

So we have $\phi = \frac{\pi n}{N+1}$ and thus

$$\frac{-m\omega^2 + 2k}{2k} = \cos\phi = \cos\frac{\pi n}{N+1}, \qquad \omega_n^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N+1)}.$$

16.1.2. Second solution.

From the Lagrangian we find the equations of motion

$$\ddot{x}_j = -\frac{k}{m}(2x_j - x_{j+1} - x_{j-1}), \qquad x_0 = x_{N+1} = 0.$$

We look for the solution in the form

$$x_j = \sin(\beta j)\sin(\omega t), \qquad \sin\beta(N+1) = 0.$$

Substituting this guess into the equation we get

$$-\omega^2\sin(\beta j) = -\frac{k}{m}\left(2\sin(\beta j) - \sin\beta(j+1) - \sin\beta(j-1)\right)$$
$$= -\frac{k}{m}\Im\left(2e^{ij\beta} - e^{i(j+1)\beta} - e^{i(j-1)\beta}\right) = -\frac{k}{m}\Im e^{ij\beta}\left(2 - e^{i\beta} - e^{-i\beta}\right) = \frac{k}{m}\Im e^{ij\beta}\left(e^{i\beta/2} - e^{-i\beta/2}\right)^2$$
$$= -4\frac{k}{m}\Im e^{ij\beta}\sin^2(\beta/2) = -4\frac{k}{m}\sin(j\beta)\sin^2(\beta/2).$$

So we have

$$\omega^2 = 4\frac{k}{m}\sin^2(\beta/2),$$

but β must be such that $\sin\beta(N+1) = 0$, so $\beta = \frac{\pi n}{N+1}$, and we have

$$\omega^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N+1)}, \qquad n = 1, \dots, N$$

16.2. A guitar string.

Here we consider a guitar/piano/harp/violin/viola... string. It has a tension and it has a linear mass density. Gravity plays no role. To simplify the problem I will consider only the string in 2D.

The potential energy of a (2D) string at tension T (gravity is neglected) of shape $y(x)$ is $T\int_0^L \sqrt{1+y'^2}dx \approx \frac{T}{2}\int_0^L y'^2 dx$. The kinetic energy is $\int_0^L \frac{\rho}{2}\dot{y}^2 dx$, so the Lagrangian is

$$\mathcal{L} = \int_0^L \left(\frac{\rho}{2}\dot{y}^2 - \frac{T}{2}y'^2\right) dx, \qquad y(0) = y(L) = 0.$$

In order to find the normal modes we need to decide on the coordinates in our space of functions $y(x,t)$. We will use the standard Fourier basis $\sin kx$ and write any function as

$$y(x,t) = \sum_k A_k(t)\sin kx, \qquad \sin kL = 0$$

The constants $A_k(t)$ are the coordinates in the Fourier basis. We then have

$$\mathcal{L} = \frac{L}{2}\sum_k \left(\frac{\rho}{2}\dot{A}_k^2 - \frac{T}{2}k^2 A_k^2\right)$$

We see, that it is just a set of decoupled harmonic oscillators and k just enumerates them. The normal frequencies are

$$\omega_k^2 = \frac{T}{\rho}k^2, \qquad \omega = \sqrt{\frac{T}{\rho}}k.$$

- We also see, that the wavelength $\lambda = 2\pi/k$. So using $\omega = 2\pi f$ we find that $\lambda f = \sqrt{T/\rho}$. So the speed of these waves is

$$c^2 = T/\rho.$$

LECTURE 17

Motion of a rigid body. Kinematics. Kinetic energy. Momentum. Tensor of inertia.

17.1. Kinematics.

17.1.1. Vector of angular velocity.

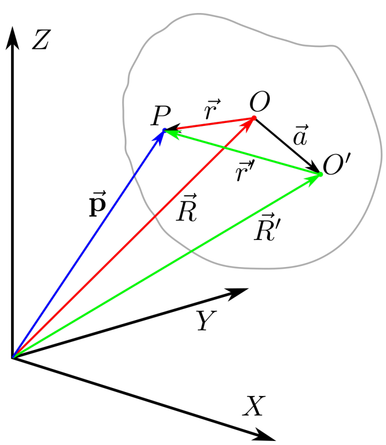


Figure 1

We will use two different system of coordinates XYZ — fixed, or external inertial system of coordinates, and xyz the moving, or internal system of coordinates which is attached to the body itself and moves with it.

Before we start I want to remind that for any vector $\vec{r}$ which is fixed in the *internal* frame of references we have

$$\dot{\vec{r}} = \vec{\Omega} \times \vec{r},$$

where Ω is the vector of angular velocity. Multiplying this equation by dt — time interval — we get

$$d\vec{r} = d\vec{\phi} \times \vec{r}$$

- Notice, that ϕ is not a vector, while $d\vec{\phi}$ is.

Let's $\vec{R}$ be radius vector of some arbitrary point O of a body with respect to the external frame of reference, $\vec{r}$ be the radius vector of any point P of the body with respect to the point O, and $\vec{\mathbf{p}}$ the radius vector of the point P with respect to the external frame of reference: $\vec{\mathbf{p}} = \vec{R} + \vec{r}$, see figure. For any infinitesimal displacement $d\vec{\mathbf{p}}$ of the point P we have

$$d\vec{\mathbf{p}} = d\vec{R} + d\vec{r} = d\vec{R} + d\vec{\phi} \times \vec{r}.$$

Or dividing by dt we find the velocity of the point P as

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}, \quad \text{where} \quad \vec{v} = \frac{d\vec{\mathbf{p}}}{dt}, \quad \vec{V} = \frac{d\vec{R}}{dt}, \quad \vec{\Omega} = \frac{d\vec{\phi}}{dt}.$$

What we see here is

- $\vec{v}$ is the velocity of the point P in the external frame of reference.
- $\vec{V}$ is the velocity of the point O in the external frame of reference.
- $\vec{\Omega}$ is the *vector* of the angular velocity.

However, as was defined above the vector $\vec{\Omega}$ may depend on the choice of the point O. As the definition used this point.

In the previous calculation the point O was an arbitrary point. So for any other point O' with a radius vector $\vec{R}' = \vec{R} + \vec{a}$ we find the radius vector of the point P to be $\vec{r}' = \vec{r} - \vec{a}$, and we must have

$$\vec{v} = \vec{V}' + \vec{\Omega}' \times \vec{r}'.$$

Now substituting $\vec{r} = \vec{r}' + \vec{a}$ into $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$ we get

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \vec{r}'.$$

The velocity $\vec{v}$ in both calculations is the velocity of the of the *same* point P of the body measured in the same external frame of reference. So both calculations must give the same result. We then conclude that

$$\vec{V}' = \vec{V} + \vec{\Omega} \times \vec{a}, \qquad \vec{\Omega}' = \vec{\Omega}.$$

- The last equation shows, that the vector of angular velocity is the same and does not depend on the choice of the point O. So $\vec{\Omega}$ can be called the angular velocity of the body.
- In particular, this means that one can deal with vector $\vec{\Omega}$ as with any other vector.

17.1.2. Instantaneous axis of rotation.

If at some instant the vectors $\vec{V}$ and $\vec{\Omega}$ are perpendicular for some choice of O, then they will be perpendicular for any other O': $\vec{\Omega} \cdot \vec{V} = \vec{\Omega} \cdot \vec{V}'$. Then it is possible to find a set of points (a line) such that their velocity is zero. It means to solve the equation for $\vec{a}$

$$\vec{V} + \vec{\Omega} \times \vec{a} = 0.$$

These are three inhomogeneous linear equations for the components of the vector $\vec{a}$.

If we take the dot product of the above equation with $\vec{\Omega}$, we find that $\vec{\Omega} \cdot \vec{V} = 0$, this is the requirement that the above equation has a solution (rhs must be orthogonal to the zero modes).

If we multiply the above equation by $\vec{V}$ "vectorly" we find

$$0 = \vec{\Omega}(\vec{V} \cdot \vec{a}) - \vec{a}(\vec{V} \cdot \vec{\Omega}) = \vec{\Omega}(\vec{V} \cdot \vec{a}),$$

so $\vec{V} \cdot \vec{a} = 0$. So the vector $\vec{a}$ must have the form

$$\vec{a} = \alpha \vec{\Omega} \times \vec{V} + \beta \vec{\Omega}.$$

Finally, substituting this form of $\vec{a}$ into the main equation we get

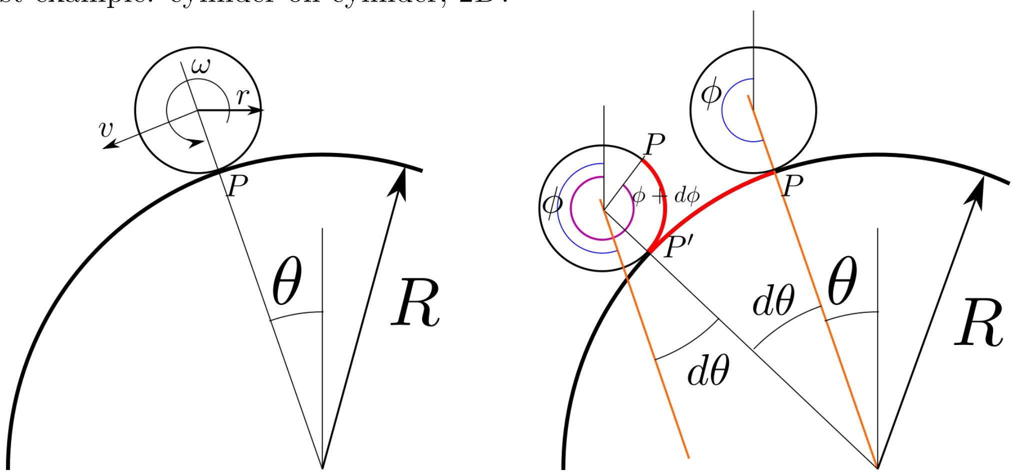
$$0 = \vec{V} + \alpha \vec{\Omega} \times [\vec{\Omega} \times \vec{V}] + \beta \vec{\Omega} \times \vec{\Omega} = \vec{V} + \alpha \vec{\Omega}(\vec{\Omega} \cdot \vec{V}) - \alpha \vec{V} \Omega^2 = \vec{V}\left(1 - \alpha \Omega^2\right)$$

So we have $\alpha \Omega^2 = 1$ and β can be any number.

$$\vec{a} = \frac{\vec{\Omega} \times \vec{V}}{\Omega^2} + \beta \vec{\Omega}.$$

This equation describes a line (as β is arbitrary) parallel to $\vec{\Omega}$. This line/axis (it may be outside of the body) has zero velocity. The rest of the body simply rotates around this axis. This line is called "instantaneous axis of rotation". (In the general case (when $\vec{\Omega} \cdot \vec{V} \neq 0$) the instantaneous axis can be made parallel to $\vec{V}$.)

- At any *instant* the velocities of all points of a body can be described as rotation with respect to some *instantaneous* axis of rotation plus translation along this axis.
- In general both the magnitude and the direction of $\vec{\Omega}$ are changing with time, so is the "*instantaneous axis of rotation*".
- If one finds two points which have zero velocity at this *instant*, then the *instantaneous* axis of rotation is the straight line through these two points. This way one figures out the direction of the vector $\vec{\Omega}$.
- First example: cylinder on cylinder, $2D$.



Let's consider a cylinder of radius r moving without slipping on top of the cylinder of radius R. We want to use the angle θ as a coordinate. We thus need to express the angular velocity ω of the cylinder r through $\dot{\theta}$. There are two ways to approach this problem (for shortness I will call the cylinders cylinder r and cylinder R):

  – The easy way. Look at the figure on the left. Due to non-slipping condition the point P of the cylinder r is not moving. So the *instantaneous* axis of rotation goes through this point. Then the velocity of the center of the cylinder r is $v = \omega r$. The same velocity is also $v = \dot{\theta}(R + r)$ as the center is moving on a circle of radius $R + r$. So we have $v = \omega r = \dot{\theta}(R + r)$, or

$$\omega = \dot{\theta}\frac{R + r}{r}.$$

  – The hard way. Look at the figure on the right. What is depicted is the change of the position of the cylinder r after time interval dt. The angle θ is changed by $d\theta$. The cylinder r also rotated by the angle $d\phi$. It is *very* important to fix the direction from which the angle ϕ is measured! On the figure it is measured from the vertical line. The non-slipping condition means that the length of the red arc PP' on the cylinder r must be equal to the length of the red arc PP' on the cylinder R. The red arc on the cylinder R has length $Rd\theta$. In order to find the length of the red arc on the cylinder r we draw the orange line parallel to the initial orange line. Now from the figure it is clear, that the angle of the red arc on the cylinder r is $d\phi - d\theta$, so that the length of this arc is $r(d\phi - d\theta)$. So no slipping condition gives $Rd\theta = r(d\phi - d\theta)$. Or $d\phi = d\theta\frac{R+r}{r}$. Dividing this by dt we get the previous result.

- Second example: a cone on a plane, $3D$.

17.2. Kinetic energy.

The total kinetic energy of a body is the sum of the kinetic energies of its parts. Lets take the origin of the moving system of coordinates to be in the center of mass. Then

$$K = \frac{1}{2}\sum m_\alpha v_\alpha^2 = \frac{1}{2}\sum m_\alpha \left(\vec{V} + \vec{\Omega}\times\vec{r}_\alpha\right)^2 = \frac{1}{2}\sum m_\alpha \vec{V}^2 + \sum m_\alpha \vec{V}\cdot\vec{\Omega}\times\vec{r}_\alpha + \frac{1}{2}\sum m_\alpha \left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$
$$= \frac{MV^2}{2} + \vec{V}\cdot\vec{\Omega}\times\sum m_\alpha\vec{r}_\alpha + \frac{1}{2}\sum m_\alpha\left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$

For the center of mass $\sum m_\alpha \vec{r}_\alpha = 0$ and we have

$$K = \frac{MV^2}{2} + \frac{1}{2}\sum m_\alpha\left(\vec{\Omega}^2\vec{r}_\alpha^{\,2} - (\vec{\Omega}\cdot\vec{r}_\alpha)^2\right) = \frac{MV^2}{2} + \frac{I_{ij}\Omega^i\Omega^j}{2},$$

where

$$I_{ij} = \sum m_\alpha\left(\delta_{ij}\vec{r}_\alpha^{\,2} - r_\alpha^i r_\alpha^j\right).$$

I_{ij} is the tensor of inertia. This tensor is symmetric and positive definite.

17.3. Angular momentum

The origin is at the center of mass. So we have

$$\vec{M} = \sum m_\alpha\vec{r}_\alpha\times\vec{v}_\alpha = \left(\sum m_\alpha\vec{r}_\alpha\right)\times\vec{V} + \sum m_\alpha\vec{r}_\alpha\times(\vec{\Omega}\times\vec{r}_\alpha) = \sum m_\alpha\left(r_\alpha^2\vec{\Omega} - \vec{r}_\alpha(\vec{r}_\alpha\cdot\vec{\Omega})\right)$$

Writing this in components we have

$$M_i = \sum m_\alpha\left(\delta_{ij}\vec{r}_\alpha^{\,2} - r_\alpha^i r_\alpha^j\right)\Omega^j$$

or

$$M_i = I_{ij}\Omega^j.$$

- In general the direction of angular momentum $\vec{M}$ and the direction of the angular velocity $\vec{\Omega}$ do not coincide.

17.4. Tensor of inertia.

Tensor of inertia is a symmetric tensor of rank two. As any such tensor it can be reduced to a diagonal form by an appropriate choice of the moving axes. Such axes are called the principal axes of inertia. The diagonal components I_1, I_2, and I_3 are called the principal moments of inertia.

- Notice, that these axes are "attached" to the body and thus rotate with the body.

In this axes the kinetic energy is simply

$$K = \frac{I_1\Omega_1^2}{2} + \frac{I_2\Omega_2^2}{2} + \frac{I_3\Omega_3^2}{2}.$$

(a) If all three principal moments of inertia are different, then the body is called "asymmetrical top".
(b) If two of the moments coincide and the third is different, then it is called "symmetrical top".
(c) If all three coincide, then it is "spherical top".

For any plane figure if z is perpendicular to the plane, then $I_1 = \sum m_\alpha y_\alpha^2$, $I_2 = \sum m_\alpha x_\alpha^2$, and $I_3 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = I_1 + I_2$. If symmetry demands that $I_1 = I_2$, then $\frac{1}{2} I_3 = I_1$. Example: a disk, a square.

If the body is a line, then (if z is along the line) $I_1 = I_2$, and $I_3 = 0$. Such system is called "rotator".

LECTURE 18
Motion of a rigid body. Rotation of a symmetric top. Euler angles.

18.1. Tensor of inertia.

Tensor of inertia is a symmetric tensor of rank two. As any such tensor it can be reduced to a diagonal form by an appropriate choice of the moving axes. Such axes are called the principal axes of inertia. The diagonal components I_{xx}, I_{yy}, and I_{zz} (they also called I_x, I_y, and I_z) are called the principal moments of inertia.

- Notice, that these axes are "attached" to the body and thus rotate with the body.

In this axes the kinetic energy is simply

$$K = \frac{I_x\Omega_x^2}{2} + \frac{I_y\Omega_y^2}{2} + \frac{I_z\Omega_z^2}{2},$$

where Ω_x, Ω_y, and Ω_z are projections of the vector $\vec{\Omega}$ on the *internal* axes.

Terminology:

(a) If all three principal moments of inertia are different, then the body is called "asymmetrical top".
(b) If two of the moments coincide and the third is different, then it is called "symmetrical top". If $I_x = I_y$, then the axes x and y can be chosen arbitrarily (but orthogonal to each other) in the plane orthogonal to the axis z! This leads to great simplifications!
(c) If all three coincide, then it is "spherical top".

