Advanced Mechanics II. Phys 303

Artem G. Abanov

Contents

LECTURE 1
Review of Phys 302. Newtonian formulation.

1.0.1. Introduction

- Syllabus. Exams. Homeworks. Grades. Office hours etc.
- Structure (tentative) of the course.
- Questions, interruptions etc.
- Life is good and physics is great!

1.0.2. Newtonian formulation

- Inertial frame of references.
- $$\vec{F} = m\vec{a}$$
- $$\vec{F}_{12} = -\vec{F}_{21}$$
- Write (draw) down all the forces that act on a body. Remember, that the force is always a result of INTERaction.
- Chose a Cartesian system of coordinates in the inertial frame of reference.
- Write down the components of the forces in the chosen system of coordinates.
- For each component write down the equation of motion
$$F_i = ma_i, \qquad \dot{\vec{p}} = \vec{F}.$$
- Solve the resulting system of generally nonlinear differential equations.
- Use the initial conditions in order to find the motion $\vec{r}(t)$.

Pros:

- Very straight forward and intuitive.
- Very general – the nature of the forces does not matter, as long as you know them.

Cons:

- The symmetries and corresponding conservation laws are hidden.
- Difficult to use in anything but the inertial frame and Cartesian coordinates. (fictitious forces etc)
- Very quickly becomes cumbersome. Easy to make mistakes.

Examples. Wedge. Wedge with friction. Oscillator. Pendulum.

1.1. Conservation laws.

- Momentum conservation law.
  - Center of mass motion.

Example: Rocket motion, Inelastic collision.

$$Mv = (M + dM)(v + dv) + (V_0 - v)dM, \qquad \frac{dv}{V_0} = -\frac{dM}{M}, \qquad v_f - v_i = -V_0 \log \frac{M_f}{M_i}$$

- Angular momentum conservation law.

- Torque
$$\vec{\tau} = \vec{r} \times \vec{F}$$
- Angular momentum
$$\vec{J} = \vec{r} \times \vec{p}$$
- Central internal forces.
-
$$\dot{\vec{J}} = \vec{\tau}_{ex}.$$

Example: A bullet in a disc.
Example: A bullet in a disc-like wheel with no friction. What height should the bullet strike for the wheel to roll without slipping?

LECTURE 2
Review of Phys 302. Energy conservation.

Example: At what point should the stick strike so that the striking hand feel good?

- Energy conservation law.
  - Work
    $$\mathcal{A} = \int_A^B \vec{F} \cdot d\vec{r}$$
    It depends on the path from A to B.
  - Kinetic energy.
    $$\mathcal{A} = \int \vec{F} \cdot d\vec{r} = \int m\frac{d\vec{v}}{dt} \cdot \vec{v}dt = \int m\vec{v} \cdot d\vec{v} = \Delta\frac{m\vec{v}^2}{2}.$$
  - Conservative forces.
    $$\vec{F} = -\frac{\partial U}{\partial \vec{r}}$$
  - Potential energy U. If forces depend on coordinates only it does not mean that the force is conservative and the function U exists.
  - On a closed contour the work of a conservative force equals zero.
  - For a conservative force the work does not depend on the path.
  - Total energy
    $$E = \frac{m\vec{v}^2}{2} + U.$$
    is conserved
    $$\frac{dE}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{\partial U}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} = \left(m\vec{a} - \vec{F}\right) \cdot \vec{v} = 0$$

Examples: A wall with rope and a cart. Elastic 1D collision.

Examples of both Energy and momentum conservation: Elastic collision in 2D (case of equal masses.)

Example of 1D motion under conservative force.

$$\int_{x_0}^{x_f} \frac{dx}{\sqrt{E - U(x)}} = \pm\sqrt{\frac{2}{m}}(t - t_0), \qquad x(t = t_0) = x_0, \qquad E = \frac{mv_0^2}{2} + U(x_0).$$

Example: $1D$, the graph $U(x)$.

LECTURE 3
Review of Phys 302. Lagrangian and Hamiltonian formulations.

3.1. Lagrangian formulation.

- Action
$$S = \int_A^B L dt$$
- Hamilton principle. Minimum of action. Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t)$. Euler-Lagrange equation.
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$
-
$$L = K - U$$
- Generalized momentum (canonical)
$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$
- Conservation of generalized momentum (ignorable coordinates).
- Conservation of energy – no explicit time dependence in the Lagrangian.

Cons:

- Only conservative forces.

Pros:

- General coordinates.
- Only one scalar function L needs to be constructed. Easier.
- Symmetries are more transparent.

Examples:

- Pendulum.

The technique of minimizing a functional is not used in mechanics exclusively. There are a lot of problems where such techniques are useful. The conservation lows will also be applicable there, but will, in general, have different meaning.

3.2. Hamiltonian formulation

- Phase space $(\{q_i\}, \{p_i\})$.
- Poisson brackets $\{p_i, q_j\}$.
- Hamiltonian $H(\{q_i\}, \{p_i\})$.
- Hamiltonian equation of motion:

$$\dot{p}_i = \{H, p_i\}, \qquad \dot{q}_i = \{H, q_i\}$$

- In canonical coordinates and momenta

$$\{p_i, q_j\} = \delta_{i,k}, \qquad \{f, g\} = \sum_i \left(\frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right).$$

-

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

-

$$H(p_i, q_i) = \sum_i p_i \dot{q}_i - L, \qquad \dot{p}_i = \frac{\partial L}{\partial \dot{q}_i}$$

- Motion in 2D in a central field.

$$L = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} - U(r)$$

$$\frac{d}{dt} m\dot{r} = mr\dot{\phi}^2 - \frac{\partial U}{\partial r}, \qquad \frac{d}{dt} mr^2\dot{\phi} = 0$$

So we the angular momentum

$$L_\phi = mr^2\dot{\phi}$$

is conserved. We can use then $\dot{\phi} = \frac{L_\phi}{mr^2}$ and write

$$\frac{d}{dt} m\dot{r} = \frac{L_\phi^2}{mr^3} - \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{L_\phi^2}{2mr^2} + U \right)$$

This is a motion in $1D$ in the effective central potential

$$U_{eff}(r) = U(r) + \frac{L_\phi^2}{2mr^2}.$$

We then know the solution

$$\frac{dr}{\sqrt{E - U_{eff}(r)}} = \pm\sqrt{\frac{2}{m}} dt, \qquad dt = \frac{mr^2}{L_\phi} d\phi$$

or

$$\pm d\phi = \frac{L_\phi}{\sqrt{2m}} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}$$

- Kepler orbits. Let's use the gravitational potential energy

$$U(r) = -\frac{GMm}{r},$$

then we have

$$U_{eff} = -\frac{GMm}{r} + \frac{L_\phi^2}{2mr^2}$$

and

$$\phi - \phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{r} \frac{dr'}{r'^2} \frac{1}{\sqrt{E - \frac{GMm}{r'} + \frac{L_\phi^2}{2mr'^2}}}$$

- Scattering angle.

LECTURE 4
Probability density. Disintegration of a particle.

4.1. Probability density.

- Probability density.
  - Probability.
  - Probability density.
  - Positive definite.
  - Normalization.
  - Averaging: $\langle v^2 \rangle = \int v^2 \rho(v) dv$, or more generally $\langle f(v) \rangle = \int f(v)\rho(v) dv$.
  - $1D$ quantum harmonic oscillator with potential energy $U(x) = \frac{m\omega^2 x^2}{2}$: the wave function of the ground state is $\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$. The probability density for the coordinate is
  $$\rho(x) = |\psi(x)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{\hbar}x^2}, \qquad \int_{-\infty}^{\infty} \rho(x) dx = 1.$$
  Average position in the ground state is given by
  $$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x) dx = 0.$$
  However,
  $$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \rho(x) dx = \frac{1}{2}\frac{\hbar}{m\omega}.$$
  In particular average potential energy
  $$\langle U(x) \rangle = \int_{-\infty}^{\infty} U(x)\rho(x) dx = \frac{m\omega^2}{2}\langle x^2 \rangle = \frac{1}{4}\hbar\omega.$$
  We also can ask what is probability density for the potential energy U? We see, that $x = \sqrt{2U/m\omega^2}$, so $dx = \frac{dU}{\sqrt{2Um\omega^2}}$.
  $$\rho(x)dx = \rho\left(\sqrt{2U/m\omega^2}\right)\frac{dU}{\sqrt{2Um\omega^2}} = \frac{1}{\sqrt{2\pi U\hbar\omega}} e^{-\frac{U}{\hbar\omega}} dU.$$
  so
  $$\rho_U(U) = \frac{1}{\sqrt{2\pi U\hbar\omega}} e^{-\frac{U}{\hbar\omega}}, \qquad \int_0^{\infty} \rho_U(U) dU = 1.$$

- Change of variables.
  Let's say we have a probability density $\rho(v)$ to find a speed between v and $v+dv$. We want to find the probability density to find a kinetic energy between K and $K+dK$. The probability is
  $$dp = \rho(v)dv$$
  The kinetic energy is $K = \frac{mv^2}{2}$, or $v = \sqrt{2K/m}$, so $dv = \frac{dK}{m\sqrt{2K/m}}$, and
  $$dp = \rho(\sqrt{2K/m})\frac{1}{m\sqrt{2K/m}}dK, \qquad \rho_K(K) = \rho(\sqrt{2K/m})\frac{1}{m\sqrt{2K/m}}.$$
- Notice the change of differential!
- Uniform distribution.

4.2. Disintegration of a particle.

- Disintegration of a single particle.
  - In the center of mass system of reference
    $$m_1 v_{C1} + m_2 v_{C2} = 0, \qquad \frac{m_1 v_{C1}^2}{2} + \frac{m_2 v_{C2}^2}{2} = \epsilon,$$
    $$\frac{m_1 v_{C1}^2}{2} + \frac{m_1^2 v_{C1}^2}{2m_2} = \frac{m_1 v_{C1}^2}{2}\left(1 + \frac{m_1}{m_2}\right) = \epsilon$$
  - In the laboratory system of reference
    $$\vec{v}_{L1} = \vec{V} + \vec{v}_{C1}.$$
  - Kinematics show
    $$\sin(\theta_{L,max}) = \frac{v_{C1}}{V}, \quad \text{if} \quad V > v_{C1}$$
    and
    $$\tan\theta_L = \frac{v_{C1}\sin(\theta)}{v_{C1}\cos(\theta) + V},$$
    or
    $$\cos\theta = -\frac{V}{v_{C1}}\sin^2(\theta_L) \pm \cos(\theta_L)\sqrt{1 - \frac{V^2}{v_{C1}^2}\sin^2(\theta_L)}.$$
    For $v_{C1} > V$ the result is one-to-one, we must take the $+$ sign, so that $\theta(\theta_L = 0) = 0$, for $v_{C1} < V$ the result is not one-to-one: for a single θ_L in laboratory frame, there are two θs in the center of mass frame.
- Disintegration of many particles.
  - We are watching only particle number 1.
  - In each disintegration we know the speed v_{C1}. The speed in the center of mass reference frame is the same for all disintegrating particles.
  - The direction of the vector $\vec{v}_{C1}$ is arbitrary. We assume that there is no preferential direction (the way the initial particle was set up) and any direction of $\vec{v}_{C1}$ is equally probable.

– All detectors are on a sphere of radius R. In the center of mass ref. frame the probability to find the particle 1 in the solid angle $d\Omega$ is

$$dp = \frac{R^2 d\Omega}{4\pi R^2} = \frac{\sin(\theta) d\theta d\phi}{4\pi}$$

– The probability density to find the velocity $\vec{v_{C1}}$ direction between the angle θ and $\theta + d\theta$ is (we do not care about the angle ϕ.)

$$dp = d\theta \int_0^{2\pi} \frac{\sin(\theta) d\phi}{4\pi} = \frac{1}{2}\sin(\theta) d\theta.$$

(One should check that the total is 1.)

– In the laboratory reference frame $\vec{v}_{L1} = \vec{V} + \vec{v}_{C1}$, so

$$v_{L1}^2 = V^2 + v_{C1}^2 + 2V v_{C1} \cos(\theta).$$

where θ is the angle in the center of mass ref. frame.