18.2. Kinematics.

We consider a body freely (no forces applied) moving and rotating. As there are no forces the center of mass must move with a constant velocity. So if we consider the motion in the external frame of references which moves with the same velocity, in this frame of references the center of mass will not be moving, or its velocity will be zero. It means that at every instant the instantaneous axis of rotation will go through the center of mass.

So if there are no forces one can always consider just a rotation of a body around an axis which goes through the center of mass.

As there are no forces acting on the body we can immediately state that

- Kinetic energy is conserved.
- Angular momentum is conserved.

Let's consider the motion in more detail in some specific cases.

- Spherical top: $I_x = I_y = I_z = I$ and $I^{ij} = I\delta^{ij}$ for *any* choice of internal axes. So for any direction we have $\vec{M} = I\vec{\Omega}$. So if $\vec{M}$ is conserved, then $\vec{\Omega}$ is a constant. The body simply rotates around the initial axes.
- Arbitrary top rotating around one of its principal axes, say x, Then $\vec{\Omega} = \Omega\vec{x}$ and $\vec{M} = I_x\Omega\hat{x}$. The axis x is not moving $\dot{\hat{x}} = \vec{\Omega} \times \hat{x} = 0$. So if $\vec{M}$ is conserved, then $\vec{\Omega}$ is conserved, and the body keeps rotating around the axis x which is also not changing.
- Symmetric top rotating around an arbitrary axis.

The last case requires some work.

18.3. Free symmetric top.

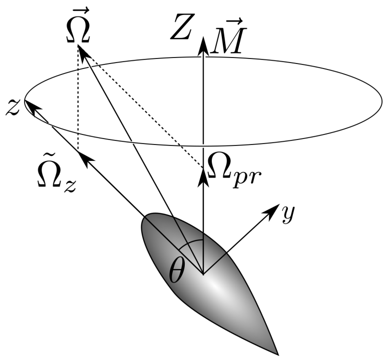


Figure 1

Consider a free rotation of a symmetric top $I_x = I_y \neq I_z$, where x, y, and z are the principal axes. The direction of the angular momentum does not coincide with the direction of any principle axes. Let's say, that the angle between $\vec{M}$ and the moving axes z at some instant is θ. We chose as the axis y the one that is in plane with the two vectors $\vec{M}$ and $\hat{z}$.

During the motion the total angular momentum is conserved.

The whole motion can be thought of as two rotations one the rotation of the body around the axes z and the other, called precession, is the rotation of the axis z around the direction of the vector $\vec{M}$ – we call its direction $\hat{Z} = \vec{M}/|\vec{M}|$.

We can think of vector $\vec{\Omega}$ in two different ways

$$\vec{\Omega} = \hat{z}\Omega_z + \hat{y}\Omega_y \tag{18.1}$$

$$\vec{\Omega} = \frac{\vec{M}}{M}\Omega_{pr} + \hat{z}\tilde{\Omega}_z \tag{18.2}$$

- Notice, that Ω_z is not the same as $\tilde{\Omega}_z$.

and angular momentum

$$\vec{M} = \Omega_y I_y \hat{y} + \Omega_z I_z \hat{z}, \tag{18.3}$$

Multiplying (18.3) by $\hat{z}$ (at this instant of time) we get

$$\Omega_z = \frac{M_z}{I_z} = \frac{M}{I_z}\cos\theta.$$

In order to find the angular velocity of precession we multiply (18.2) and (18.3) by $\hat{y}$ and get

$$\Omega_y = \frac{\hat{y}\cdot\vec{M}}{M}\Omega_{pr} \qquad \text{and} \qquad \hat{y}\cdot\vec{M} = I_y\Omega_y$$

- From the second equation above we see, that $\Omega_y = \frac{M_y}{I_y} = \frac{M}{I_y}\sin\theta$.

Substituting $\hat{y} \cdot \vec{M}$ from the second equation into the first

$$\Omega_y = \frac{I_y}{M} \Omega_y \Omega_{pr}.$$

This equation has two solutions $\Omega_y = 0$ – which corresponds to $\vec{M} \parallel \hat{z}$, or if $\Omega_y \neq 0$

$$\Omega_{pr} = \frac{M}{I_y}.$$

Also multiplying (18.2) by $\hat{z}$ we find

$$\tilde{\Omega}_z = \Omega_z - \frac{M_z}{M} \Omega_{pr} = M \left(\frac{1}{I_z} - \frac{1}{I_y} \right) \cos\theta.$$

18.4. Euler's angles

The orientation of a rigid body is described by three angles. There are different ways to parametrize orientation. Here we consider what is called Euler's angles.

The fixed coordinates are XYZ, the moving coordinates xyz. The plane xy intersects the plane XY along the line ON called the line of nodes.

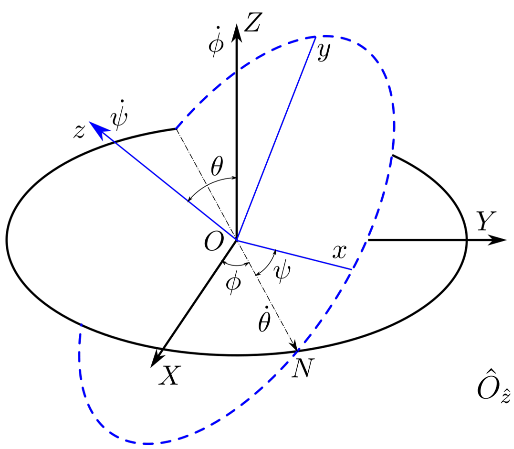


Figure 2

The angle θ is the angle between the Z and z axes. The angle ϕ is the angle between the X axes and the line of nodes, and the angle ψ is the angle between the x axes and the line of nodes.

Let's consider what we need to do to orient a body according to the given Euler angles.

Initially the axes XYZ and xyz coincide. Let's denote $\hat{O}_{\hat{\xi}}(\alpha)$ a rotation around a unit vector $\hat{\xi}$ on the angle α. Then in order to get the orientation on the picture we need to perform three separate rotations

$$\hat{O}_{\hat{z}}(\psi) \circ \hat{O}_{\hat{Z}}(\phi) \circ \hat{O}_{\hat{X}}(\theta)$$

What it means is that one needs (read above from right to left — it is an operator. Also remember, that xyz is attached to the body)

- Rotate the body around axis X by angle θ.
  After that the axes Z and z have an angle θ between them. The axes X, x, and the line of nodes coincide.
- Rotate the body around axis Z by angle ϕ.
  After that the axes Z and z still have an angle θ between them. The axis x and the line of nodes still coincide, but there is an angle ϕ between the axis X and the line of nodes.
- Rotate the body around axis z by angle ψ.
  After that the axes Z and z still have an angle θ between them. The angle between the axis X and the line of nodes is still ϕ. But there is an angle ψ between the axis x and the line of nodes.

The angle θ is from 0 to π, the ϕ and ψ angles are from 0 to 2π.

Another way of looking at Euler angles is to recognize, that the angles θ and ϕ are our normal angles of spherical coordinates. They determine the direction of the axis z. After that we rotate the body around the axis z by angle ψ.

These three Euler angles θ, ϕ, and ψ completely determine the orientation of a body. By knowing this orientation we can compute the potential energy of the body.

We also need to express the kinetic energy through our coordinates (θ, ϕ, and ψ) and their velocities ($\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$).

If xyz are the principle axes of inertia, then the kinetic energy is simply $K = \frac{1}{2}I_x\Omega_x^2 + \frac{1}{2}I_y\Omega_y^2 + \frac{1}{2}I_z\Omega_z^2$. Where Ω_x, Ω_y, and Ω_z are projections of the vector $\vec{\Omega}$ on the *internal* axes xyz. So we need to express Ω_x, Ω_y, and Ω_z through θ, ϕ, and ψ and their velocities $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$.

Let's consider changing the angle θ to $\theta + d\theta$. It is clear, that we need to rotate the body by an angle $d\theta$ around the line of nodes ON. So the vector $d\vec{\theta}$ has a length $d\theta$ and is directed along the line of nodes ON. Then the vector $\vec{\dot{\theta}}$ has a length $\dot{\theta}$ and is directed along the line of nodes ON.

The same way the vector $d\vec{\phi}$ has a length $d\phi$ and is directed along Z. Then the vector $\vec{\dot{\phi}}$ has the length $\dot{\phi}$ and is directed along Z. The vector $d\vec{\psi}$ has a length $d\psi$ and is directed along z. Then the vector $\vec{\dot{\psi}}$ has the length $\dot{\psi}$ and is directed along z.

It is important to recognize, that vector $\vec{\Omega}$ is simply the sum

$$\vec{\Omega} = \vec{\dot{\theta}} + \vec{\dot{\phi}} + \vec{\dot{\psi}}.$$

So we need write the components of the vectors $\vec{\dot{\theta}}$, $\vec{\dot{\phi}}$, and $\vec{\dot{\psi}}$ in the internal frame:

(a) The **vector** $\vec{\dot{\theta}}$ is along the line of nodes, so its components along $\hat{x}$, $\hat{y}$, and $\hat{z}$ are (I'll write the z component first just for convenience)

$$\dot{\theta}_z = 0$$
$$\dot{\theta}_x = \dot{\theta}\cos\psi$$
$$\dot{\theta}_y = -\dot{\theta}\sin\psi$$

(b) The **vector** $\vec{\dot{\phi}}$ is along the Z direction, so its component along $\hat{z}$, $\hat{x}$, and $\hat{y}$.

$$\dot{\phi}_z = \dot{\phi}\cos\theta$$
$$\dot{\phi}_x = \dot{\phi}\sin\theta\sin\psi$$
$$\dot{\phi}_y = \dot{\phi}\sin\theta\cos\psi$$

(c) The **vector** $\vec{\dot{\psi}}$ is along the z direction, so its component along $\hat{z}$, $\hat{x}$, and $\hat{y}$.

$$\dot{\psi}_z = \dot{\psi}$$
$$\dot{\psi}_x = 0$$
$$\dot{\psi}_y = 0$$

We now collect all angular velocities along each axis as $\Omega_x = \dot{\theta}_x + \dot{\phi}_x + \dot{\psi}_x$ etc. and find

$$\Omega_x = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$
$$\Omega_y = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$
$$\Omega_z = \dot{\phi}\cos\theta + \dot{\psi} \tag{18.4}$$

These equations allow us:

- if we know the principal moments of inertia, to express the kinetic energy $K = \frac{1}{2}I_x\Omega_x^2 + \frac{1}{2}I_y\Omega_y^2 + \frac{1}{2}I_z\Omega_z^2$ in terms of the derivative of the coordinates and coordinates θ, ϕ, and ψ. Or
- first solve problem in the moving system of coordinates, find Ω_x, Ω_y, and Ω_z, and then calculate $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$.

18.5. Free symmetric top again.

Consider the symmetric top again $I_y = I_x$. We take Z to be the direction of the angular momentum. We can take the axis x coincide with the line of nodes. Then $\psi = 0$ (but $\dot{\psi} \neq 0$!), and from (18.4) we have

$$\begin{aligned}
\Omega_x &= \dot{\theta} \\
\Omega_y &= \dot{\phi}\sin\theta \\
\Omega_z &= \dot{\phi}\cos\theta + \dot{\psi}
\end{aligned}$$

The components of the angular momentum are

$$\begin{aligned}
M_x &= I_x\Omega_x = I_x\dot{\theta} \\
M_y &= I_y\Omega_y = I_y\dot{\phi}\sin\theta \\
M_z &= I_z\Omega_z
\end{aligned}$$

On the other hand

$$\begin{aligned}
M_x &= 0 \\
M_y &= M\sin\theta \\
M_z &= M\cos\theta
\end{aligned}$$

Comparing those we find

$$\dot{\theta} = 0, \qquad \Omega_{pr} = \dot{\phi} = \frac{M}{I_y}, \qquad \Omega_z = \frac{M}{I_z}\cos\theta.$$

Now using $\Omega_z = \dot{\phi}\cos\theta + \dot{\psi}$ and the result for $\dot{\phi}$ we find

$$\tilde{\Omega}_z = \dot{\psi} = M\left(\frac{1}{I_z} - \frac{1}{I_y}\right)\cos\theta.$$

LECTURE 19
Symmetric top in gravitational field.

19.1. The Lagrangian.

We want to consider the motion of the symmetric top ($I_x = I_y$) whose lowest point is fixed. We call this point O.

- As point O is fixed, the instantaneous axis of rotation goes through it.

It then makes sense to chose the coordinate frames as shown in the figure with the origin at point O.

The Euler angles θ, ϕ, and ψ fully describe the orientation of the top. The angles are unconstrained and change $0 < \theta < \pi$, $0 < \psi, \phi < 2\pi$. So the Euler angles are good generalized coordinates

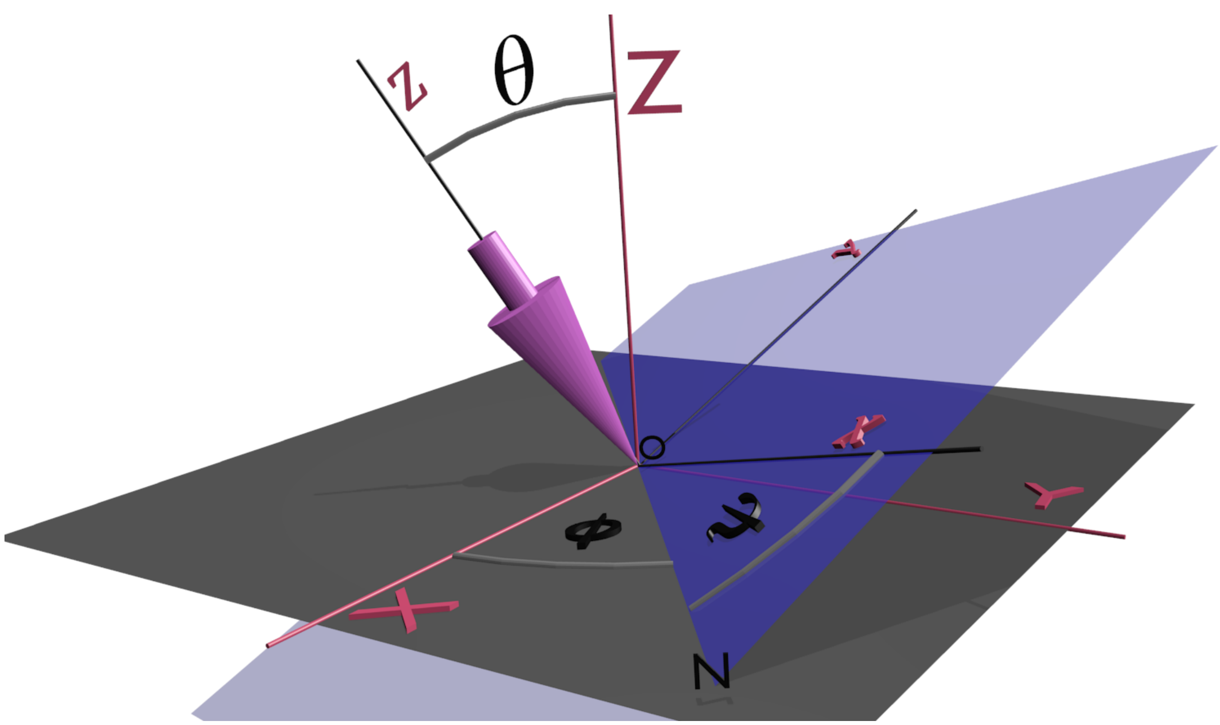


Figure 1

- The line of nodes is an intersection between XY and xy planes. It is the line ON on the figure.

Instead of defining the tensor of inertia with respect to the center of mass, we now need to define it with respect to the point O. The principal axes trough this point are parallel to the ones through the center of mass. The principal moment I_z does not change under such shift, the principal moment with respect to the axes x and y become by $I = I_x + ml^2$, where l is the distance from the point O to the center of mass.

Now we can simply use the formulas derived in the previous lecture to express the components of the vector $\vec{\Omega}$ in the internal xyz frames of reference through the our generalized coordinates (Euler angles) and their velocities.

$$\begin{aligned} \Omega_x &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\ \Omega_y &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\ \Omega_z &= \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}$$

The kinetic energy of the symmetric top is

$$K = \frac{I_z}{2}\Omega_z^2 + \frac{I}{2}\left(\Omega_x^2 + \Omega_y^2\right) = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

- Notice that the kinetic energy is greatly simplified due to the fact that the principle moments of inertia along x and y equal to each other.
- It is because of that symmetry the kinetic energy only depends on the combination $\Omega_x^2 + \Omega_y^2$.
- Which leads to the fact, that the kinetic energy does not depend on ψ. (It does depend on $\dot{\psi}$.)

The potential energy is simply $mgl\cos\theta$, so the Lagrangian is

$$L = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta.$$

19.2. The solution.

In principle we can write the equations of motion from the Lagrangian. These will be three second order nonlinear differential equations. These equations are needed to be supplemented by six initial conditions:

$$\begin{aligned} \theta(t=0) &= \theta_0, & \dot{\theta}(t=0) &= \dot{\theta}_0 \\ \phi(t=0) &= \phi_0, & \dot{\phi}(t=0) &= \dot{\phi}_0 \\ \psi(t=0) &= \psi_0, & \dot{\psi}(t=0) &= \dot{\psi}_0 \end{aligned}$$

However, there is a simpler way!

We see that the Lagrangian does not depend on ϕ and ψ – this is only correct for the symmetric top. The corresponding momenta $M_Z = \frac{\partial L}{\partial \dot{\phi}}$ and $M_3 = \frac{\partial L}{\partial \dot{\psi}}$ are conserved.

$$M_3 = I_z(\dot{\psi} + \dot{\phi}\cos\theta), \qquad M_Z = (I\sin^2\theta + I_z\cos^2\theta)\dot{\phi} + I_z\dot{\psi}\cos\theta.$$

It is instructive to see what symmetries are responsible for these conservation laws.