– The kinetic energy of the particle 1 in the laboratory ref. frame is

$$K_L = \frac{m_1 v_{L1}^2}{2} = \frac{m_1 V^2}{2} + \frac{m_1 v_{C1}^2}{2} + m_1 V v_{C1} \cos(\theta),$$

$$dK = -m_1 V v_{C1} \sin(\theta) d\theta.$$

–

$$dp = \frac{1}{2 m_1 V v_{C1}} dK.$$

The uniform distribution: $\rho_K = \frac{1}{2 m_1 V v_{C1}}$

– The maximum kinetic energy is $K_{max} = \frac{m_1 (V + v_{C1})^2}{2}$. For $V > v_{C1}$ the minimum is $K_{min} = \frac{m_1 (V - v_{C1})^2}{2}$.

– One can check that

$$\int_{K_{min}}^{K_{max}} \rho_K dK = 1$$

LECTURE 5
Scattering cross-section.

- Set up of a scattering problem. Experiment, detector, etc.
- Energy. Impact parameter. The scattering angle. Impact parameter as a function of the scattering angle $\rho(\theta)$.
- Flux of particle. Same energy, different impact parameters, different scattering angles.
- The scattering problem, n — the flux, number of particles per unit area per unit time. dN the number of particles scattered between the angles θ and $\theta + d\theta$ per unit time. A suitable quantity do describe the scattering
$$d\sigma = \frac{dN}{n}.$$
It has the units of area and is called differential cross-section.
- If we know the function $\rho(\theta)$, then only the particles which are in between $\rho(\theta)$ and $\rho(\theta + d\theta)$ are scattered at the angle between θ and $\theta + d\theta$. So $dN = n2\pi\rho d\rho$, or
$$d\sigma = 2\pi\rho d\rho = 2\pi\rho \left|\frac{d\rho}{d\theta}\right| d\theta$$
(The absolute value is needed because the derivative is usually negative.)
- Often $d\sigma$ refers not to the scattering between θ and $\theta + d\theta$, but to the scattering to the solid angle $d\omega = 2\pi \sin\theta d\theta$. Then
$$d\sigma = \frac{\rho}{\sin\theta} \left|\frac{d\rho}{d\theta}\right| d\omega$$

Examples

- Cross-section for scattering of particles from a perfectly rigid sphere of radius R.
  - The scattering angle $\theta = 2\phi$.
  - $R\sin\phi = \rho$, so $\rho = R\sin(\theta/2)$.
  - $$\sigma = \frac{\rho}{\sin\theta} \left|\frac{d\rho}{d\theta}\right| d\omega = R^2 = \frac{1}{4}R^2 d\omega$$
  - Independent of the incoming energy. The scattering does not probe what is inside.

  - The total cross-section area is
$$\sigma = \int d\sigma = \frac{1}{4}R^2 2\pi \int_0^\pi \sin\theta d\theta = \pi R^2$$

- Cross-section for scattering of particles from a spherical potential well of depth U_0 and radius R.
  - Energy conservation
$$\frac{mv_0^2}{2} = \frac{mv^2}{2} - U_0, \qquad v = v_0\sqrt{1+\frac{2U_0}{mv_0^2}} = v_0\sqrt{1+U_0/E}$$
  - Angular momentum conservation
$$v_0 \sin\alpha = v\sin\beta, \qquad \sin\alpha = n(E)\sin\beta, \qquad n(E) = \sqrt{1+U_0/E}$$
  - Scattering angle
$$\theta = 2(\alpha-\beta)$$
  - Impact parameter
$$\rho = R\sin\alpha$$
  - So we have
$$\frac{\rho}{R} = n\sin(\alpha-\theta/2) = n\sin\alpha\cos(\theta/2) - n\cos\alpha\sin(\theta/2) = n\frac{\rho}{R}\cos(\theta/2) - n\sqrt{1-\rho^2/R^2}\sin(\theta/2)$$
$$\rho^2 = R^2\frac{n^2\sin^2(\theta/2)}{1+n^2-2n\cos(\theta/2)}.$$
  - The differential cross-section is
$$d\sigma = \frac{R^2n^2}{4\cos(\theta/2)}\frac{(n\cos(\theta/2)-1)(n-\cos(\theta/2))}{(1+n^2-2n\cos(\theta/2))^2}d\omega$$
  - Differential cross-section depends on E/U_0, where E is the energy of incoming particles. By measuring this dependence we can find U_0 from the scattering.
  - The scattering angle changes from 0 ($\rho = 0$) to θ_{max}, where $\cos(\theta_{max}) = 1/n$ (for $\rho = R$). The total cross-section is the integral
$$\sigma = \int_0^{\theta_{max}} d\sigma = \pi R^2.$$
    It does not depend on energy or U_0.
- Return to the rigid sphere but with U_0.

LECTURE 6
Rutherford's formula.

6.1. Rutherford experiment.

- What is the question?
- Experiment set up.
- Expected result from Thomson model.
- Obtained result.

6.2. Rutherford formula.

Consider the scattering of a particle of initial velocity v_∞ from the central force given by the potential energy $U(r)$.

- The energy is

$$E = \frac{mv_\infty^2}{2}.$$

- The angular momentum is given by

$$L_\phi = mv_\infty \rho,$$

  where ρ is the impact parameter.
- The trajectory is given by

$$\pm(\phi - \phi_0) = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{r} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}, \qquad U_{eff}(r) = U(r) + \frac{L_\phi^2}{2mr^2}$$

where r_0 and ϕ_0 are some distance and angle on the trajectory.

At some point the particle is at the closest distance r_0 to the center. The angle at this point is ϕ_0 (the angle at the initial infinity is zero.) Let's find the distance r_0. As the energy and the angular momentum are conserved and at the closest point the velocity is perpendicular to the radius we have

$$E = \frac{mv_0^2}{2} + U(r_0), \qquad L_\phi = mr_0v_0.$$

so we find that the equation for r_0 is

$$U_{eff}(r_0) = E.$$

This is, of course, obvious from the picture of motion in the central field as a one dimensional motion in the effective potential $U_{eff}(r)$.

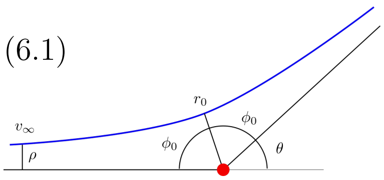


Figure 1. Rutherford experiment.

The angle ϕ_0 is then given by

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - U_{eff}(r)}}. \tag{6.1}$$

From geometry the scattering angle θ is given by the relation

$$\theta + 2\phi_0 = \pi. \tag{6.2}$$

So we see, that for a fixed v_0 the energy E is given, but the angular momentum L_ϕ depends on the impact parameter ρ. The equation (6.1) then gives the dependence of ϕ_0 on ρ. Then the equation (6.2) gives the dependence of the scattering angle θ on the impact parameter ρ. If we know that dependence, we can calculate the scattering cross-section.

$$d\sigma = \frac{\rho}{\sin\theta} \left| \frac{d\rho}{d\theta} \right| d\omega$$

Example: Coulomb interaction. Let's say that we have a repulsive Coulomb interaction

$$U = \frac{\alpha}{r}, \qquad \alpha > 0$$

In this case the geometry gives

$$\theta = \pi - 2\phi_0.$$

Let's calculate ϕ_0

$$\phi_0 = \frac{L_\phi}{\sqrt{2m}} \int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}}$$

where r_0 is the value of r, where the expression under the square root is zero.

Let's take the integral

$$\int_{r_0}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{L_\phi^2}{2mr^2}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E - \alpha x - x^2 \frac{L_\phi^2}{2m}}} = \int_0^{1/r_0} \frac{dx}{\sqrt{E + \frac{\alpha^2 m}{2L_\phi^2} - \frac{L_\phi^2}{2m}(x + \frac{\alpha m}{L_\phi^2})^2}}$$

$$= \sqrt{\frac{2m}{L_\phi^2}} \int_0^{1/r_0} \frac{dx}{\sqrt{\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4} - (x + \frac{\alpha m}{L_\phi^2})^2}}$$

changing $\sqrt{\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4}} \sin\psi = x + \frac{\alpha m}{L_\phi^2}$ we find that the integral is

$$\sqrt{\frac{2m}{L_\phi^2}} \int_{\psi_1}^{\pi/2} d\psi,$$

where $\sin(\psi_1) = \frac{\alpha m}{L_\phi^2} \left(\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4} \right)^{-1/2}$ So we find that

$$\phi_0 = \pi/2 - \psi_1$$

or

$$\cos\phi_0 = \sin\psi_1 = \frac{\alpha m}{L_\phi^2} \left(\frac{2mE}{L_\phi^2} + \frac{\alpha^2 m^2}{L_\phi^4} \right)^{-1/2}.$$

Using $L_\phi = \rho\sqrt{2mE}$ this gives

$$\sin\frac{\theta}{2} = \frac{\alpha}{2E}\left(\rho^2 + \frac{\alpha^2}{4E^2}\right)^{-1/2}$$

or

$$\frac{\alpha^2}{4E^2}\cot^2\frac{\theta}{2} = \rho^2$$

The differential cross-section then is

$$d\sigma = \frac{d\rho^2}{d\theta}\frac{1}{2\sin\theta}d\omega = \left(\frac{\alpha}{4E}\right)^2\frac{1}{\sin^4(\theta/2)}d\omega$$

- Notice, that the total cross-section diverges at small scattering angles.

LECTURE 7
Rutherford's formula. Intro in oscillations.

7.1. Rutherford formula.

Rutherford's formula.

$$d\sigma = \frac{d\rho^2}{d\theta}\frac{1}{2\sin\theta}d\omega = \left(\frac{\alpha}{4E}\right)^2\frac{1}{\sin^4(\theta/2)}d\omega$$

Divergence of the forward scattering in the ideal case – the cross-section of the beam.

- The beam. How do you characterize it?
- What is measured?
- The statistics. How much data we need to collect to get certainty of our results?
- The beam again. Interactions.
- Final state interactions.
- The forward scattering diverges.
- The cut off of the divergence is given by the size of the atom.
- Back scattering. Almost no dependence on θ.
- Energy dependence $1/E^2$.
- Plot $d\sigma$ as a function of $1/(4E)^2$, expect a straight line at large $1/(4E)^2$.
- The slope of the line gives α^2.
- What is the behavior at very large E? What is the crossing point?
- The crossing point tells us the size of the nucleus $d\sigma = \frac{R^2}{4}d\omega$.

7.2. Small oscillations.

Problem with one degree of freedom: $U(x)$. The Lagrangian is

$$L = \frac{m\dot{x}^2}{2} - U(x).$$

The equation of motion is

$$m\ddot{x} = -\frac{\partial U}{\partial x}$$

If the function $U(x)$ has an extremum at $x = x_0$, then $\left.\frac{\partial U}{\partial x}\right|_{x=x_0} = 0$. Then $x = x_0$ is a (time independent) solution of the equation of motion.

Consider a small deviation from the solution $x = x_0 + \delta x(t)$. Assuming that δx stays small during the motion we have

$$U(x) = U(x_0 + \delta x) \approx U(x_0) + U'(x_0)\delta x + \frac{1}{2}U''(x_0)\delta x^2 = U(x_0) + \frac{1}{2}U''(x_0)\delta x^2$$

The equation of motion becomes

$$m\ddot{\delta x} = -U''(x_0)\delta x$$

- If $U''(x_0) > 0$, then we have small oscillations with the frequency

$$\omega^2 = \frac{U''(x_0)}{m}$$

  This is a stable equilibrium.
- If $U''(x_0) < 0$, then the solution grows exponentially, and at some point our approximation becomes invalid. The equilibrium is unstable.

Look at what it means graphically.

7.2.1. Examples.

- $U(x) = \frac{kx^2}{2} + \frac{\gamma x^4}{4}$, where $k > 0$ and $\gamma > 0$.
- $U(x) = -\frac{kx^2}{2} + \frac{\gamma x^4}{4}$, where $k > 0$ and $\gamma > 0$.

Look at what it means graphically.

7.2.2. Noise and dissipation.

Generality: consider a system with infinitesimally small dissipation and external perturbations. The perturbations will kick it out of any unstable equilibrium. The dissipation will bring it down to a stable equilibrium. It may take a very long time.