- The conservation of M_Z is due to the symmetry of the system under the rotation around Z axis.

- The conservation of M_3 is due to the symmetry of the system under the rotation around z axis – it only exists for the symmetric top!

The energy is also conserved

$$E = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mgl\cos\theta.$$

- The energy conservation is due to time translation invariance.

The values of M_Z, M_3, and E are given by the initial conditions.

So we have three unknown functions $\theta(t)$, $\phi(t)$, and $\psi(t)$ and three conserved quantities. The conservation laws then completely determine the whole motion.

- I again want to emphasize, that we have one extra conservation law because the top is symmetric. Without this symmetry we would have to solve the second order nonlinear differential equations.

The equations for M_Z and M_3 can be considered as two linear equations for two velocities $\dot{\phi}$ and $\dot{\psi}$. We can solve these equations and express $\dot{\phi}$ and $\dot{\psi}$ through M_Z, M_3, and θ.

$$\dot{\phi} = \frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$
$$\dot{\psi} = \frac{M_3}{I_z} - \cos\theta\frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$

We then substitute the values of the $\dot{\phi}$ and $\dot{\psi}$ into the expression for the energy and find

$$E' = \frac{1}{2}I\dot{\theta}^2 + U_{eff}(\theta),$$

where

$$E' = E - \frac{M_3^2}{2I_z} - mgl, \qquad U_{eff}(\theta) = \frac{(M_Z - M_3\cos\theta)^2}{2I\sin^2\theta} - mgl(1-\cos\theta).$$

This is an equation of motion for a $1D$ motion, so we get

$$t = \sqrt{\frac{I}{2}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{E' - U_{eff}(\theta')}}.$$

where θ_0 is given by the initial condition $\theta(t=0) = \theta_0$. The integral written above is an elliptic integral. Taking this integral (and inverting) we will find $\theta(t)$. Knowing this function we will find $\dot{\phi}(t)$ and $\dot{\psi}(t)$. Integrating these velocities over time (and using the initial conditions $\phi(t=0) = \phi_0$ and $\psi(t=0) = \psi_0$) we will find $\phi(t)$ and $\psi(t)$. So we solve the problem.

The result will be expresses in some elliptic functions. However, analyzing, the motion as we usually analyze the $1D$ motion reveals all the features of the solution without much work.

The effective potential energy goes to infinity when $\theta \to 0, \pi$. The function θ oscillates between θ_{min} and θ_{max} which are the solutions of the equation $E' = U_{eff}(\theta)$. These oscillations are called *nutations*. As $\dot{\phi} = \frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$ the motion depends on whether $M_Z - M_3\cos\theta$ changes sign in between θ_{min} and θ_{max}.

19.3. Stability.

We can find a condition for the stable rotation about the Z axes. For such rotation $M_3 = M_Z = M$, so the effective potential energy is

$$U_{eff} = \frac{M^2}{2I}\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} - 2mgl\sin^2(\theta/2) \approx \left(\frac{M^2}{8I} - \frac{1}{2}mgl\right)\theta^2,$$

where the last is correct for small θ. We see, that the rotation is stable if $M^2 > 4Imgl$, or, as $M = I_z\Omega_z$

$$\Omega_z^2 > \frac{4Imgl}{I_z^2}.$$

LECTURE 20

Rolling coin. An example of the rigid body dynamics.

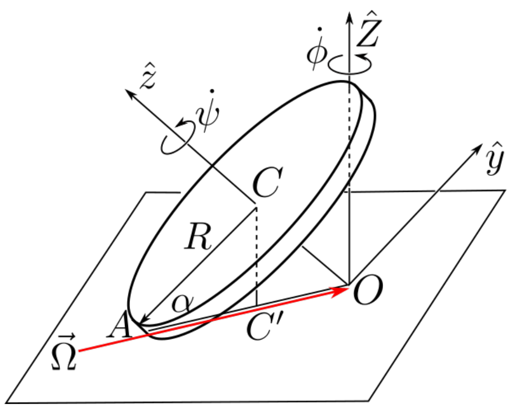


Figure 1

In this lecture we consider the dynamics of the following object:

A uniform thin disc (a coin) of mass m and radius R rolls without slipping on a horizontal plane. The disc makes an angle α with the plane, the center of the disc C moves around the horizontal circle with a constant speed v. The axis of symmetry of the disc CO intersects the horizontal plane in the point O. The point O is stationary.

- Intuitively clear, that this problem is overdetermined. We want to see why.

First, some geometrical facts

$$|OC| = R\tan\alpha, \qquad |OA| = \frac{R}{\cos\alpha}, \qquad |OC'| = |OC|\sin\alpha = R\frac{\sin^2\alpha}{\cos\alpha}, \qquad |CC'| = R\sin\alpha.$$

This is a symmetric top, so we can chose the internal x and y axes anywhere in the plane orthogonal to the internal z axis.

We chose the internal system of coordinates xyz as shown on the figure. In this system the principal moments of inertia are

$$I_z = \frac{1}{2}mR^2, \qquad I_y = I_x = \frac{1}{4}mR^2 + m|OC|^2 = mR^2\left(\frac{1}{4} + \tan^2\alpha\right)$$

20.1. Kinematics.

<u>*Simple way.*</u> According to the problem statement the points O and A are stationary at this instant of time. So the are on the *instantaneous* axis of rotation. It means that the vector $\vec\Omega$ is along this axis.

The point C has a velocity v. For any point $\vec r$ of a rotating body the velocity is $\vec v = \vec\Omega\times\vec r$. So we see, that $v = \Omega|CC'|$, or

$$\Omega = \frac{v}{R}\frac{1}{\sin\alpha}.$$

So we know both the direction and the magnitude of the vector $\vec\Omega$.

In the internal system of coordinates xyz we then have

$$\Omega_z = -\Omega\sin\alpha = -\frac{v}{R}, \qquad \Omega_y = \Omega\cos\alpha = \frac{v}{R}\frac{\cos\alpha}{\sin\alpha}, \qquad \Omega_x = 0.$$

Euler angles. We can find the same result from the Euler angles. As this is symmetric top, we can set $\psi = 0$, but we need to know θ, $\dot\theta$, $\dot\phi$, and $\dot\psi$. According to the figure the Euler angle $\theta = \alpha$, so $\dot\theta = 0$. Our equation for Ω_x, Ω_y, and Ω_z read

$$\begin{aligned}\Omega_x &= 0\\ \Omega_y &= \dot\phi\sin\alpha\\ \Omega_z &= \dot\phi\cos\alpha + \dot\psi\end{aligned}$$

The vector $\vec\Omega = \Omega_x\hat x + \Omega_y\hat y + \Omega_z\hat z$.

We know the velocities of points C and A: $\vec v_C = v\hat x$ and $\vec v_A = 0$ — this is no-slipping condition. According to the general formula $\dot{\vec r} = \vec\Omega\times\vec r$ we find $\vec v_C = \vec\Omega\times\vec{OC}$ and $\vec v_A = \vec\Omega\times\vec{OA}$. Using $\vec{OC} = |OC|\hat z$ and $\vec{OA} = |OC|\hat z - R\hat y$ we compute

$$\begin{aligned}\vec v_C &= \left(\hat y\dot\phi\sin\alpha + \hat z(\dot\phi\cos\alpha + \dot\psi)\right)\times|OC|\hat z = \hat x|OC|\dot\phi\sin\alpha = v\hat x\\ \vec v_A &= \left(\hat y\dot\phi\sin\alpha + \hat z(\dot\phi\cos\alpha + \dot\psi)\right)\times(|OC|\hat z - R\hat y) = \hat x\left(|OC|\dot\phi\sin\alpha + R(\dot\phi\cos\alpha+\dot\psi)\right) = 0\end{aligned}$$

From the first equation we find

$$\dot\phi = \frac{v}{|OC|\sin\alpha} = \frac{v}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

From the second equation (remember, this is our no-slipping condition) we find

$$\dot\psi = -\dot\phi\left(\frac{|OC|}{R}\sin\alpha + \cos\alpha\right) = -\frac{\dot\phi}{\cos\alpha} = -\frac{v}{R}\frac{1}{\sin^2\alpha}$$

(the $-$ sign is important here!)

Using these relations for Ω_x, Ω_y, and Ω_z we find

$$\begin{aligned}\Omega_x &= 0\\ \Omega_y &= \frac{v}{R}\frac{\cos\alpha}{\sin\alpha}\\ \Omega_z &= -\frac{v}{R}\end{aligned}$$

These are exactly the results we got earlier.

20.2. Dynamics.

The main dynamic equations are $\frac{d\vec M}{dt} = \vec\tau$ and $\vec F = m\vec a$. It is very important to recognize that these equations must be written in the external/inertial frame of reference.

Let's start with the angular momentum.

In the internal system of coordinates the angular momentum at this instant of time is

$$\vec M = I_z\Omega_z\hat z + I_y\Omega_y\hat y.$$

From the external point of view, this is a vector of constant magnitude which rotates around the $\hat Z$ axis with angular velocity $\dot\phi$. So we write

$$\dot{\vec M} = \dot\phi\hat Z\times\vec M.$$

Now the expression $\hat{Z} \times \vec{M}$ has no differentiation over time, it is simply the vector product of two vectors. So it can be computed in any frame of references. We compute the RHS at this instant of time

$$\dot{\vec{M}} = \dot{\phi}\hat{Z} \times (I_z\Omega_z\hat{z} + I_y\Omega_y\hat{y}) = \hat{x}\dot{\phi}\left(-I_y\Omega_y\cos\alpha + I_z\Omega_z\sin\alpha\right)$$

Using our kinematic relations we get

$$\dot{\vec{M}} = -\frac{v^2}{R^2}\frac{\cos\alpha}{\sin^3\alpha}\left(I_z\sin^2\alpha + I_y\cos^2\alpha\right)\hat{x}.$$

There are three forces that act on the coin: the gravity mg applied to the point C, pointing down; the normal force N applied to the point A and pointing up; and the friction force F applied to the point A and pointing towards point O. As the center of mass does not move in the Z direction, the normal force and the gravity must compensate each other, so $N = mg$ (N is up).

We want to compute the total torque with respect to point O acting on the coin at this instant of time. The torque of a force $\vec{F}$ applied at point $\vec{r}$ is $\vec{\tau} = \vec{r} \times \vec{F}$. So the torque of the friction force is zero. The torque of the gravity is $\vec{\tau}_g = -\vec{OC} \times mg\hat{Z} = \hat{x}|OC'|mg = \hat{x}Rmg\frac{\sin^2\alpha}{\cos\alpha}$. The torque of the normal force is $\vec{\tau}_N = -mg|OA|\hat{x}$. So the total torque is

$$\vec{\tau} = Rmg\left(\frac{\sin^2\alpha}{\cos\alpha} - \frac{1}{\cos\alpha}\right)\hat{x} = -mgR\cos\alpha\hat{x}.$$

Notice, that this result would be much easier to obtain if we simply computed the torques with respect to point A, but this is not a trivial statement, as point A is not inertial.

Thus we have

$$\frac{v^2}{R^2}\frac{\cos\alpha}{\sin^3\alpha}\left(I_z\sin^2\alpha + I_y\cos^2\alpha\right) = mgR\cos\alpha$$

Substituting here the values of I_y and I_z we get

$$\frac{1}{4}\frac{v^2}{R}\frac{\cos\alpha}{\sin^3\alpha}\left(1 + 5\sin^2\alpha\right) = g\cos\alpha.$$

- Notice, that $\alpha = \pi/2$ (or $\cos\alpha = 0$) is a solution for any v and R – as expected.

For $\cos\alpha \neq 0$ we get

$$\frac{1}{4}\frac{v^2}{R}\left(1 + 5\sin^2\alpha\right) = g\sin^3\alpha.$$

- So v, R, and α cannot be arbitrary!!!

20.3. Friction force.

The center of mass of the coin moves around the circle of radius $|OC'|$ with velocity v, so its acceleration is $\frac{v^2}{|OC'|} = \frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}$ The force that provides this acceleration is the friction force, so

$$F = m\frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

However, this force cannot be larger than $\mu N = \mu Mg$, so we have

$$\mu g > \frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

Using the previous result $\frac{v^2}{Rg} = \frac{4\sin^3\alpha}{1+5\sin^2\alpha}$ we get

$$\frac{4\cos\alpha\sin\alpha}{1+5\sin^2\alpha} < \mu$$

20.4. Lagrangian approach to the rolling coin.

This part of the lecture is for self-study/fun.

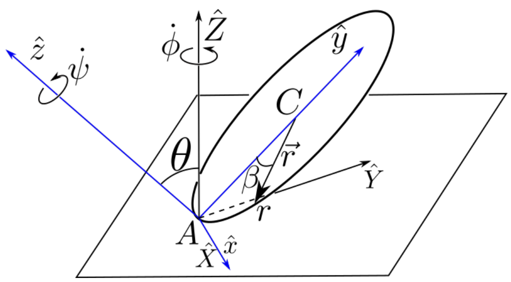


Figure 2

In order to study the full Lagrangian dynamics of the coin we first must assume, that the non-slipping condition is valid at all times – otherwise we will have dissipation and the Lagrangian method would not work. Second we should not assume that the angle θ (or α) is constant in time, so the kinetic energy term will have $\dot{\theta}$.

As point A has zero velocity I want to use it as the coordinate origin (the instantaneous axis of rotation goes through it). So instead of the figure 1 I will use figure 2.

If a coin has the principle moment of inertia around the perpendicular axes through the center of mass I (for a uniform disc it is $I = mR^2/2$) then for our internal axes

$$I_x = I/2 + mR^2$$
$$I_y = I/2$$
$$I_z = I + mR^2$$

I will assume, that at this particular point of time $\psi = 0$ and $\phi = 0$, but naturally $\dot{\psi} \neq 0$, and $\dot{\phi} \neq 0$ (putting $\phi = 0$ has no consequences, but it simplifies the figure.) According to our relations

$$\Omega_x = \dot{\theta}$$
$$\Omega_y = \dot{\phi}\sin\theta$$
$$\Omega_z = \dot{\phi}\sin\theta + \dot{\psi}$$

The most tricky and nontrivial part is to use the non-slipping condition. Naively it is automatically satisfied, as the point A is the origin then if we compute its velocity it will be zero. This is very misleading.

Lets compute the velocity of a point r (see figure) on the rim. This point is given by the vector $R\hat{y} + \vec{r}$ — see figure. The velocity is

$$\vec{v}_r = \vec{\Omega} \times (R\hat{y} + \vec{r})$$

The vector $\vec{r} = -R\cos\beta\hat{y} + R\sin\beta\hat{x}$, so we have

$$\begin{aligned}\vec{v}_r &= (\Omega_x\hat{x} + \Omega_y\hat{y} + \Omega_z\hat{z}) \times (R(1-\cos\beta)\hat{y} + R\sin\beta\hat{x}) \\ &= R\left[\dot{\theta}(1-\cos\beta) - \dot{\phi}\sin\theta\sin\beta\right]\hat{z} - R(1-\cos\beta)(\dot{\phi}\cos\theta + \dot{\psi})\hat{x} + R\sin\beta(\dot{\phi}\cos\theta + \dot{\psi})\hat{y}\end{aligned}$$

Now I want to project this velocity on the external XYZ axes, by simply $v_X = \hat{X} \cdot \vec{v}_r$ etc. The result is

$$\begin{aligned} v_Z &= R\dot{\theta}(1-\cos\beta)\cos\theta + R\dot{\psi}\sin\beta\sin\theta \\ v_X &= R(1-\cos\beta)(\dot{\phi}\cos\theta + \dot{\psi}) \\ v_Y &= -R\dot{\theta}(1-\cos\beta) + R\sin\beta\left[\dot{\phi} + \dot{\psi}\cos\theta\right] \end{aligned}$$

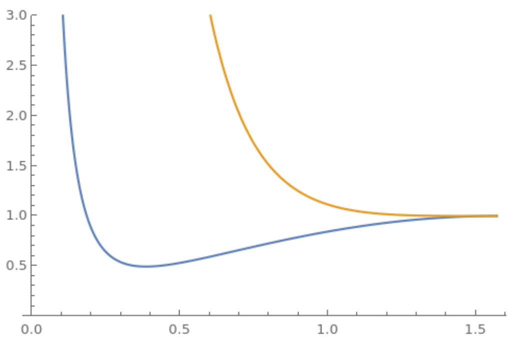


Figure 3

Let's consider very small β. No slipping means that the coin does not scraping on the plane. It means that the projection of the velocity of a point on a rim which is very close to A on the plane must not have a component of the order of β at small β. This requirement gives

$$\dot{\phi} + \dot{\psi}\cos\theta = 0$$

Notice, that it is the same relation that we had before! Using $\dot{\psi} = -\frac{\dot{\phi}}{\cos\theta}$ we find

$$\begin{aligned} \Omega_x &= \dot{\theta} \\ \Omega_y &= \dot{\phi}\sin\theta \\ \Omega_z &= -\dot{\phi}\frac{\sin^2\theta}{\cos\theta} \end{aligned}$$

- Notice, that $\frac{-\Omega_z}{\Omega_y} = \tan\theta$. So the vector $\vec{\Omega}$ is lying in the table's plane.