After that the response of the system to small enough perturbations will be defined by the small oscillations around the equilibrium

LECTURE 8
Oscillations. Many degrees of freedom.

8.1. $\omega = 0$ case.

8.2. Many degrees of freedom.

Consider two equal masses in $1D$ connected by springs of constant k to each other and to the walls.

There are two coordinates: x_1 and x_2.

There are two modes $x_1 - x_2$ and $x_1 + x_2$.

The potential energy of the system is

$$U(x_1, x_2) = \frac{kx_1^2}{2} + \frac{k(x_1 - x_2)^2}{2} + \frac{kx_2^2}{2}$$

The Lagrangian

$$L = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{kx_1^2}{2} - \frac{k(x_1 - x_2)^2}{2} - \frac{kx_2^2}{2}$$

The equations of motion are

$$m\ddot{x}_1 = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -2kx_2 + kx_1$$

These are two second order differential equations. Total they must have four solutions. Let's look for the solutions in the form

$$x_1 = A_1 e^{i\omega t}, \qquad x_2 = A_2 e^{i\omega t}$$

then

$$-\omega^2 m A_1 = -2kA_1 + kA_2$$
$$-\omega^2 m A_2 = -2kA_2 + kA_1$$

or

$$(2k - m\omega^2)A_1 - kA_2 = 0$$
$$(2k - m\omega^2)A_2 - kA_1 = 0$$

or

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

In order for this set of equations to have a non trivial solution we must have

$$\det \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} = 0, \qquad (2k - m\omega^2)^2 - k^2 = 0, \qquad (k - m\omega^2)(3k - m\omega^2) = 0$$

There are two modes with the frequencies

$$\omega_a^2 = k/m, \qquad \omega_b^2 = 3k/m$$

and corresponding eigen vectors

$$\begin{pmatrix} A_1^a \\ A_2^a \end{pmatrix} = A^a \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} A_1^b \\ A_2^b \end{pmatrix} = A^b \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The general solution then is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_a t + \phi_a) + A^b \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_b t + \phi_b)$$

What will happen if the masses and springs constants are different?

Repeat the previous calculation for arbitrary m_1, m_2, k_1, k_2, k_3.

General scheme.

LECTURE 9
Oscillations. Many degrees of freedom.

9.1. Example of $\omega = 0$.

A rail and two objects with masses m and M connected by a spring k.

9.2. Example of mode disappearing.

A rail a wall and to objects with masses m and M connected by a spring k with each other and by the spring k through mass m with the wall. Consider the case $m = 0$ and $m \to 0$.

9.3. General situation.

Let's consider a general situation in detail. We start from an arbitrary Lagrangian

$$L = K(\{\dot{q}_i\}, \{q_i\}) - U(\{q_i\})$$

Very generally the kinetic energy is zero if all velocities are zero. It will also increase if any of the velocities increase.

It is assumed that the potential energy has a minimum at some values of the coordinates $q_i = q_{i0}$. Let's first change the definition of the coordinates $x_i = q_i - q_{0i}$. We rewrite the Lagrangian in these new coordinates.

$$L = K(\{\dot{x}_i\}, \{x_i\}) - U(\{x_i\})$$

We can take the potential energy to be zero at $x_i = 0$, also as $x_i = 0$ is a minimum we must have $\partial U/\partial x_i|_{\{x_i\}=0} = 0$.

Let's now assume, that the motion has very small amplitude. We then can use Taylor expansion in both $\{\dot{x}_i\}$ and $\{x_i\}$ up to the second order.

The time reversal invariance demands that only even powers of velocities can be present in the expansion. Also as the kinetic energy is zero if all velocities are zero, we have $K(0, \{x_i\}) = 0$, so we have

$$K(\{\dot{x}_i\}, \{x_i\}) \approx \frac{1}{2} \sum_{i,j} \left.\frac{\partial K}{\partial \dot{x}_i \partial \dot{x}_j}\right|_{\dot{x}=0, x=0} \dot{x}_i \dot{x}_j = \frac{1}{2} k_{ij} \dot{x}_i \dot{x}_j,$$

where the constant matrix k_{ij} is symmetric and positive definite.

For the potential energy we have

$$U(\{x_i\}) \approx \frac{1}{2}\sum_{i,j} \left.\frac{\partial U}{\partial x_i \partial x_j}\right|_{x=0} x_i x_j = \frac{1}{2} u_{ij} x_i x_j,$$

where the constant matrix u_{ij} is symmetric. If $x = 0$ is indeed a minimum, then the matrix u_{ij} is also positive definite.

The Lagrangian is then

$$L = \frac{1}{2} k_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} u_{ij} x_i x_j$$

where k_{ij} and u_{ij} are just constant matrices. The Lagrange equations are

$$k_{ij} \ddot{x}_j = -u_{ij} x_j$$

We are looking for the solutions in the form

$$x_j^a = A_j^a e^{i\omega_a t},$$

then

$$\left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a = 0, \tag{9.1}$$

where the summation over the index j is assumed.

In order for this linear equation to have a nontrivial solution we must have

$$\det\left(\omega_a^2 k_{ij} - u_{ij}\right) = 0$$

After solving this equation we can find all N of eigen/normal frequencies ω_a and the eigen/normal modes of the small oscillations A_i^a.

We can prove, that all ω_a^2 are positive (if U is at minimum.) Let's substitute the solutions ω_a and A_j^a into equation (9.1), multiply it by $(A_i^a)^*$ and sum over the index i.

$$\sum_{ij} \left(\omega_a^2 k_{ij} - u_{ij}\right) A_j^a A_i^* = 0.$$

From here we see

$$\omega_a^2 = \frac{\sum_{ij} u_{ij} A_j^a A_i^*}{\sum_{ij} k_{ij} A_j^a A_i^*}$$

As both matrices k_{ij} and u_{ij} are symmetric and positive definite, we have the ratio of two positive real numbers in the RHS. So ω_a^2 must be positive and real.

Examples

- Problem with three masses on a ring. Symmetries. Zero mode.
- Two masses, splitting of symmetric and anitsymmetric modes.

LECTURE 10
Oscillations with parameters depending on time. Kapitza pendulum.

- Oscillations with parameters depending on time.

$$L = \frac{1}{2}m(t)\dot{x}^2 - \frac{1}{2}k(t)x^2.$$

The Lagrange equation

$$\frac{d}{dt}m(t)\frac{d}{dt}x = -k(t)x.$$

We change the definition of time

$$m(t)\frac{d}{dt} = \frac{d}{d\tau}, \qquad \frac{d\tau}{dt} = \frac{1}{m(t)}$$

then the equation of motion is

$$\frac{d^2x}{d\tau^2} = -mkx.$$

So without loss of generality we can consider an equation

$$\ddot{x} = -\omega^2(\tau)x$$

- We call Ω the frequency of change of ω.
- Different time scales. Three different cases: $\Omega \gg \omega$, $\Omega \ll \omega$, and $\Omega \approx \omega$.

10.1. Kapitza pendulum $\Omega \gg \omega$

10.1.1. Vertical displacement.

- Set up of the problem.
- Time scales difference.
- Expected results.

The coordinates

$$\begin{matrix} x = l\sin\phi \\ y = l(1-\cos\phi) + \xi \end{matrix}, \qquad \begin{matrix} \dot{x} = l\dot{\phi}\cos\phi \\ \dot{y} = l\dot{\phi}\sin\phi + \dot{\xi} \end{matrix}$$

The Lagrangian

$$L = \frac{ml^2}{2}\dot\phi^2 + ml\dot\phi\dot\xi\sin\phi + mgl\cos\phi$$

The equation of motion

$$\ddot\phi + \frac{\ddot\xi}{l}\sin\phi = -\omega^2\sin\phi, \qquad \omega^2 = g/l$$

Look for the solution

$$\phi = \phi_0 + \theta, \qquad \bar\theta = 0$$

- What does averaging means. Separation of the time scales. Time T such that $\Omega^{-1} \ll T \ll \omega^{-1}$.

We expect θ to be small, but $\dot\theta$ and $\ddot\theta$ are NOT small. The equation then is

$$\ddot\phi_0 + \ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 = -\omega^2\sin\phi_0 - \omega^2\theta\cos\phi_0 \tag{10.1}$$

The frequency of the function ϕ_0 is small, so the fast oscillating functions must cancel each other. So

$$\ddot\theta + \frac{\ddot\xi}{l}\sin\phi_0 + \frac{\ddot\xi}{l}\theta\cos\phi_0 = -\omega^2\theta\cos\phi_0.$$

Neglecting term proportional to small θ we get

$$\theta = -\frac{\xi}{l}\sin\phi_0.$$

As $\bar\xi = 0$, the requirement $\bar\theta = 0$ fixes the other terms coming from the integration.

Now we take the equation (10.1) and average it over the time T.

$$\ddot\phi_0 + \frac{\overline{\theta\ddot\xi}}{l}\cos\phi_0 = -\omega^2\sin\phi_0$$

We now have

$$\overline{\theta\ddot\xi} = -\overline{\xi\ddot\xi}\frac{1}{l}\sin\phi_0, \qquad \overline{\xi\ddot\xi} = \frac{1}{T}\int_0^T \xi\ddot\xi dt = -\frac{1}{T}\int_0^T (\dot\xi)^2 dt = -\overline{(\dot\xi)^2}$$

Our equation then is

$$\ddot\phi_0 = -\left(\omega^2\sin\phi_0 + \frac{\overline{(\dot\xi)^2}}{2l^2}\sin 2\phi_0\right) = -\frac{\partial}{\partial\phi_0}\left(-\omega^2\cos\phi_0 - \frac{\overline{(\dot\xi)^2}}{4l^2}\cos 2\phi_0\right)$$

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2\cos\phi_0 - \frac{\overline{(\dot\xi)^2}}{4l^2}\cos 2\phi_0$$

The equilibrium positions are given by

$$\frac{\partial U}{\partial\phi_0} = \omega^2\sin\phi_0 + \frac{\overline{(\dot\xi)^2}}{2l^2}\sin 2\phi_0 = 0, \qquad \sin\phi_0\left(\omega^2 + \frac{\overline{(\dot\xi)^2}}{l^2}\cos\phi_0\right) = 0$$

We see, that if $\frac{\omega^2 l^2}{\overline{(\dot\xi)^2}} < 1$, a pair of new solutions appears.

The stability is defined by the sign of

$$\frac{\partial^2 U}{\partial\phi_0^2} = \omega^2\cos\phi_0 + \frac{\overline{(\dot\xi)^2}}{l^2}\cos 2\phi_0$$

One see, that

- $\phi_0 = 0$ is always a stable solution.
- $\phi_0 = \pi$ is unstable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} > 1$, but becomes stable if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$ are unstable.

For ϕ_0 close to π we can introduce $\phi_0 = \pi + \tilde{\phi}$

$$\ddot{\tilde{\phi}} = -\omega^2 \left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1 \right) \tilde{\phi}$$

We see, that for $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$, the oscillations in the upper point have the frequency

$$\tilde{\omega}^2 = \omega^2 \left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1 \right)$$

Remember, that above calculation is correct if Ω of the ξ is much larger than ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot{\xi})^2} \approx \Omega^2 \xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime ($\overline{(\dot{\xi})^2}/l^2\omega^2 \sim 1$) is at

$$\Omega^2 \sim \omega^2 \frac{l^2}{\xi_0^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

LECTURE 11
Oscillations with parameters depending on time. Kapitza pendulum. Horizontal case.

Let's consider a shaken pendulum without the gravitation force acting on it. The fast shaking is given by a fast time dependent vector $\vec{\xi}(t)$. This vector defines a direction in space. I will call this direction $\hat{z}$, so $\vec{\xi}(t) = \hat{z}\xi(t)$.

The amplitude ξ is small $\xi \ll l$, where l is the length of the pendulum, but the shaking is very fast $\Omega \gg \omega$, the frequency of the pendulum motion (without gravity it is not well defined, but we will keep in mind that we are going to include gravity later.)