Now we can write the kinetic energy. And as the potential energy is simply $mgR\sin\theta$ we get

$$L = \frac{1}{2}\left[\dot{\phi}^2\left((I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}\right)\sin^2\theta + \dot{\theta}^2\left(\frac{I}{2} + mR^2\right)\right] - mgR\sin\theta.$$

This Lagrangian does not depend on ϕ! It also conserves energy. So we have two conserved quantity

$$\begin{aligned} M &= \frac{\partial L}{\partial\dot{\phi}} = \dot{\phi}\left((I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}\right)\sin^2\theta \\ E &= \frac{1}{2}\left[\dot{\phi}^2\left((I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}\right) + \dot{\theta}^2\left(\frac{I}{2} + mR^2\right)\right] + mgR\sin\theta. \end{aligned}$$

Expressing $\dot{\phi}$ from the first equation

$$\dot{\phi} = \frac{1}{\sin^2\theta}\frac{M}{(I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}}$$

and substituting it to the second, we find

$$E = \frac{1}{2}\dot{\theta}^2\left(\frac{I}{2} + mR^2\right) + \frac{1}{2}\frac{1}{\sin^2\theta}\frac{M^2}{(I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}} + mgR\sin\theta.$$

This is just $1D$ motion in the effective potential

$$U_{eff}(\theta) = \frac{1}{2}\frac{1}{\sin^2\theta}\frac{M^2}{(I+mR^2)\frac{\sin^2\theta}{\cos^2\theta} + \frac{I}{2}} + mgR\sin\theta$$

One can simplify it a bit if one uses $I = mR^2/2$. In this case

$$U_{eff}(\theta) = \frac{2M^2}{mR^2}\frac{\cos^2\theta}{\sin^2\theta}\frac{1}{1+5\sin^2\theta} + mgR\sin\theta$$

A plot of this function for small M (blue for $2M^2/m^2gR^3 = 3$) and for large M (orange for $2M^2/m^2gR^3 = 1/30$) is shown on figure 3. One can clearly see, that for large initial M the coin will straighten up, and for small it will wobble (provided the energy in the right range.)

If we solve for $\theta(t)$ we will know $\vec{\Omega}$, hence, as the velocity of the point A is zero we will know the velocity of the center of mass. Knowing this velocity we will be able to find the coin's trajectory as it rolls on the plane.

LECTURE 21
Motion of a rigid body. Euler equations. Stability of asymmetric top.

21.1. Euler equations.

A few lectures before (see section 18.3) we considered free motion of a symmetric top. Due to the additional symmetry this motion can be exactly solved. In this section we will derive the equation of motion for a free asymmetric top.

As there no forces acting on the top, we can always think that it rotates around an axis which goes through the center of mass (there is always an external frame of reference where the center of mass does not move.) As there are no torques acting on the top the main equation is simply the angular momentum conservation law.

$$\dot{\vec{M}} = 0,$$

where $\vec{M}$ is the vector of angular momentum. However, one has to be careful with this equation. Namely, one has to be careful with the time differentiation. In this equation it is assumed that the time derivative should be taken in the external/inertial frame of references.

Let's write the vector $\vec{M}$ in the following form

$$\vec{M} = I_x \Omega_x \hat{x} + I_y \Omega_y \hat{y} + I_z \Omega_z \hat{z},$$

where I_x, I_y, and I_z are principal moments of inertial, $\hat{x}$, $\hat{y}$, and $\hat{z}$ are principle axes of inertia, and Ω_x, Ω_y, and Ω_z are the projections of the vector of angular velocity $\vec{\Omega}$ on the principle axes at this particular instant of time.

In the next instant of time the vector $\vec{\Omega}$ itself will change, but also the orientation of the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$ will also change – remember we are observing the motion from the external frame of references. So when taking the time derivative we need to differentiate both the components Ω_x, Ω_y, and Ω_z, and the vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$.

$$\dot{\vec{M}} = I_x \dot{\Omega}_x \hat{x} + I_y \dot{\Omega}_y \hat{y} + I_z \dot{\Omega}_z \hat{z} + I_x \Omega_x \dot{\hat{x}} + I_y \Omega_y \dot{\hat{y}} + I_z \Omega_z \dot{\hat{z}}$$

All three vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ are constant vectors in the internal frame of references – they are the internal frame of references. As for any such vector we can write

$$\dot{\hat{x}} = \vec{\Omega} \times \hat{x}, \qquad \dot{\hat{y}} = \vec{\Omega} \times \hat{y}, \qquad \dot{\hat{z}} = \vec{\Omega} \times \hat{z}.$$

(after all, this is the definition of $\vec{\Omega}$)

So we have:

$$0 = \dot{\vec{M}} = I_x\dot{\Omega}_x\hat{x} + I_y\dot{\Omega}_y\hat{y} + I_z\dot{\Omega}_z\hat{z} + I_x\Omega_x\vec{\Omega}\times\hat{x} + I_y\Omega_y\vec{\Omega}\times\hat{y} + I_z\Omega_z\vec{\Omega}\times\hat{z}.$$

Multiplying the above equation by $\hat{x}$, will find

$$0 = I_x\dot{\Omega}_x + I_y\Omega_y\vec{\Omega}\cdot[\hat{y}\times\hat{x}] + I_z\Omega_z\vec{\Omega}\cdot[\hat{z}\times\hat{x}],$$

or, as $[\hat{y}\times\hat{x}] = -\hat{z}$ and $[\hat{z}\times\hat{x}] = \hat{y}$ – remember the principle axes are orthogonal to each other, we can write

$$0 = I_x\dot{\Omega}_x - I_y\Omega_y\left(\vec{\Omega}\cdot\hat{z}\right) + I_z\Omega_z\left(\vec{\Omega}\cdot\hat{y}\right)$$

finally

$$I_x\dot{\Omega}_x = (I_y - I_z)\Omega_y\Omega_z.$$

Analogously for $\hat{y}$ and $\hat{z}$, and we get the Euler equations:

$$\begin{aligned}
I_x\dot{\Omega}_x &= (I_y - I_z)\Omega_y\Omega_z \\
I_y\dot{\Omega}_y &= (I_z - I_x)\Omega_z\Omega_x \\
I_z\dot{\Omega}_z &= (I_x - I_y)\Omega_x\Omega_y
\end{aligned}$$

(these equation turn into each other under cyclic permutation of x, y, and z indexes.) These three equation are called the Euler equations.

One can immediately see, that the energy is conserved. In order to do that one simply multiplies the first equation by Ω_x, the second equation by Ω_y, and the third equation by Ω_z. Then one sums the equations up. The sum of the right hand sides (after the corresponding multiplication) is zero. So we have

$$I_x\dot{\Omega}_x\Omega_x + I_y\dot{\Omega}_y\Omega_y + I_z\dot{\Omega}_z\Omega_z = 0,$$

which is the same as

$$\frac{d}{dt}\left(\frac{I_x\Omega_x^2}{2} + \frac{I_y\Omega_y^2}{2} + \frac{I_z\Omega_z^2}{2}\right) = 0.$$

So the kinetic energy is constant.

- The Euler equations are three non-linear <u>first order</u> differential equations.
- As such one needs only three initial conditions for the complete solution.
- We know, however, that the complete solution must depend on six initial conditions (three degrees of freedom, each takes initial position and initial velocity). The question is: where do the other three initial conditions go?
- Imaging, that we have solved the Euler equations with some initial conditions for Ω_x, Ω_y, and Ω_z. Then we will know $\Omega_x(t)$, $\Omega_y(t)$, and $\Omega_z(t)$ — the components of the angular velocity (in the internal frame) as functions of time.
- But in order to describe the motion in the external frame we need to know how the (external) coordinates depend on time.
- If we use the Euler angles as the coordinates, then we need to solve three more first order non-linear differential equations:

$$\begin{aligned}
\Omega_x &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\
\Omega_y &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\
\Omega_z &= \dot{\phi}\cos\theta + \dot{\psi}.
\end{aligned}$$

In the left had side are now the known functions. The full solution will require three more initial conditions.

21.1.1. Symmetric top again.

The symmetric top is a particular case of the asymmetric one. For a symmetric top, taking $I_y = I_x$ we find

$$
\begin{aligned}
I_x\dot{\Omega}_x &= -(I_z - I_x)\Omega_y\Omega_z \\
I_x\dot{\Omega}_y &= (I_z - I_x)\Omega_z\Omega_x \\
I_z\dot{\Omega}_z &= 0
\end{aligned}
$$

So we have $\Omega_z = \text{const.}$, then denoting $\omega = \Omega_z \frac{I_z - I_x}{I_x}$ — this is just a constant — we get

$$
\begin{aligned}
\dot{\Omega}_x &= -\omega\Omega_y \\
\dot{\Omega}_y &= \omega\Omega_x
\end{aligned}
$$

The solution is

$$\Omega_x = A\cos\omega t, \qquad \Omega_y = A\sin\omega t.$$

So the vector $\vec{\Omega}$ rotates around the z axis with the frequency ω. Notice, that the vector $\vec{\Omega}$ rotates in the internal frame of reference! So if you are sitting on top of this top you will see the world around you rotating around some axes, but the axis around which the world is rotating will also rotate around the $\hat{z}$ axis (this is the axis in <u>your</u> frame of references) with the frequency ω.

The vector $\vec{M}$ will also rotate with frequency ω — this is the picture in the moving frame of reference.

This is the picture of the rotation as seen from a person on top of the top. Previously, we described the rotation as it is viewed from the outside. Let's check that these to pictures describe the same motion.

Using Euler angles we can write with respect to some external system of coordinates.

$$
\begin{aligned}
\Omega_x &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi = A\cos\omega t \\
\Omega_y &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi = A\sin\omega t \\
\Omega_z &= \dot{\phi}\cos\theta + \dot{\psi} = \Omega_z
\end{aligned}
$$

Multiplying the first equation by $\cos\psi$, the second by $\sin\psi$ and subtracting one from another we get

$$\dot{\theta} = A\cos(\psi + \omega t).$$

This is written in arbitrary external frame of references. Previously, we have used a specific external frame in which the angular momentum was along the $\hat{Z}$ axis. In order to compare the two motions we need to use the same external frame.

It is not obvious how to do that. But we remember, that previously we had θ as a constant. So let's chose such an external frame where $\dot{\theta} = A\cos(\psi + \omega t) = 0$. Such frame does not necessarily exist, so we must check that this guess is consistent with the rest of the full set of equations.

The requirement $\dot{\theta} = 0$ means

$$\psi = \pi/2 - \omega t.$$

The first two equations then give the same relation

$$\dot{\phi}\sin\theta = A$$

and the third one gives

$$\dot{\phi}\cos\theta = \Omega_z + \omega = \Omega_z \frac{I_z}{I_x},$$

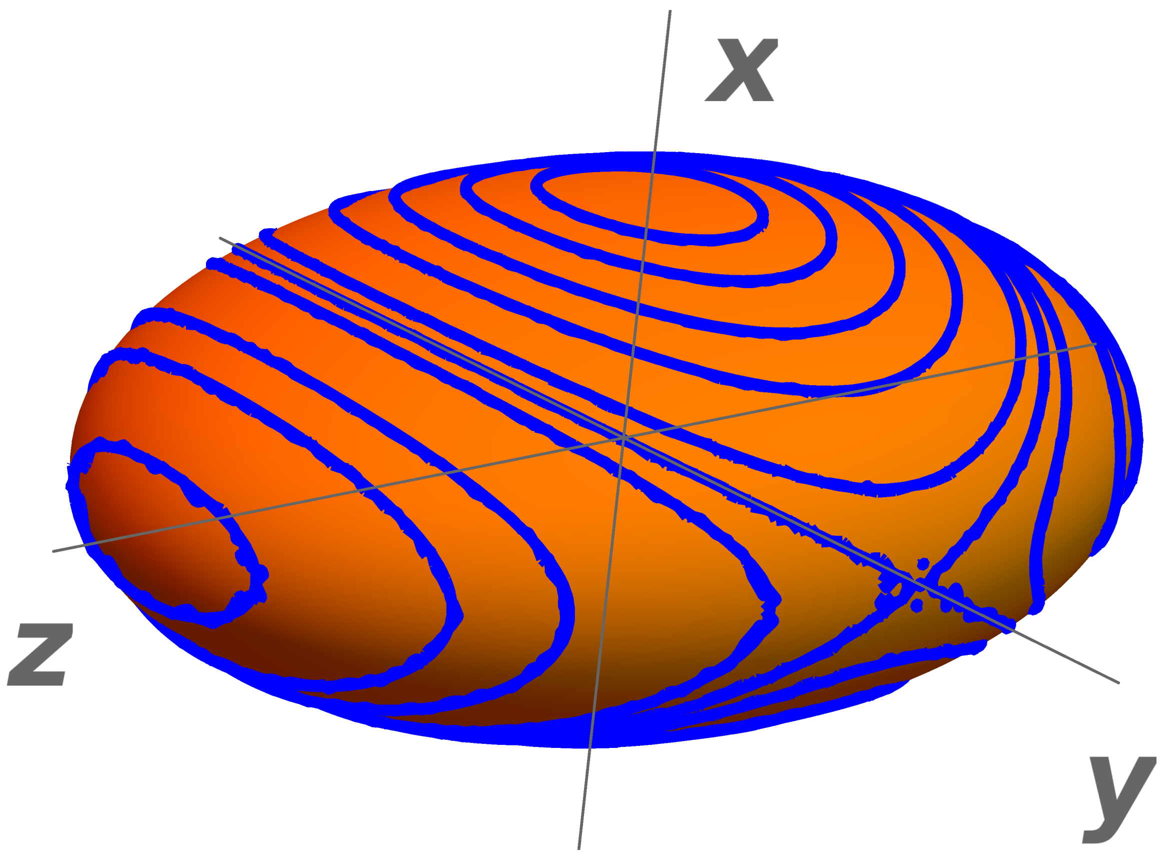
so we see, that from these two equations $\tan\theta = \frac{AI_x}{\Omega_z I_z}$ is indeed a constant, so $\dot{\theta} = 0$ is consistent. Moreover, as $A = \sqrt{\Omega_x^2 + \Omega_y^2} = \Omega_\perp$ we see, that $\tan\theta = \frac{I_x \Omega_\perp}{I_z \Omega_z} = \frac{M_\perp}{M_z}$ as it should be, because $\vec{M}$ is constant.

We also see, that

$$\Omega_{pr} = \dot{\phi} = \frac{\Omega_z I_z}{\cos\theta}\frac{1}{I_x} = \frac{M_z}{\cos\theta}\frac{1}{I_x} = \frac{M}{I_x}$$

21.2. Stability of the free rotation of a asymmetric top.

- Different meaning of stability. Static stability and dynamic stability.



Conservation of energy and the magnitude of the total angular momentum read

$$\frac{I_x\Omega_x^2}{2} + \frac{I_y\Omega_y^2}{2} + \frac{I_z\Omega_z^2}{2} = E$$
$$I_x^2\Omega_x^2 + I_x^2\Omega_x^2 + I_x^2\Omega_x^2 = M^2$$

In terms of the components of the angular momentum these equations read

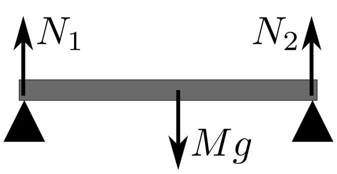
$$\frac{M_x^2}{2I_x} + \frac{M_y^2}{2I_y} + \frac{M_z^2}{2I_z} = E$$
$$M_x^2 + M_y^2 + M_z^2 = M^2$$

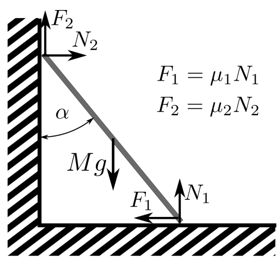
The first equation describes an ellipsoid with the semiaxes $\sqrt{2I_xE}$, $\sqrt{2I_yE}$, and $\sqrt{2I_zE}$. The second equation describes a sphere of a radius M. The initial conditions give us E and M, the true solution must satisfy the conservation lows at all times. So the vector $\vec{M}$ will lie on the lines of intersection of the ellipsoid, and sphere. Notice, how different these lines are.

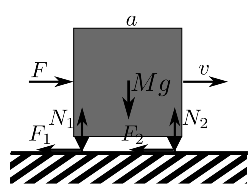
Watch the video on the effect `https://youtu.be/NJLdW4DHRcA`.

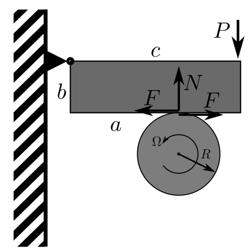
LECTURE 22
Statics.

22.1. Statics.









Static conditions:

- Sum of all forces is zero. $\sum \vec{F}_i = 0$.
- Sum of all torques is zero: $\sum \vec{r}_i \times \vec{F}_i = 0$.

If the sum of all forces is zero, then the torque condition is independent of where the coordinate origin is.

$$\sum(\vec{r}_i + \vec{a}) \times \vec{F}_i = \sum \vec{r}_i \times \vec{F}_i + \vec{a} \times \sum \vec{F}_i$$

Examples

- A bar on two supports. The force equation

$$-Mg + N_1 + N_2 = 0.$$

The torque equation (with respect to the support 1)

$$Mgl/2 - N_2 l = 0.$$

So the solution is $N_1 = N_2 = Mg/2$.

If we write the torque equation with respect to the support 2. We get

$$-Mgl/2 + N_1 l = 0$$

Together with the force equation it gives the same result $N_1 = N_2 = Mg/2$ Notice, that the we get the third equation by simply multiplying the first equation by l and adding to it the second equation.