Let's now use a non inertial frame of reference connected to the point of attachment of the pendulum. In this frame of reference there is a artificial force which acts on the pendulum. This force is

$$\vec{f} = -\ddot{\xi} m \hat{z}.$$

If the pendulum makes an angle ϕ with respect to the axis $\hat{z}$, then the torque of the force $\vec{f}$ is $\vec{\tau} = \vec{r} \times \vec{f}$, its magnitude $\tau = lf \sin\phi$ — the positive direction is defined by the positive direction of the angle. So the equation of motion

$$ml^2\ddot{\phi} = lm\ddot{\xi}\sin\phi, \qquad \ddot{\phi} = \frac{\ddot{\xi}}{l}\sin\phi$$

Now we split the angle onto slow motion described by ϕ_0 – a slow function of time, and fast motion $\theta(t)$ a fast oscillating function of time such that $\bar{\theta} = 0$.

We then have

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0 + \theta)$$

Notice the non linearity of the RHS.

As $\theta \ll \phi_0$, we can use the Taylor expansion

$$\ddot{\phi}_0 + \ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0) + \frac{\ddot{\xi}\theta}{l}\cos(\phi_0) \tag{11.1}$$

Double derivatives of θ and ξ are very large, so in the zeroth order we can write

$$\ddot{\theta} = \frac{\ddot{\xi}}{l}\sin(\phi_0), \qquad \theta = \frac{\xi}{l}\sin(\phi_0).$$

Now averaging the equation (11.1) in the way described in the previous lecture we get

$$\ddot{\phi}_0 = \frac{\overline{\ddot{\xi}\theta}}{l}\cos(\phi_0) = \frac{\overline{\ddot{\xi}\xi}}{l^2}\sin(\phi_0)\cos(\phi_0)$$

or

$$\ddot{\phi}_0 = \frac{\overline{\ddot{\xi}\theta}}{l}\cos(\phi_0) = -\frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0)$$

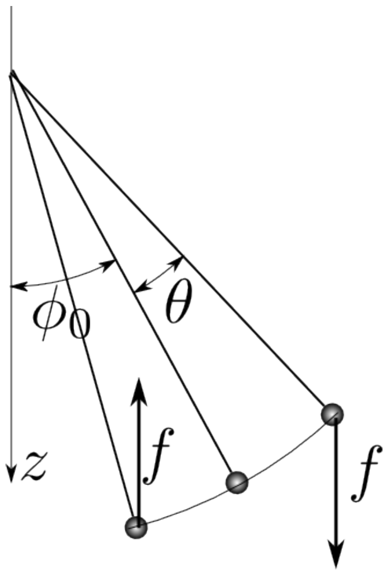


Figure 1. The Kapitza pendulum.

What is happening is illustrated on the figure. If ξ is positive, then $\ddot{\xi}$ is negative, so the torque is negative and is larger, because the angle $\phi = \phi_0 + \theta$ is larger. So the net torque is negative!

11.0.1. Vertical.

Now we can get the result from the previous lecture. We just need to add the gravitational term $-\omega^2 \sin\phi_0$.

$$\ddot{\phi}_0 = -\omega^2 \sin\phi_0 - \frac{\overline{\dot{\xi}^2}}{l^2}\sin(\phi_0)\cos(\phi_0).$$

So we have a motion in the effective potential field

$$U_{eff} = -\omega^2 \cos\phi_0 - \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0$$

The equilibrium positions are given by

$$\frac{\partial U}{\partial \phi_0} = \omega^2 \sin\phi_0 + \frac{\overline{(\dot{\xi})^2}}{2l^2}\sin 2\phi_0 = 0, \qquad \sin\phi_0\left(\omega^2 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_0\right) = 0$$

We see, that if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$, a pair of new solutions appears.

The stability is defined by the sign of

$$\frac{\partial^2 U}{\partial \phi_0^2} = \omega^2 \cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{l^2}\cos 2\phi_0$$

One see, that

- $\phi_0 = 0$ is always a stable solution.
- $\phi_0 = \pi$ is unstable for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} > 1$, but becomes stable if $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$.
- The new solutions that appear for $\frac{\omega^2 l^2}{\overline{(\dot{\xi})^2}} < 1$ are unstable.

For ϕ_0 close to π we can introduce $\phi_0 = \pi + \tilde{\phi}$

$$\ddot{\tilde{\phi}} = -\omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)\tilde{\phi}$$

We see, that for $\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} > 1$ the frequency of the oscillations in the upper point have the frequency

$$\tilde{\omega}^2 = \omega^2\left(\frac{\overline{(\dot{\xi})^2}}{l^2\omega^2} - 1\right)$$

Remember, that above calculation is correct if Ω of the ξ is much larger then ω. If ξ is oscillating with the frequency Ω, then we can estimate $\overline{(\dot{\xi})^2} \approx \Omega^2\xi_0^2$, where ξ_0 is the amplitude of the motion. Then the interesting regime is at

$$\Omega^2 > \omega^2\frac{l^2}{\xi^2} \gg \omega^2.$$

So the interesting regime is well withing the applicability of the employed approximations.

11.0.2. Horizontal.

If ξ is horizontal, the it is convenient to redefine the angle $\phi_0 \to \pi/2 + \phi_0$, then the shake contribution changes sign and we get

$$U_{eff} = -\omega^2\cos\phi_0 + \frac{\overline{(\dot{\xi})^2}}{4l^2}\cos 2\phi_0$$

The equilibrium position is found by

$$\frac{\partial U_{eff}}{\partial\phi_0} = \sin\phi_0\left(\omega^2 - \frac{\overline{(\dot{\xi})^2}}{l^2}\cos\phi_0\right).$$

Let's write U_{eff} for small angles, then (dropping the constant.)

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{24}\left(4\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} - 1\right)\phi_0^4$$

If $\frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2} \approx 1$, then

$$U_{eff} \approx \frac{\omega^2}{2}\left(1 - \frac{\overline{(\dot{\xi})^2}}{\omega^2 l^2}\right)\phi_0^2 + \frac{\omega^2}{8}\phi_0^4.$$

- Spontaneous symmetry braking.

LECTURE 12
Oscillations with parameters depending on time. Foucault pendulum.

The opposite situation, when the change of parameters is very slow – adiabatic approximation.