- A uniform ladder of a length l in a corner.

$$\begin{aligned} \text{Force equation, } x\text{-component:} &\qquad -F_1 + N_2 = 0 \\ \text{Force equation, } y\text{-component:} &\qquad -Mg + F_2 + N_1 = 0 \\ \text{Torque equation with respect to point 1:} &\qquad Mg\frac{l}{2}\sin\alpha - N_2 l\cos\alpha + F_2 l\sin\alpha = 0 \\ \text{Friction force:} &\qquad F_2 = \mu_2 N_2, \quad F_1 = \mu_1 N_1 \end{aligned}$$

We have five equations and only 4 unknowns. The system is over-determined! It is no surprise, as in reality the friction force conditions are inequalities. We then can pose the following question: At which angle α_c the ladder starts to slide? As the ladder is sliding, the friction forces must be at maximum and we can us $F_2 = \mu_2 N_2$ and $F_1 = \mu_1 N_1$. But then we have one more unknown – α_c. So we have the set of equations

$$\begin{aligned} &-F_1 + N_2 = 0 \\ &-Mg + F_2 + N_1 = 0 \\ &Mg\frac{l}{2}\sin\alpha_c - N_2 l\cos\alpha_c + F_2 l\sin\alpha_c = 0 \\ &F_2 = \mu_2 N_2 \\ &F_1 = \mu_1 N_1 \end{aligned}$$

This is just a set of linear equations. The solution is

$$\begin{aligned} N_2 &= Mg\frac{\mu_1}{1+\mu_1\mu_2} \\ N_1 &= \frac{Mg}{1+\mu_1\mu_2} \\ \tan\alpha_c &= \frac{2\mu_1}{1+3\mu_1\mu_2} \end{aligned}$$

  – Again, one can chose any other point to write the torque equation.

- A block with two legs moving with a constant velocity on the floor with μ_1 and μ_2 coefficients of friction.

  As the velocity is constant everything works exactly the same

$$\begin{aligned} \text{Force equation, } x\text{-component:} &\qquad F - F_1 - F_2 = 0 \\ \text{Force equation, } y\text{-component:} &\qquad -Mg + N_2 + N_1 = 0 \\ \text{Torque equation with respect to point 1:} &\qquad F\frac{a}{2} + Mg\frac{a}{2} - N_2 a = 0 \\ \text{Friction force:} &\qquad F_2 = \mu_2 N_2, \quad F_1 = \mu_1 N_1 \end{aligned}$$

we have 5 equations and 5 unknowns. Solving the equations we find

$$F = \frac{\mu_1 + \mu_2}{2 + \mu_1 - \mu_2} Mg$$

- A brake. We neglect the block's weight. Consider the equilibrium of the block. I do not know the forces on the hinge (I can compute them, but I will not need to) I will write the torque equation with respect to the hinge

$$aN = Fb + Pc.$$

Also $F = \mu N$, so we have

$$N = \frac{c}{a - \mu b} P, \qquad F = \frac{c}{a - \mu b} \mu P$$

Notice, that

- at some ratio of a and b the force changes sign. It is obviously impossible.
- Notice, that right before that the force N diverges. So does the force F. So the wheel cannot rotate.
- Notice, that this divergence happens at any non-zero P no matter how small. However, for $P = 0$ the result is obviously $N = F = 0$.
- There is a huge difference between $P = 0$ and $P \to 0$ cases.

A problem for students in class:

- A bar on three supports.

22.2. Elastic deformations.

- Continuous media. Scales.
- Small, only linear terms.
- No nonelastic effects.
- Static.
- Isothermal.

Definition of derivatives.

LECTURE 23
Strain and Stress.

When we deform an object there are a bunch of internal forces that appears inside the object. We need to write the relation/equation between the deformation and these forces. In order to do that we need to first figure out how to describe the deformation and the internal forces.

The deformation and the internal forces are described by the strain and the stress tensors respectively.

In this lecture we define both tensors and discuss their meaning.

23.1. Einstein notations

First we start with some preliminary mathematics.

- Einstein notations. Divergence etc. difference between u_{ii}^2 and u_{ij}^2.
- We only work in Euclidean space, so there is no reason to distinguish co- and contravariant indexes.

Example.

Consider an arbitrary 3×3 tensor u_{ij} (this is just 3×3 matrix). It can be written as

$$u_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right) + \frac{1}{2}\left(u_{ij} - u_{ji}\right) = u_{ij}^s + u_{ij}^a$$

The first term is a symmetric tensor, the second term is antisymmetric tensor. Any antisymmetric 3×3 tensor in $3D$ can be written as

$$u_{ij}^a = \epsilon^{ijk} l_k$$

For some vector $\vec{l}$. It is easy to prove simply by multiplying the above equation by ϵ^{ijn} (remember! I use Einstein notation. This multiplication also involves summation over repeated indexes.)

$$\epsilon^{ijn} u_{ij}^a = \epsilon^{ijn} \epsilon^{ijk} l_k = 2\delta_{nk} l_k = 2l_n.$$

So any 3×3 tensor can be expressed as

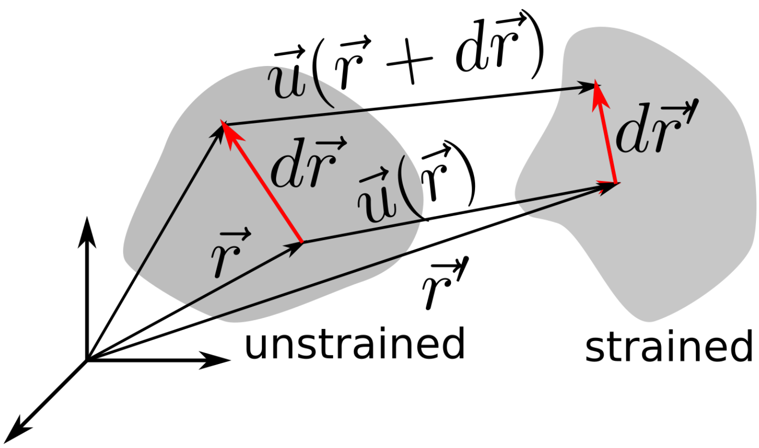
$$u_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right) + \epsilon^{ijk} l_k.$$

A side note:

- This construction works only in $3D$. Only in $3D$ the number of independent elements of a 3×3 antisymmetric tensor is 3 – exactly the number of components of a vector.

- In $4D$ the number of independent components of a 4×4 antisymmetric tensor is 6. So if we want to present it using $3D$ vectors, we would need two such vectors. In electrodynamics for the electromagnetic tensor (which is antisymmetric) these are vectors of electric and magnetic fields.

23.2. Strain



We need do find a way to describe strain in a body. In equilibrium there is a certain distance between different points of a body. Strain appears when this distance changes. In particular, <u>no strain</u> appears if

- we uniformly translate a body;
- we uniformly rotate a body.

We can describe the change of the position of a point of a body by a vector field $\vec{u}(\vec{r})$. However, this vector field in general will also describe the parallel translations and rotations of the body. We need to find a way to exclude these contributions.

Let the unstrained lattice be given positions x_i (I use x_i as coordinates of a vector $\vec{r}$ in some external system of coordinates) and the strained lattice be given positions $x'_i = x_i + u_i$. It is tempting then to write

$$dx'_i = dx_i + \frac{\partial u_i}{\partial x_j} dx_j$$

and use the tensor $\tilde{u}_{ij} = \frac{\partial u_i}{\partial x_j}$ as a measure of strain. It will sure exclude the parallel translation, as parallel translation means that $\vec{u}$ is the same for all points. However, we can write the above as

$$dx'_i = dx_i + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_j + \epsilon^{ijk} dx_j l_k$$

Now we see, that the last part is just a rotation $[\vec{r} \times \vec{l}]_i$ around vector $-\vec{l}$, so it does not describe the deformation. As such it must be excluded.

Let's then do it in a more systematic way. The distance dl between two points in the unstrained lattice is given by $dl^2 = dx_i^2$. The distance dl'^2 between two points in the strained lattice is given by

$$\begin{aligned} dl'^2 = dx_i'^2 = (dx_i + du_i)^2 &= dx_i^2 + 2dx_i du_i + du_i^2 \\ &= dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k} dx_j dx_k \\ &= dl^2 + 2u_{ik} dx_i dx_k, \end{aligned} \tag{23.1}$$

where

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right) \tag{23.2}$$

Notice,

- The tensor u_{ij} gives the change of distance between two nearby points – so it does describe the deformation.
- We specifically defined the tensor u_{ij} to be symmetric, so the rotations are excluded (the tensor has no antisymmetric part).
- The tensor u_{ij} may be different in every point of the body. It then defines a tensor field $u_{ij}(\vec{r})$.

Normally we will take only the case of small strains and consider only linear approximation. Small strains do not mean, that $\vec{u}$ is small, but it does mean that the derivatives $\partial u_i/\partial x_j$ are small. Then we can use

$$u_{ik} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \tag{23.3}$$

In the *linear* approximation we can also write

$$dx_i' = (\delta_{ij} + u_{ij})\, dx_j.$$

This can be checked by using the above to compute dl'^2 and see, that the result coincides with (23.1) in the *linear* approximation.

We can locally diagonalize the real symmetric u_{ik}, and get orthogonal basis set. In that local frame $(1, 2, 3)$ have $dx_1' = dx_1(1 + u_{11})$, etc. Hence the new volume is given by

$$\begin{aligned} dV' = dx_1' dx_2' dx_3' &\approx dx_1 dx_2 dx_3 (1 + u_{11} + u_{22} + u_{33}) \\ &= dV(1 + u_{ii}), \end{aligned} \tag{23.4}$$

where u_{ii} is the trace of the tensor. From the linear algebra we know, that the trace is invariant to the coordinate system used. Hence the fractional change in the volume is given by

$$\frac{\delta(dV)}{dV} = u_{ii}. \tag{23.5}$$

We also see, that in the *linear* approximation

$$du_i = dx_i' - dx_i = u_{ij} dx_j,$$

or

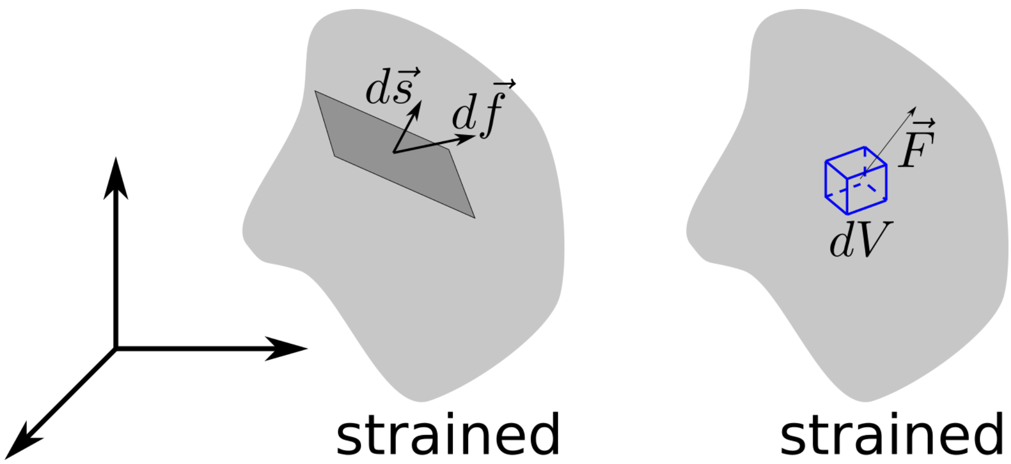
$$u_i(\vec{r}) = \int_{\vec{r}_0,\Gamma}^{\vec{r}} u_{ij} dx_j.$$

where $\vec{r}_0$ is some point which we think as not moving under the deformation (Such point can always be found, as we can always add a parallel translation and rotation to any deformation) and Γ is an arbitrary path from the point $\vec{r}_0$ to the point $\vec{r}$.

- If one using this formula to find the shape of the strained body, one has to remember that this will give the shape up to a uniform rotation and uniform translation.

Let me explain why we need a tensor to describe strain. The strain describes the shift of the relative position of two nearby points. The shift of the relative position is a vector, but the nearby points are also described by a vector. So the strain must connect two vectors — it, thus, must be a second rank tensor.

23.3. Stress



If we take a body and split it on very many small (infinitesimal) volumes. These volumes will interact with each other, exerting forces on each other. These forces are described by another 3×3 tensor called stress tensor.

- The forces are considered to be short range.

Let me explain why we need a tensor to describe these forces. The idea is to consider an infinitesimal square inside a body and think about the force $d\vec{f}$ (infinitesimal) which acts on it <u>from one side</u>. The force is a vector. But this vector depends on the orientation (and area) of the square, which is given by another vector $d\vec{s}$. So the stress connects two vectors – it, thus, must be a second rank tensor.

Consider a volume V which is embedded in a large deformed media.

- We are considering one particular point of time. At this moment we do not assume, that the body is at equilibrium.

There are a lot of forces that act on this volume from the media due to internal stresses of the media. These forces are acting on the surface of the volume V.Let's use $\mathcal{F}_i$ to denote the <u>total</u> force which acts on the volume V. Then we have:

$$\mathcal{F}_i = \int_V \frac{d\mathcal{F}_i}{dV} dV = \int_V F_i dV. \tag{23.6}$$

The meaning of the formula is that the total force $\mathcal{F}_i$ which acts on the volume V is the sum of the forces which act on all small pieces dV of the volume V.

However, because the forces are short-range it should also be possible to write the total force as sum of all forces acting on the boundary of the volume V, or as an integral over the surface elements $dS_i = n_i dS$ of the surface ∂V of the volume V, where $\hat{n}$ is the outward normal (L&L use df_i for the surface element). Thus we expect that

$$\mathcal{F}_i = \int_{\partial V} \sigma_{ij} dS_j \tag{23.7}$$

for some σ_{ij}. (The notation ∂V just denotes the surface of a volume V.) Thinking of it as a set of three vectors (labeled by i) with vector index j, we can apply Gauss's Theorem to rewrite this as

$$\mathcal{F}_i = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV, \tag{23.8}$$

so comparison of the two volume integrals gives

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{23.9}$$

What we found is the following. If we know the stress tensor $\sigma_{ij}(\vec{r})$ — it is stress tensor field — then

- The force acting on a small volume dV at position $\vec{r}$ is given by

$$df_i = \frac{\partial \sigma_{ij}}{\partial x_j} dV.$$

- The force which acts on a infinitesimal surface $d\vec{s}$ from one side is given by

$$df_i = \sigma_{ij} ds_j.$$

In equilibrium when only the internal stresses act the total force acting on any piece dV must be zero, so the equilibrium condition is

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0.$$

If there is a long-range force, such as gravity acting, with force $F_i^g = \rho g_i$, where ρ is the mass density and g_i is the gravitational field, then in equilibrium $F_i + F_i^g = 0$. This latter case is important for objects with relatively small elastic constant per unit mass, because then they must distort significantly in order to support their weight.

When no surface force is applied, the stress at the surface is zero. When there is a surface force P_i per unit area, this determines the stress force $\sigma_{ij}\hat{n}_j$, so

$$P_i = \sigma_{ij}\hat{n}_j \tag{23.10}$$

If the surface force is a pressure, then $P_i = -P\hat{n}_i = \sigma_{ij}\hat{n}_j$. The only way this can be true for any $\hat{n}$ is if

$$\sigma_{ij} = -P\delta_{ij}. \tag{23.11}$$

Just as the force due to the internal stresses should be written as a surface integral, so should the torque. Each of the three torques is an antisymmetric tensor, so we consider

$$\begin{aligned} M_{ik} &= \int (F_i x_k - F_k x_i) dV = \int \left(\frac{\partial \sigma_{ij}}{\partial x_j} x_k - \frac{\partial \sigma_{kj}}{\partial x_j} x_i \right) dV \\ &= \int \left(\frac{\partial(\sigma_{ij} x_k)}{\partial x_j} - \frac{\partial(\sigma_{kj} x_i)}{\partial x_j} - (\sigma_{ik} - \sigma_{ki}) \right) dV \\ &= \int (\sigma_{ij} x_k - \sigma_{kj} x_i) dS_j - \int (\sigma_{ik} - \sigma_{ki}) dV. \end{aligned} \tag{23.12}$$

To eliminate the volume term we require that

$$\sigma_{ik} = \sigma_{ki}. \tag{23.13}$$

LECTURE 24
Work, Stress, and Strain.

In the last lecture we introduced strain and stress tensors.

If a deformation of a body is described by a vector field $\vec{u}(\vec{r})$ – a point $\vec{r}$ of an unstrained body is shifted by vector $\vec{u}$ – then, in the *linear* approximation the strain tensor is given by

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In the *linear* approximation, it has the following properties

(a) $u_{ij} = u_{ji}$ – this is correct for non-linear also.
(b) $\frac{d\delta V}{dV} = u_{ii}$, where $u_{ii} = \mathrm{tr}\hat{u}$.
(c) $u_i(\vec{r}) = \int_{\vec{r}_0,\Gamma}^{\vec{r}} u_{ij} dx_j$.

The stress tensor σ_{ij} describes the internal elastic forces.

(a) $\sigma_{ij} = \sigma_{ji}$.
(b) The force on the <u>internal volume</u> element dV is $df_i = \frac{\partial \sigma_{ij}}{\partial x_j} dV$.
(c) The force on the <u>surface</u> element $d\vec{s}$ is $df_i = \sigma_{ij} ds_j$.

So

- if we know strain tensor u_{ij} we know the deformation u_i (in the linear approximation),

$$u_i(\vec{r}) = \int_{\vec{r}_0,\Gamma}^{\vec{r}} u_{ij} dx_j.$$

  and if we know the deformation u_i we know the strain tensor.

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

- If we know stress tensor σ_{ij} we know all the forces on any piece of volume dV:

$$df_i = \frac{\partial \sigma_{ij}}{\partial x_j} dV$$

  or any piece of surface (from one side) $d\vec{s}$:

$$df_i = \sigma_{ij} ds_j.$$

  In particular, we know all the forces on the surface of our object.

On the other hand, if we know all the forces on the surface of an object, and the object is at equilibrium, then we need to solve the equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \qquad \sigma_{ij} = \sigma_{ji},$$

with the boundary condition that $\sigma_{ij} ds_j$ gives the know forces on the surface of the object. Then by knowing the forces we will know the stress tensor σ_{ij}.

What we now need to complete the problem is the connection between the strain u_{ij} and stress σ_{ij} tensors. If we know this connection, then by knowing deformation u_i of the object, we will know what forces on the surface of the object which are required for such a deformation. Also if we know what forces are acting on the surface we will know the deformation of the object.

Different materials deform differently under the same forces, so the connection between the two tensors is material dependent.

This lecture we want to find a connection between the stress and the strain tensors. We are working in a linear and local approximation, so the connection must be linear and have a form

$$u^{ij} = D^{ijkl} \sigma^{kl},$$

where D^{ijkl} is a material dependent local tensor of the fourth rank.

- An arbitrary fourth rank tensor in $3D$ has 81 independent elements.
- In our case the tensor D^{ijkl} must be symmetric with respect to exchange $i \leftrightarrow j$ and $k \leftrightarrow l$. This reduces the number of independent elements to 36.
- However, any crystal has addition symmetries which will reduce the number of elements even further.
- It is known how to write any tensor which respects any symmetry.
- The higher the symmetry, the more restrictions we have, the simplest the tensor will be.
- The simplest tensor appears when the symmetry is the highest — isotropic material.
- Crystals are not isotropic, but a lot of materials (amorphous, mulicrystals etc.) are isotropic.

In addition, the requirement of stability of the equilibrium (when $u^{ij} = 0$) must lead to some inequalities between the elements of the tensor D. As it is a tensor of the fourth rank it is very tedious to analyze. So instead we will work around it.

24.1. Work against Internal Stresses

Let's imagine an experiment when we want to slightly change the field $\vec{u}(\vec{r})$ while keeping the shape of the object intact. In order to do that we have to do work against the internal stresses. The force we need to apply to a piece of volume dV is $-F_i dV$ (where F_i is the force due to the internal stresses) So we need to do the work $\delta R_{our} = -F_i \delta u_i dV$. The internal forces then do the work $\delta R = F_i \delta u_i dV$. Hence the total work done by the internal stresses is given by

$$\delta W = \int \delta R dV = \int F_i \delta u_i dV = \int \frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i dV$$

$$= \int \frac{\partial(\sigma_{ij}\delta u_i)}{\partial x_j} dV - \int \sigma_{ij} \frac{\partial(\delta u_i)}{\partial x_j} dV. \tag{24.1}$$

If we transform the first integral to a surface integral, by Gauss's Theorem, and take $\delta u_i = 0$ on the surface — we fix the boundary, we do not change the shape of the object, then we eliminate the first term. If we use the symmetry of σ_{ik} and the small-amplitude form of the strain (*linear* approximation), then the last term can be rewritten so that we deduce that

$$\delta R = -\sigma_{ik}\delta u_{ki}. \tag{24.2}$$

24.1.1. Thermodynamics

We now assume the system to be in thermodynamic equilibrium. Using the energy density $d\epsilon$ and the entropy density s, the first law of thermodynamics gives

$$d\epsilon = T ds - dR = T ds + \sigma_{ik} du_{ki}. \tag{24.3}$$

Defining the free energy density $F_F = \epsilon - Ts$ we have

$$dF_F = -s dT + \sigma_{ik} du_{ki}. \tag{24.4}$$

In the next section we consider the form of the free energy density as a function of T and u_{ik}: $F_F(T, u_{ij})$. Then we will use

$$\sigma_{ij} = \frac{\partial F_F}{\partial u_{ij}}$$

From the form of dF_F we see that this derivative must be taken at fixed temperature $dT = 0$ — this is exactly what we need, as we consider only isothermal processes

24.2. Elastic Energy

The elastic equations must be linear, as this is the accuracy which we work with. The free energy density then must be quadratic in the strain tensor. We thus need to construct a scalar out of the strain tensor in the second order. If we assume that the body is isotropic, then the only way to do that is:

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ik}^2. \tag{24.5}$$

- Notice, that in this approach instead of working with the tensor of the fourth rank we are working with a scalar – free energy.

Here λ and μ are the only parameters (in the isotropic case). These parameters are different for different materials.

- The elastic properties (in linear order) of any isotropic material are characterized by only two (temperature dependent) parameters!

These parameters λ and μ are called *Lamé coefficients,* and in particular μ is called the *shear modulus* or *modulus of rigidity.* Note that u_{ii} is associated with a volume change, by (23.5).

- The two terms in (24.5) are not independent, so in order to take the derivative, we must rewrite it as independent terms.

The quantity

$$\tilde{u}_{ik} = u_{ik} - \frac{1}{3}\delta_{ik}u_{jj} \tag{24.6}$$

satisfies $\tilde{u}_{ii} = 0$, and is said to describe a pure shear.

With this definition we have

$$u_{ik} = \tilde{u}_{ik} + \frac{1}{3}\delta_{ik}u_{jj} \tag{24.7}$$

$$u_{ik}^2 = \tilde{u}_{ik}^2 + \frac{2}{3}\tilde{u}_{ii}u_{kk} + \frac{1}{3}u_{jj}^2 = \tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2. \tag{24.8}$$

Hence (24.5) becomes

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu(\tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2) = F_0 + \frac{1}{2}Ku_{ii}^2 + \mu\tilde{u}_{ik}^2. \qquad (K \equiv \lambda + \frac{2}{3}\mu) \tag{24.9}$$

In this form the two elastic terms are independent of one another.

- For the elastic energy to correspond to a stable system, each of them must be positive, so $K > 0$ and $\mu > 0$.

24.2.1. Stress

Now we have the free energy as a function of *independent* variables u_{ii} and $\tilde{u}_{ij}$. So we can take the variation of the Free energy with respect to these independent variables.

On varying u_{ik} at fixed T the free energy of (24.9) changes by

$$\begin{aligned} dF =& Ku_{jj}du_{kk} + 2\mu\tilde{u}_{ik}d\tilde{u}_{ik} = Ku_{jj}du_{kk} + 2\mu\tilde{u}_{ik}(du_{ik} - \frac{1}{3}\delta_{ik}du_{jj}) \\ =& Ku_{jj}du_{kk} + 2\mu\tilde{u}_{ik}du_{ik} = Ku_{jj}\delta_{ik}du_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}\right)du_{ik} \\ =& \left[Ku_{jj}\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}\right)\right]du_{ik}, \end{aligned} \tag{24.10}$$

so comparison with (24.4) gives

$$\sigma_{ik} = Ku_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}). \tag{24.11}$$

Note that $\sigma_{jj} = 3Ku_{jj}$, so that

$$u_{jj} = \frac{\sigma_{jj}}{3K}.$$

We now use this equation for u_{jj} in (24.11), and then solve it for u_{ik}:

$$u_{ik} = \delta_{ik}\frac{\sigma_{jj}}{9K} + \frac{1}{2\mu}\left(\sigma_{ik} - \frac{1}{3}\sigma_{jj}\delta_{ik}\right). \tag{24.12}$$

In the above the first term has a finite trace and the second term has zero trace.

This is the desired result:

- knowing the stress σ_{ij} we can find the strain u_{ij} by (24.12).
- knowing the strain u_{ij} we can find the stress σ_{ij} by (24.11).

All *isotropic* materials in the linear approximation are described by just two constants.

LECTURE 25
Elastic Moduli.

Results of last lecture:The elastic properties of an isotropic material in the linear approximation are characterized by two constants K and μ. They are called elastic moduli. Both of these are positive: $\mu > 0$ and $K > 0$. μ is called shear modulus or modulus of rigidity, K is inverse (isothermal) comressibility. These moduli can be expressed through Lamé coefficients λ and μ: $K = \lambda + \frac{2}{3}\mu$, and μ is the same. The elastic moduli allows one to connect stress and strain tensors.

$$\sigma_{ik} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}). \tag{25.1}$$

$$u_{ik} = \delta_{ik}\frac{\sigma_{jj}}{9K} + \frac{1}{2\mu}\left(\sigma_{ik} - \frac{1}{3}\sigma_{jj}\delta_{ik}\right), \tag{25.2}$$

where $K > 0$ and $\mu > 0$.

Taking the trace of either equation we get

$$u_{jj} = \frac{\sigma_{jj}}{3K}. \tag{25.3}$$

This lecture is about the physical meaning of the elastic constants.

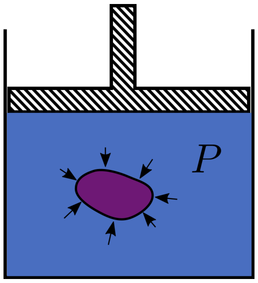
These two elastic moduli can be expressed through Young's moduli: E and σ. E is called Young's modulus, or the modulus of extension. σ is called Poisson's ratio.

The equation of equilibrium is $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$. Generally, we need to find a solution of this equation (it is really three equations, as there is a free index i) which satisfies the boundary conditions. It is a complicated problem in general. However, in some simple cases we can use our intuition to guess the solution, and then check it.

25.1. Bulk Modulus and Young's Modulus

In this part we will guess σ_{ij}, check the guess, and find the corresponding deformation.

25.1.1. Hydrostatic compression.



For hydrostatic compression the force on a small tile is always perpendicular to that tile, so the vector of force and the vector area of the tile have the exactly opposite directions. It means that $\sigma_{ik} = -P\delta_{ik}$, so (25.3) gives

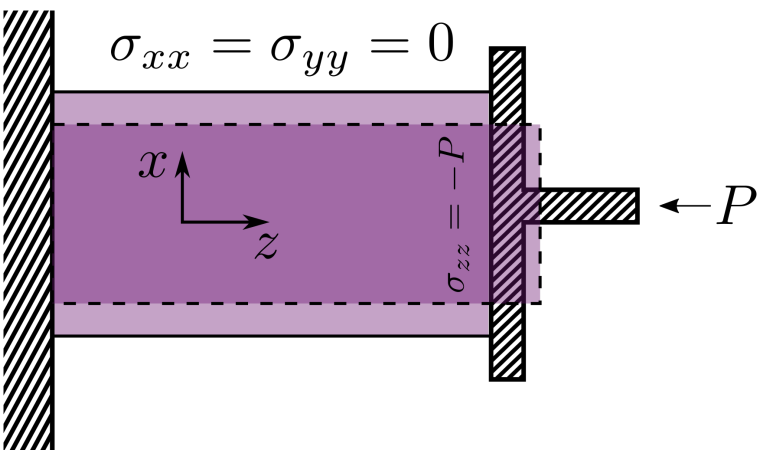
$$u_{jj} = -\frac{P}{K}. \qquad \text{(hydrostatic compression)} \tag{25.4}$$

We can think of this as being a δu_{jj} that gives a $\delta V/V$, by (23.5), due to $P = \delta P$, so

$$\frac{1}{K} = -\frac{\delta u_{jj}}{\delta P} = -\left.\frac{1}{V}\frac{\partial V}{\partial P}\right|_T. \tag{25.5}$$

So K is inverse isothermal compressibility $\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ as defined in thermodynamics.

25.1.2. Uni-direction compression.



Now let there be a compressive force per unit area P along z axis for a system with normal along z, so that $\sigma_{zz} = -P$, but $\sigma_{xx} = \sigma_{yy} = 0$, on the surface of the stick. The stick is in equilibrium, so $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$. The stick is straight and uniform. The obvious solution of the equation is $\sigma_{zz}(\vec{r}) = -P$ and all other components of σ_{ij} are zero everywhere.

We then have $\sigma_{ii} = -P$, and by (25.2) $u_{ik} = 0$ for $i \neq k$, and

$$u_{xx} = u_{yy} = \frac{P}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right), \tag{25.6}$$

$$u_{zz} = -\frac{P}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right) = -\frac{P}{E}, \qquad E \equiv \frac{9K\mu}{3K+\mu}. \tag{25.7}$$

Notice, that for positive pressure (compression) u_{zz} is always negative, as both $K > 0$ and $\mu > 0$, and hence $E > 0$.

The coefficient of P is called the *coefficient of extension.* Its inverse E is called *Young's modulus,* or the *modulus of extension.*

In particular a spring constant can be found by

$$\Delta z = \int_0^L u_{zz} dz = u_{zz} L = -\frac{PL}{E} = -\frac{L}{AE} F, \qquad k = \frac{AE}{L}$$

25.1.3. Poisson's ratio.

For the previous experiment we can define *Poisson's ratio* σ via

$$u_{xx} = -\sigma u_{zz}. \tag{25.8}$$

Then we find that

$$\sigma = -\frac{u_{xx}}{u_{zz}} = \frac{(\frac{1}{2\mu} - \frac{1}{3K})}{(\frac{1}{3K} + \frac{1}{\mu})} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}. \tag{25.9}$$

Since K and μ are positive, the maximum value for σ is $\frac{1}{2}$ and the minimum value is -1. All materials in Nature (except some) have $\sigma > 0$.

Notice, that the volume is changing by $\frac{\delta dV}{dV} = u_{ii} = u_{zz}(1 - 2\sigma)$, so if $\sigma = 1/2$ the volume does not change – incompressible liquid. The requirements that when we compress the volume cannot increase is the requirement that $\sigma < 1/2$.

Often one uses E and σ instead of K and μ. We leave it to the reader to show that

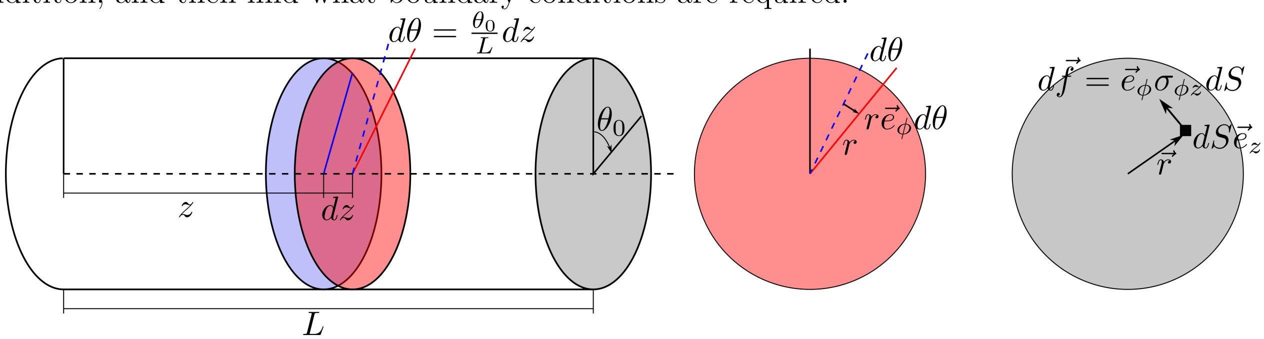
$$\lambda = \frac{E\sigma}{(1 - 2\sigma)(1 + \sigma)}, \tag{25.10}$$

$$\mu = \frac{E}{2(1 + \sigma)}, \tag{25.11}$$

$$K = \frac{E}{3(1 - 2\sigma)}. \tag{25.12}$$

25.2. Twisted rod.

In this part we will guess u_{ij}, find the corresponding σ_{ij}, check that it satisfies the equilibrium condititon, and then find what boundary conditions are required.



Let's take a circular rod of radius a and length L and twist its end by a small angle θ_0. We want to calculate the torque required for that.

- We first guess the right solution.

If z is the coordinate along the axis of the rod, then the cross-section at distance z from the left end (we are twisting the right end) is rotated by the angle $\theta(z) = \frac{z}{L}\theta_0$

Two cross-sections a distance dz from each other are twisted by the angle $\frac{\theta_0}{L}dz$ with respect to each other. So a point at distance r from the center on the cross-section at $z + dz$ is shifted by the vector $d\vec{u} = r\frac{\theta_0}{L}dz\vec{e}_\phi$ in comparison to that point in the cross-section at z. We thus see that the strain tensor is

$$u_{z\phi} = u_{\phi z} = \frac{1}{2}\frac{du_\phi}{dz} = \frac{1}{2}r\frac{\theta_0}{L}$$

and all other elements are zero.

The relation between u_{ij} and σ_{ij} is local, so we can write them in any local system of coordinates. So as the strain tensor is trace-less

$$\sigma_{z\phi} = \sigma_{\phi z} = \mu r\frac{\theta_0}{L}.$$

and all other elements are zeros.

- Notice, that for that stress tensor $\frac{\partial\sigma_{z\phi}}{\partial z} = \frac{\partial\sigma_{z\phi}}{\partial\phi} = 0$, so the condition of equilibrium is satisfied and our guess was right.

Now we calculate the torque on we need to apply to the end. To a small area dS at a point at distance r from the center. The vector of area is $d\vec{S} = dS\vec{e}_z$, it has only z component. As we know, $df_i = \sigma_{ij}dS_j$, so in this case $df_i = \sigma_{iz}dS$. But the only nonzero element of σ_{iz} is $\sigma_{\phi,z}$. So the force will have only ϕ component. So the force we need to apply to the element dS is $d\vec{f} = \sigma_{\phi z}dS\vec{e}_\phi$. The torque of this force with respect to the center is along z direction and is given by $d\vec{\tau} = \vec{r}\times d\vec{f} = r\sigma_{\phi z}dS\vec{e}_r\times\vec{e}_\phi = r\sigma_{\phi z}dS\vec{e}_z$. As all $d\vec{\tau}$ are in the same direction z, we can simply add them up. So the total torque is

$$\tau = \int r\sigma_{\phi z}dS = \int_0^a r\mu r\frac{\theta_0}{L}rdrd\phi = \mu\frac{\theta_0}{L}\int_0^a r^3drd\phi = \frac{\pi}{2}\frac{\mu}{L}a^4\theta_0.$$

So we can measure μ in this experiment by the following way

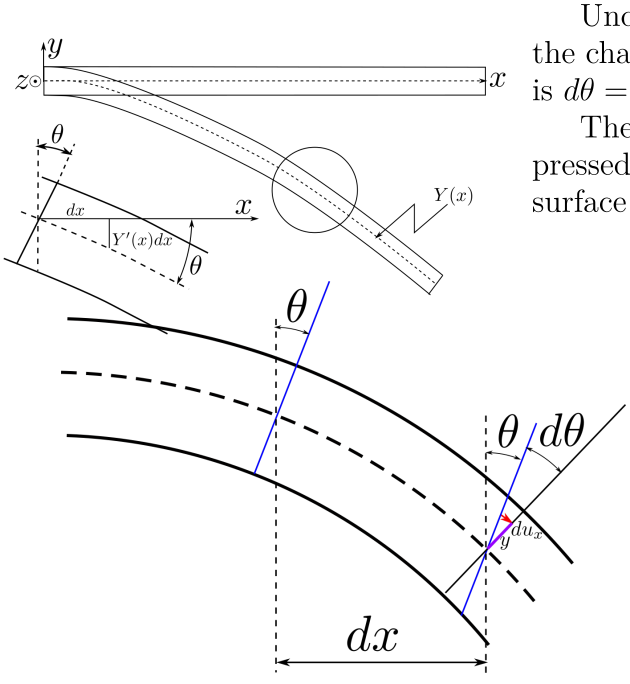
(a) Prepare rods of different radii and lengths.
(b) For each rod measure torque τ as a function of the twist angle θ.
(c) For each rod plot τ as a function of θ. Verify, that for small enough angle τ/θ does not depend on θ and is just a constant. This constant is a slope of each graph at small θ.
(d) Plot this constant as a function of $\frac{\pi a^4}{2L}$. Verify, that the points are on a straight line for small $\frac{\pi a^4}{2L}$. The slope of this line at small $\frac{\pi a^4}{2L}$ is the sheer modulus μ.
(e) One can also measure $\frac{\pi a^4}{2L}$ by measuring the frequency of oscillations of a disk on known moment of inertia hanged on a thread.

LECTURE 26
Small deformation of a beam.

Let's consider a small deformation of a (narrow) beam with rectangular cross-section under gravity.

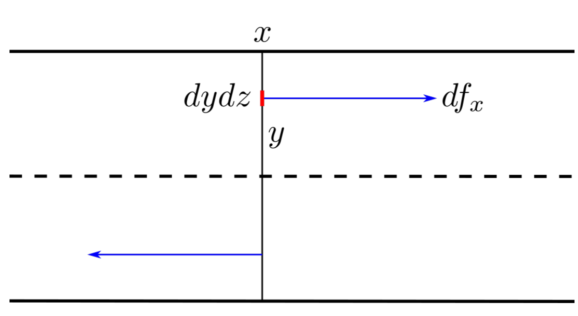
We do not want to find the full deformation of it – this is a difficult problem, and we do not need it. We want to describe an overall "shape" of the beam only. So we want to disregard the changes in cross sections, but we cannot disregard the forces that act in there.

- x coordinate is along undeformed beam, y is perpendicular to it, pointing up.
- Nothing depends on z. $\hat{z}$ points towards us.
- The beam is made of a material with known Young's modulus E.
- Part of the beam is compressed, part is stretched.
- *Neutral surface.* In any cross-section part of it (lower) is squeezed and part of it (upper) is stretched. So there is a point (in $x - y$) which is neither squeezed, nor stretched. This point for the cross-section at x has coordinate Y. So we have a neutral surface $Y(x)$. This function is what we will use to describe the shape of the beam. It is this function $Y(x)$ that we want to find.
- Deformation is small, $|Y'(x)| \ll 1$.



Under these conditions the angle $\theta(x) \approx Y'(x)$. So the change of the angle $\theta(x)$ between two near points is $d\theta = Y''(x)dx$.

The neutral surface is neither stretched, nor compressed. The line which is a distance y from this surface is stretched (compressed) in x direction by



$du_x = yd\theta = yY''dx$, so we have

$$u_{xx} = \frac{\partial u_x}{\partial x} = y\frac{\partial^2 Y(x)}{\partial x^2}.$$

- The stretching (compression) proportional to the second derivative, as the first derivative describes the uniform rotation of the beam.

There is no confining in the y or z directions, so we find that

$$\sigma_{xx} = -Eu_{xx} = -Ey\frac{\partial^2 Y(x)}{\partial x^2}.$$

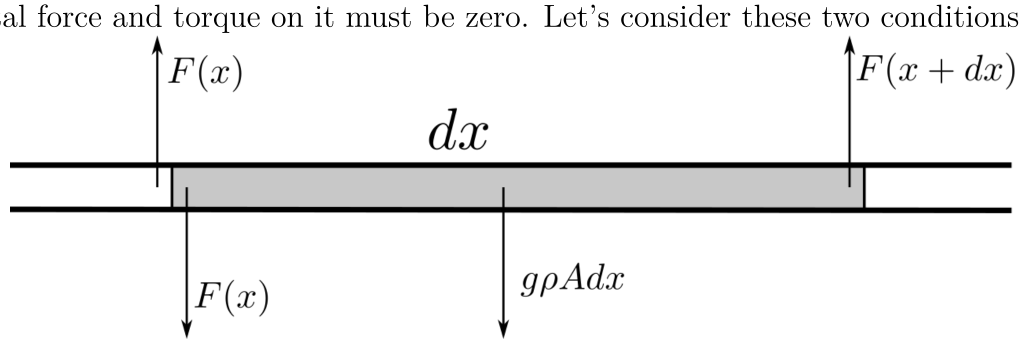
Consider a cross-section of the beam at point x. The force in the x direction of the $dydz$ element of the beam is $df_x = \sigma_{xx}dzdy$. The torque which acts from the **right** part on the **left** is in the negative $\hat{z}$ direction. Its z component is:

$$\tau(x) = \int y\sigma_{xx}dydz = -E\frac{\partial^2 Y(x)}{\partial x^2}\int y^2 dzdy = -IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad I = \frac{\int y^2 dydz}{\int dydz}.$$

(A is the cross-section area.) **This is our first equation.**

- This equation simply states how torque in a cross-section depends on the shape $Y(x)$.
- Notice, that the cross-section area A and the quantity I are just properties of an unstretched beam. They are just some constants that characterize the beam and they do not depend on its bending.

The beam is at equilibrium. So if we take a small portion of it, between x and $x+dx$, the total force and torque on it must be zero. Let's consider these two conditions one by one:

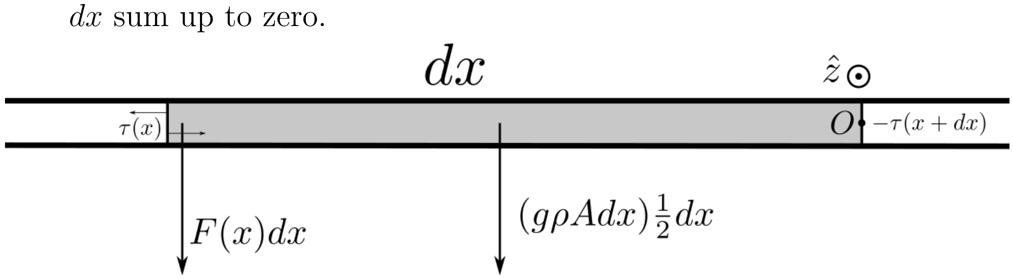


Force: Let's say that the y component of the force on the cross-section at x with which the **right** side is acting on the **left** is $F(x)$.

$$F(x+dx) - F(x) - \rho gAdx = 0, \qquad \frac{\partial F}{\partial x} = \rho gA.$$

(positive direction is up, along $\hat{y}$.) **This is our second equation.**

- This equation encodes the first equilibrium condition that the forces acting on a piece dx sum up to zero.



Torque: The total torque (with respect to the point $x + dx$ — point O in the figure) acting on this portion is

$$-\tau(x+dx) + \tau(x) + F(x)dx + \frac{1}{2}m\rho g A (dx)^2 = 0, \qquad \frac{\partial \tau}{\partial x} = F(x).$$

This is our third equation.

- This equation encodes the second equilibrium condition that the torques acting on a piece dx sum up to zero.

Now we collect all the three equations:

$$\tau(x) = -IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad \frac{\partial F}{\partial x} = \rho g A, \qquad \frac{\partial \tau}{\partial x} = F(x).$$

From these equations we find

$$\frac{\partial^2 \tau}{\partial x^2} = \frac{\partial F}{\partial x} = \rho g A, \qquad IAE\frac{\partial^4 Y(x)}{\partial x^4} = -\rho g A.$$

The general solution of this equation is simply

$$Y(x) = -\frac{\rho g}{24IE}x^4 + \frac{C_3}{6}x^3 + \frac{C_2}{2}x^2 + C_1 x + C_0.$$

$$\tau(x) = -IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad \text{along } -\hat{z} \text{ direction}$$

$$F(x) = -IAE\frac{\partial^3 Y(x)}{\partial x^3}, \qquad \text{along } +\hat{y} \text{ direction} \tag{26.1}$$

Both the force and the torque is from the **right** on the **left** side of a cross-section at x.

- The constants must be found from the boundary conditions.

26.1. A beam with free end. A diving board.

We need to determine four unknown constants. C_0, C_1, C_2, and C_3.

We take $Y = 0$ at $x = 0$ — fixing the position of one end — which gives $C_0 = 0$. Another condition is that at $x = 0$ the board is horizontal – **the end is clamped** ,

$$Y'(x=0) = 0$$

This determines $C_1 = 0$.

At the other end (distance L) both the force and the torque are zero — **it is a free end condition**, so we get the conditions

$$F(x=L) = \left.\frac{\partial^3 Y(x)}{\partial x^3}\right|_{x=L} = 0, \qquad \tau(x=L) = \left.\frac{\partial^2 Y(x)}{\partial x^2}\right|_{x=L} = 0.$$

These two conditions will define $C_3 = \frac{\rho g}{IE}L$ and $C_2 = -\frac{\rho g}{2IE}L^2$.

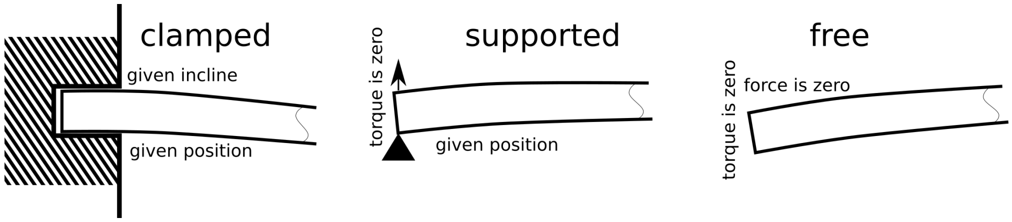
$$Y(x) = -\frac{\rho g}{24IE}x^2\left(x^2 - 4xL + 6L^2\right).$$

In particular,

$$Y(x=L) = -\frac{\rho g}{8IE}L^4.$$

- Notice the proportionality to the fourth power.
- If ρg is larger, then the free end of the beam hangs lower.
- If E is smaller — the beam is less rigid — the free end hangs lower.

26.2. Different models for the boundary conditions.



- Clamped. The position Y, as well as inclination Y' are given on that end. These are two conditions.
- Supported. There is a support at the end. So the position Y at this end is given, but the inclination is not. There is a force at the end – the force that supports the end at the given position, but the torque is zero. Again, it is two conditions at this end: The position Y, and the torque.
- Free. There is nothing at the end. So both force and torque at this end are zero. Again, it is two conditions.

As any beam has two ends, and we have two conditions at each end, we have total four conditions — exactly the number of conditions we need to find the values of four constants C_0, C_1, C_2, and C_3.

LECTURE 27
A rigid beam on three supports.

27.1. Results of the previous lecture.

The shape of the beam is given by

$$Y(x) = -\frac{\rho g}{24IE}x^4 + \frac{C_3}{6}x^3 + \frac{C_2}{2}x^2 + C_1 x + C_0.$$
$$\tau(x) = -IAE\frac{\partial^2 Y(x)}{\partial x^2}, \qquad \text{along } -\hat{z} \text{ direction}$$
$$F(x) = -IAE\frac{\partial^3 Y(x)}{\partial x^3}, \qquad \text{along } +\hat{y} \text{ direction} \tag{27.1}$$

Both the force and the torque is from the **right** on the **left** side of a cross-section at x. The positive direction of the force is up. The positive direction of the torque is toward us.

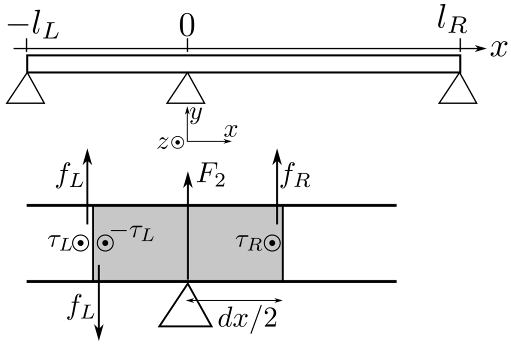
- The constants must be found from the boundary conditions.
- On each of the two ends of a beam we have two conditions (free, supported, or clamped). It is total 4 conditions — exactly as many as we need to determine 4 constants.

27.2. The force on the middle.

Consider an absolutely rigid $E = \infty$ horizontal beam with its ends fixed. Let's see how the force on the central support changes as a function of height h of this support. For $h < 0$ the force is zero. For $h > 0$ the force is infinite and $h \to 0_-$ and $h \to 0_+$ are very different. So the situation is unphysical. It means that the order of limits first $E \to \infty$ and then $h \to 0$ is wrong. We need to take the limits in the opposite order: first take $h = 0$ and then $E \to \infty$. In this order the limits are well defined. So we need to solve the static horizontal beam on three supports for large, but finite E and then take the limit $E \to \infty$ at the very end, when we already know the solution. Luckily we know how to solve this problem for large E!

The beam is of length L. The central support has a coordinate $x = 0$ and is at the distance l_L from the left end and at the distance l_R from the right end ($l_R + l_L = L$).

In the previous lecture we saw, that we need four boundary conditions to define the shape of the beam. Each end of the beam gives two conditions. However, the central support will give another set of conditions. It looks like the problem will be over-determined.



It is not so. There is one place, an infenitesimal piece around the central support, where the previous lecture's calculation fails, as in that calculation we only considered elastic forces and the gravity. Here there is another force — the force from the support — which must be included in the equilibrium condition for that one infinitesimal piece. Outside of this one infinitesimal piece we have only elastic forces and the gravity, so outside of this one piece the calculations of the previous lecture are valid.

The central support exerts a force F_2 on the beam. This force is at a single point.

- It means that there is a jump/discontinuity in the internal elastic forces at $x = 0$.
- However, for everything outside of the infinitesimal piece around the central support we can use the results of the previous lecture. So everything is piece-wise continuous on the left and on the right of $x = 0$. We then use two functions $Y_L(x)$ and $Y_R(x)$ to describe the shape to the left and to the right of the central support.

We then have the shape which is given by

$$Y_L(-l_L < x < 0) \qquad \text{and} \qquad Y_R(0 < x < l_R)$$

As all supports are at the same height we must have $Y_L(x = 0) = Y_L(x = -l_L) = Y_R(x = 0) = Y_R(x = l_R) = 0$, so

$$\begin{aligned} Y_L &= -\tfrac{\rho g}{24IE} x(x + l_L)\left(x^2 + C_1^L x + C_0^L\right) && \text{for } -l_L < x < 0 \\ Y_R &= -\tfrac{\rho g}{24IE} x(x - l_R)\left(x^2 + C_1^R x + C_0^R\right) && \text{for } 0 < x < l_R \end{aligned}$$

The form of these functions may look different from what we have used in the previous lecture. However, the statement of equilibrium which we derived in the previous lecture simply demands, that the function $Y(x)$ is a polynomial of the fourth order with the coefficient in front of x^4 be $-\frac{\rho g}{24IE}$. Both functions above are exactly of this kind with the additional requirement, that all supports are at the same height.

We thus have 4 unknown constants. We need four boundary conditions. Two boundary conditions are given by the fact, that there is no torque on the left and on the right sides of the beam: $\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=-l_L} = 0$ and $\left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=l_R} = 0$.

The other two boundary conditions must come from the central support. First, it is clear, that the beam must be smooth at $x = 0$, so $\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0}$.

To find the last boundary condition we compute the total torque on the infinitesimal element of length dx centered at $x = 0$. The total torque must be zero, so we have (Positive torque is towards us, τ_L and τ_R, are torques on a given cross-section from the right part to the left part. For f_R and f_L positive is up and are applied on a given gross-section from the right part to the left part. See figure.)

$$f_R dx/2 + f_L dx/2 + \tau_R - \tau_L = 0, \quad \text{in } dx \to 0 \text{ limit}, \quad \tau_R(0) = \tau_L(0).$$

As torque is the second derivative of Y (the left and right parts are made of the same materials), this condition means that the second derivative from the left and from the right must be the same. So the boundary conditions are

- The torque at $x = 0$ is continuous: $\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=0}$.

- The beam is smooth at $x = 0$: $\frac{\partial Y_L}{\partial x}\Big|_{x=0} = \frac{\partial Y_R}{\partial x}\Big|_{x=0}$.
- The torques on both ends are zero, $\frac{\partial^2 Y_L}{\partial x^2}\Big|_{x=-l_L} = 0$ and $\frac{\partial^2 Y_R}{\partial x^2}\Big|_{x=l_R} = 0$.

We thus have four conditions and four unknowns.

In order to find the force *from the middle support on the beam* F_2, lets again consider a small (length dx) element right on top of the middle support. The sum of all forces must be zero, so we get (see figure)

$$F_2 + f_R - f_L - \rho A g dx = 0.$$

Taking the limit $dx \to 0$ we find

$$F_2 = f_L - f_R = IAE\left(\frac{d^3 Y_R}{dx^3}\Big|_{x=0} - \frac{d^3 Y_L}{dx^3}\Big|_{x=0}\right) = -\frac{\rho g A}{4}\left(C_1^R - C_1^L - l_R - l_L\right).$$

(Check the units.) We see, that if we know $C_1^R - C_1^L$, then we will not the force we are interested in.

Let's see what the boundary conditions give one by one:

- $$C_0^L + l_L C_1^L = C_0^R - l_R C_1^R.$$
- $$l_L C_0^L = -l_R C_0^R.$$
- $$3l_R^2 + 2C_1^R l_R + C_0^R = 0, \qquad 3l_L^2 - 2C_1^L l_L + C_0^L = 0.$$

These are four linear equation for four unknowns. We only need the combination $C_1^R - C_1^L$ from them. Solving the equations we find

$$C_1^R - C_1^L = -\frac{1}{2}(l_R + l_L)\frac{l_R^2 + l_R l_L + l_L^2}{l_R l_L}.$$

and hence the force is

$$F_2 = \frac{\rho g A}{8}(l_R + l_L)\left(1 + \frac{(l_R + l_L)^2}{l_R l_L}\right) = \frac{Mg}{8}\left(1 + \frac{L^2}{l_L(L - l_L)}\right).$$

After this we find that

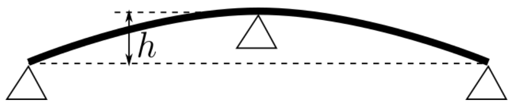
$$F_L = \frac{Mg}{8}\left(3 + \frac{l_L}{L} - \frac{L}{l_L}\right), \qquad F_R = \frac{Mg}{8}\left(3 + \frac{L - l_L}{L} - \frac{L}{L - l_L}\right).$$

In particular

- The answer does not depend on E! So the limit $E \to \infty$ is well defined!
- If $l_L = L/2$, we have $F_2 = \frac{5}{8}Mg$, $F_L = F_R = \frac{3}{16}Mg$. The guy at the center carries more than half of the total weight!
- If $l_L \to 0$ ($l_L \to L$), then F_2 and F_L (F_R) diverges. Why?

27.3. The force as a function of h.

Now let's finish this problem and compute how the force F_2 depends on the height h of the middle support. We simplify the problem by considering the middle support to be in the center.



We expect that the result for the force on the center support will be linear in h as for a spring.

- This is different from the situation of two unstretched springs. The difference is the torques that appears at bending.

So the result should have the form $F(h) = -\frac{5}{8}Mg - kh$. The spring constant k will depend on the Young modulus E. It is also clear, that if we fix the position of the ends (this is what we do for the solution) the spring constant will not depend on g, as it will be the same even without gravity. The force is always proportional to the combination EIA.

- The dimensional analysis then gives $kh \sim \frac{EIA}{l^2}\frac{h}{l}$. ($E$ has units of pressure, A is cross-section area, I has units of length square.)

The prefactor should be just a number.

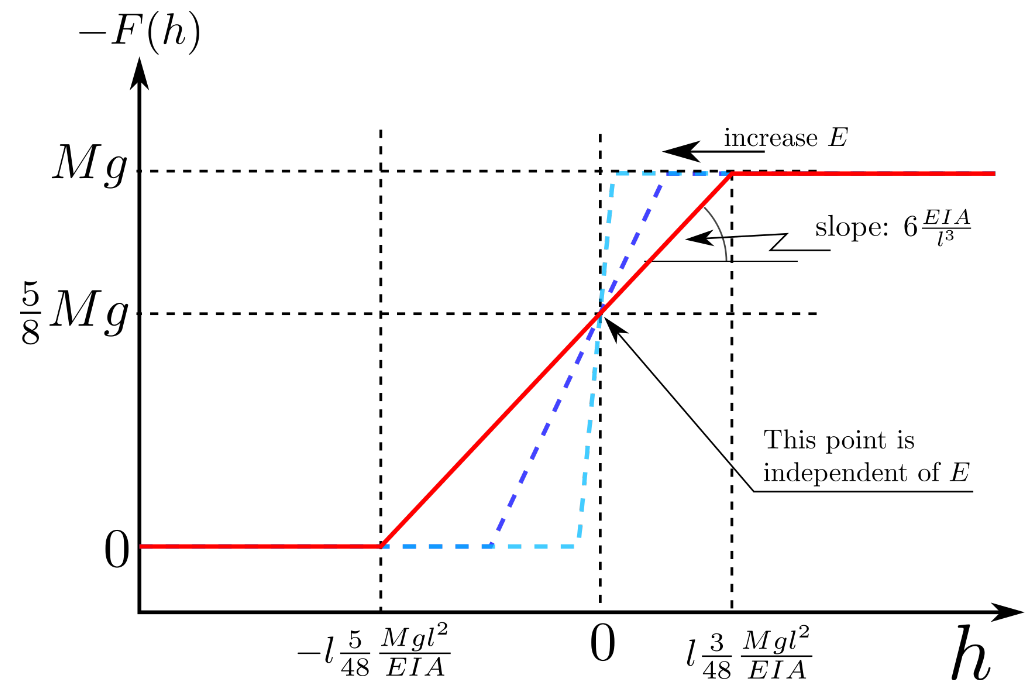
Again we have two functions $Y_L(x)$ and $Y_R(x)$ and the following boundary conditions

- At both ends we must have $Y = 0$, so
$$Y_L(x=-l) = 0, \qquad Y_R(x=l) = 0.$$
- At $x = 0$ we must have $Y = h$ for both parts, so
$$Y_L(x=0) = h, \qquad Y_R(x=0) = h.$$
- The torque is continuous at the center
$$\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=0}.$$
- The beam is smooth at the center
$$\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0},$$
- The torques at the ends are zero.
$$\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=-l} = 0, \qquad \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=l} = 0,$$

The force *on the support from the beam* is given by (different sign then before)
$$F = -IAE\left(\left.\frac{d^3 Y_R}{dx^3}\right|_{x=0} - \left.\frac{d^3 Y_L}{dx^3}\right|_{x=0}\right).$$
The first two conditions are satisfied by the functions of the form
$$Y_L(x) = -\frac{\rho g}{24IE}(x+l)\left(x^3 + C_2^L x^2 + C_1^L x - \frac{24IE}{\rho g l}h\right)$$
$$Y_R(x) = -\frac{\rho g}{24IE}(x-l)\left(x^3 + C_2^R x^2 + C_1^R x + \frac{24IE}{\rho g l}h\right)$$
The rest four conditions are enough to determine four unknown constants. As the result we have for the force *on the support*
$$F(h) = -\frac{5}{8}Mg - 6\frac{EIA}{l^2}\frac{h}{l}.$$

It has the expected form. One can see, that

$$F(h=0) = -\frac{5}{8}Mg$$
$$F = 0, \quad \text{for } h = -l\tfrac{5}{48}\tfrac{Mgl^2}{EIA}$$
$$F = -Mg, \quad \text{for } h = l\tfrac{3}{48}\tfrac{Mgl^2}{EIA}.$$



27.4. N supports.

Let's see if this scheme will work if we have N supports: two at the ends of a beam and $N-2$ somewhere in between.

- We will have $N-1$ functions Y, each requires 4 constants, so we need $4(N-1)$ conditions.
- We have two conditions on both ends — 4 conditions.
- On each of the $N-2$ supports in the middle we have:
  - Two conditions that the functions on the left and on the right have the same value — 2 conditions.
  - The smoothness — match of the first derivatives on the left to the that of on the right — 1 condition.
  - The torque condition — match of the second derivatives on the left to the that of on the right — 1 condition.
  - Total 4 conditions on each of the middle support.
- So total for all middle supports we have $4(N-2)$ conditions.
- Adding the conditions on the ends we have $4(N-2)+4 = 4(N-1)$ conditions.
- The total number of conditions equals to the total number of the unknowns!

So the scheme which we derived will work for any number of supports.

LECTURE 28
Hydrodynamics of Ideal Fluid: Mass conservation and Euler equation.

28.1. Hydrostatics.

For the statics of liquid we can use the elastic theory. The main difference between the solid body and the liquid is that the liquid has zero sheer coefficient. In this case the equation

$$\sigma_{ik} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}).$$

tells us that the stress tensor is diagonal and we can use $\sigma_{ij} = -P\delta_{ij}$. The constant P is called pressure. We then have

$$\frac{\delta V}{V} = u_{ii} = \frac{\sigma_{ii}}{3K} = -\frac{P}{K}.$$

The constant K is then given by the equation of state for the liquid.

The equilibrium condition

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho g_i,$$

gives

$$\frac{\partial P}{\partial x_i} = -\rho g_i, \qquad \vec{\nabla} P = -\rho \vec{g}$$

So $P = \rho g h$.

Consider a small volume dV. The force which acts on it is the weight $\rho\vec{g}dV$ and the force of the hydrostatic pressure. We see, that the force of the hydrostatic pressure is $d\vec{f} = -dV\vec{\nabla}P$.

28.2. Hydrodynamics

- Separation of scales.
- Separation of time scales.
- Universality.

Ideal fluid means that there is no viscosity.

28.2.1. Mass conservation.

The liquid is now moving. Mass current: amount of mass dM through an area dS during time dt is $dM = Idt$, I is proportional to dS and depends on the orientation, so $I = \vec{j} \cdot d\vec{S}$. $\vec{j}$ is the current density and is

$$\vec{j} = \rho\vec{v}.$$

Mass conservation, consider a small volume dV

- during time dt the amount of mass in the volume changes by $\delta m = -dt \oint \vec{j} \cdot d\vec{S} = -dt \int \vec{\nabla} \cdot \vec{j} dV$.
- The change of mass is $dt \int \dot{\rho} dV$.
- As it is correct for any volume we have

$$\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0.$$

This is called continuity equation. It represents the fact that the mass cannot appear or disappear. It will also be correct for any conserved quantity with the correct definition of "current density".

28.2.2. Another Euler equation.

We can describe the flow of liquid in two different ways:

- Describe the position and the velocity of the "liquid particles" as the function of time.
- Introduce the fields $\rho(\vec{r}, t)$, $P(\vec{r}, t)$, and $\vec{v}(\vec{r}, t)$ of density, pressure, and velocity and describe the dynamics of these fields.

Describe the two point of views.

- the field $\vec{v}(\vec{r}, t)$ describes the velocity at the point $\vec{r}$ at time t. It is NOT a velocity of an object! it's time derivative $\frac{\partial \vec{v}}{\partial t}$ is NOT an acceleration of an object. We cannot use the Newton's laws for it.

Instead we must consider a small volume dV at point $\vec{r}$ at time t. This volume has mass ρdV and is an object to which we can apply the Newton's law $\vec{F} = m\vec{a}$.

- The force which acts on this volume is $-dV\nabla P$.
- Considering the vector field $\vec{v}(\vec{r}, t)$ as given at each point and at each time(!):
  - At time t our volume dV has the velocity $\vec{v}(\vec{r}, t)$.
  - At time $t + dt$ this volume/object will shift to the position $\vec{r}' = \vec{r} + \vec{v}(\vec{r}, t)dt$.
  - So its velocity at time $t + dt$ is

$$\vec{v}(\vec{r}', t + dt) = \vec{v}(\vec{r} + \vec{v}(\vec{r}, t)dt, t + dt) \approx \vec{v}(\vec{r}, t) + (\nabla_i \vec{v})v_i dt + \frac{\partial \vec{v}}{\partial t} dt.$$

  - So during time dt the velocity of the volume/object dV has changed by $dv = dt(\vec{v} \cdot \nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t} dt$.
  - Its acceleration then is $\vec{a} = \frac{dv}{dt} = (\vec{v} \cdot \nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t}$.
- Now we can write $\vec{F} = m\vec{a}$:

$$-dV\nabla P = \rho dV \left((\vec{v} \cdot \nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t} \right)$$

The equation of the vector field $\vec{v}(\vec{r}, t)$ time evolution (Euler equation) is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} P$$

In case there is gravity there is extra force $\rho dV \vec{g}$ on this volume/object, so the equation is modified

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} P + \vec{g}.$$

This equation together with the continuity equation *and the equation of state* are the full set of equations which must be supplied with the boundary conditions.

- The equation of state – how ρ depends on P (and may be temperature) – is what distinguishes one liquid from another.
- All together we have three equations (five in components) for three (again five in components) fields $\vec{v}$, P, and ρ.

LECTURE 29
Hydrodynamics of Ideal Fluid: Incompressible fluid, potential flow.

29.1. Incompressible liquid.

In case of incompressible liquid the equation of state is particularly simple: the density is constant. So we have

$$\vec{\nabla}\cdot\vec{v}=0,\qquad \frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\vec{\nabla}\left(\frac{P}{\rho}+\Phi_g\right).$$

In this case we can use the following trick (more formal justification can be found in the cutout): we will be looking for the solution in the form

$$\vec{v}=\nabla\phi.$$

Notice, that this implies that $\mathrm{curl}\vec{v}=0$.

The continuity equation then gives

$$\Delta\phi=0.$$

This is so called potential flow. We need to supplement this equation with the boundary conditions. The simplest one is that on each boundary the component of the fluid velocity perpendicular to the boundary equals to the component of the boundary velocity perpendicular to the boundary.

Now substituting $\vec{v}=\vec{\nabla}\phi$ into the Euler equation and using $(\vec{v}\cdot\vec{\nabla})\vec{v}=\partial^i\phi\partial^i\partial^j\phi=\partial^i\phi\partial^j\partial^i\phi=\frac{1}{2}\partial^j(\partial^i\phi)^2=\nabla\frac{1}{2}v^2$ we find

$$\vec{\nabla}\left(\frac{\partial\phi}{\partial t}+\frac{P}{\rho}+\Phi_g+\frac{1}{2}v^2\right)=0,$$

Using the formula

$$\vec{v}\times\mathrm{curl}\vec{v}=\frac{1}{2}\vec{\nabla}v^2-(\vec{v}\cdot\vec{\nabla})\vec{v}$$

we can rewrite the Euler equation as

$$\frac{\partial\vec{v}}{\partial t}-\vec{v}\times\mathrm{curl}\vec{v}=-\vec{\nabla}\left(\frac{P}{\rho}+\Phi_g+\frac{1}{2}v^2\right).$$

If we now take curl of both sides we'll get

$$\frac{\partial}{\partial t}\mathrm{curl}\vec{v}-\mathrm{curl}(\vec{v}\times\mathrm{curl}\vec{v})=0$$

Notice, that this equation is identically satisfied if $\mathrm{curl}\vec{v}=0$. Which in turn identically satisfied by $\vec{v}=\vec{\nabla}\phi$ for some function ϕ.

and finally (notice, that the function f does not depend on the coordinate and thus must be given by the boundary conditions) we get the equation for P.

$$\frac{\partial\phi}{\partial t}+\frac{P}{\rho}+\Phi_g+\frac{1}{2}v^2=f(t)$$

29.2. Potential flow around a moving sphere

Consider a sphere of radius R moving with the velocity $\vec{u}$ in the ideal incompressible fluid. The flow of the fluid around the sphere is potential, so we need to solve the equation

$$\Delta\phi = 0, \qquad \vec{n}\cdot\vec{v}|_{\text{on sphere}} = \vec{n}\cdot\vec{u}, \qquad \vec{v}|_{r\to\infty} \to 0.$$

where the boundary conditions demand that the normal component of the fluid on the sphere equals the normal component of the element of the sphere.

The function ϕ is the scalar. It must linearly depend on the velocity $\vec{u}$ as both the Laplace equation and the boundary conditions are linear. This is analogous to the dipole field in the electrostatics, so the solution must be of the form

$$\phi = a\vec{u}\cdot\vec{\nabla}\frac{1}{r},$$

where a is an arbitrary constant which must be found from the boundary conditions. This is the field produced by the dipole $\vec{d} = a\vec{u}$, so the velocity (electric field) is

$$\vec{v} = \frac{a}{r^3}\left[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}\right].$$

So on the sphere surface we have

$$\vec{v}\cdot\vec{n}|_{r=R} = \frac{2a}{R^3}(\vec{u}\cdot\vec{n})$$

and we see, that $a = \frac{R^3}{2}$ and

$$\phi = -\frac{R^3}{2r^2}\vec{u}\cdot\vec{n}, \qquad \vec{v} = \frac{R^3}{2r^3}\left[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}\right].$$

In order to calculate the pressure use $\frac{\partial\phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}v^2 = \frac{P_0}{\rho}$. We then need to calculate $\frac{\partial\phi}{\partial t}$. In order to do we must remember, that the sphere is moving, so we need to think about the potential ϕ as a function of the position of the center of the sphere $\vec{r}_0$ and its velocity $\vec{u}$: $\phi(\vec{r}-\vec{r}_0, u)$, then we have

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\vec{u}}\cdot\dot{\vec{u}} - \vec{u}\cdot\nabla\phi.$$

We then find

$$P = P_0 + \frac{1}{8}\rho u^2(9\cos^2\theta - 5) + \frac{1}{2}\rho R\vec{n}\cdot\frac{d\vec{u}}{dt}.$$

We can calculate the total force acting on the sphere

$$\vec{F} = \oint P d\vec{S}.$$

The integration of the first two terms in P gives zero. For the last term we find

$$F_i = \frac{1}{2}\rho R\frac{du_j}{dt}4\pi R^2\overline{n_j n_i} = \frac{1}{2}\frac{4\pi}{3}\rho R^3\frac{du_i}{dt}.$$

(it is clear that $\overline{n_j n_i}$ must be diagonal. Also $\overline{n_x n_x} = \overline{n_y n_y} = \overline{n_z n_z}$ and $\overline{n_i n_i} = 1$) So we find that

$$\vec{F} = \frac{1}{2}\rho R\frac{du_j}{dt}4\pi R^2\overline{n_j n_i} = \frac{2\pi}{3}\rho R^3\frac{d\vec{u}}{dt}.$$

Notice:

- Without the viscosity the force is zero if the velocity of the sphere does not change.

- The liquid just effectively changes the mass of the sphere by the value

$$\frac{1}{2}\frac{4}{3}\pi R^3\rho.$$

  Half of the expelled mass.