In rotation

$$\dot{\vec{r}} = \vec{\Omega} \times \vec{r}.$$

In our local system of coordinate (not inertial) a radius-vector is

$$\vec{r} = x\vec{e}_x + y\vec{e}_y.$$

So

$$\dot{\vec{r}} = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + x\vec{\Omega} \times \vec{e}_x + y\vec{\Omega} \times \vec{e}_y$$

I chose the system of coordinate such that $e_x \perp \vec{\Omega}$. Then

$$\vec{v}^2 = \dot{x}^2 + \dot{y}^2 + y^2\Omega^2\cos^2\theta + \Omega^2 x^2 + 2\Omega(x\dot{y} - y\dot{x})\cos\theta$$

For a pendulum we have

$$x = l\phi\cos\psi, \quad y = l\phi\sin\psi$$

so

$$\dot{x}^2 + \dot{y}^2 = l^2\dot{\phi}^2 + l^2\phi^2\dot{\psi}^2$$
$$x\dot{y} - y\dot{x} = l^2\phi^2\dot{\psi}$$

and

$$v^2 = l^2\dot{\phi}^2 + l^2\phi^2\dot{\psi}^2 + 2\Omega l^2\phi^2\dot{\psi}\cos\theta + \Omega^2 l^2\phi^2(\cos^2\psi + \sin^2\psi\cos^2\theta)$$

The Lagrangian then is (the gravitation potential energy is $mgl(1-\cos\phi) \approx \frac{1}{2}mgl\phi^2 = \frac{1}{2}ml^2\omega^2\phi^2$, where $\omega^2 = g/l$)

$$L = \frac{mv^2}{2} + mgl\cos\phi = \frac{mv^2}{2} - \frac{1}{2}ml^2\omega^2\phi^2$$

- In fact it is not exact as the centripetal force is missing. However, this force is of the order of Ω^2 and we will see, that the terms of that order can be ignored.

and the Lagrangian equations ($\omega^2 = g/l$)

$$\ddot{\phi} = -\omega^2\phi + \phi\dot{\psi}^2 + 2\Omega\phi\dot{\psi}\cos\theta + \Omega^2\phi(\sin^2\psi\cos^2\theta + \cos^2\psi)$$
$$2\phi\dot{\phi}\dot{\psi} + \phi^2\ddot{\psi} + 2\phi\dot{\phi}\Omega\cos\theta = -\frac{1}{2}\Omega^2\phi^2\sin 2\psi\sin^2\theta$$

We will see, that $\dot{\psi} \sim \Omega$. Then neglecting all terms of the order of Ω^2 we find

$$\ddot{\phi} = -\omega^2\phi$$
$$\dot{\psi} = -\Omega\cos\theta$$

The total change of the angle ψ for the period is

$$\Delta\psi = \Omega T\cos\theta = 2\pi\cos\theta.$$

- Geometrical meaning.

LECTURE 13
Oscillations with parameters depending on time. Foucault pendulum. General case.

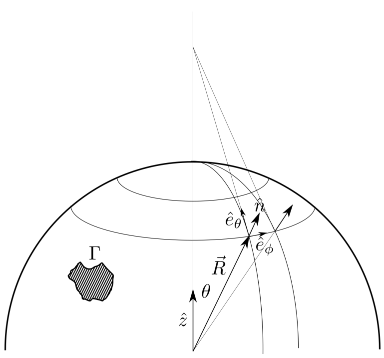


Figure 1

We want to move a pendulum around the world along some closed trajectory. The question is what angle the plane of oscillations will turn after we return back to the original point?

We assume that the earth is not rotating.

We assume that we are moving the pendulum slowly.

First of all we need to decide on the system of coordinates. For our the simple case we can do it in the following way.

(a) We choose a global unit vector $\hat{z}$. The only requirement is that the z line does not intersect our trajectory.

(b) After that we can introduce the angles θ and ϕ in the usual way. (strictly speaking in order to introduce ϕ we also need to introduce a global vector $\hat{x}$, thus introducing a full global system of coordinates.)

(c) In each point on the sphere we introduce it's own system/vectors of coordinates $\hat{e}_\phi$, $\hat{e}_\theta$, and $\hat{n}$, where $\hat{n}$ is along the radius-vector $\vec{R}$, $\hat{e}_\phi$ is orthogonal to both $\hat{n}$ and $\hat{z}$, and $\hat{e}_\theta = \hat{n} \times \hat{e}_\phi$.

We then have

$$\hat{e}_\theta^2 = \hat{e}_\phi^2 = \hat{n}^2 = 1, \qquad \hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\theta \cdot \hat{n} = \hat{e}_\phi \cdot \hat{n} = 0.$$

Let's look how the coordinate vectors change when we change a point where we siting. So let as change our position by a small vector $d\vec{r}$. The coordinate vectors then change by $\hat{e}_\theta \to \hat{e}_\theta + d\hat{e}_\theta$, etc. Where $d\hat{e}_\theta$, $d\hat{e}_\phi$, and $d\hat{n}$ will be proportional to $d\vec{r}$. We then see that

$$\hat{e}_\theta \cdot d\hat{e}_\theta = \hat{e}_\phi \cdot d\hat{e}_\phi = \hat{n} \cdot d\hat{n} = 0, \qquad \hat{e}_\theta \cdot d\hat{e}_\phi + d\hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\theta \cdot d\hat{n} + d\hat{e}_\theta \cdot \hat{n} = \hat{e}_\phi \cdot d\hat{n} + d\hat{e}_\phi \cdot \hat{n} = 0.$$

or

$$d\hat{e}_\theta = a\hat{e}_\phi + b\hat{n}$$
$$d\hat{e}_\phi = -a\hat{e}_\theta + c\hat{n}$$
$$d\hat{n} = -b\hat{e}_\theta - c\hat{e}_\phi$$

Where coefficients a, b, and c are linear in $d\vec{r}$.

Let's now assume, that our $d\vec{r}$ is along the vector $\hat{e}_\phi$. Then it is clear, that $d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi$, and $d\hat{e}_\theta = -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi$.

If $d\vec{r}$ is along the vector $\hat{e}_\theta$, then $d\hat{e}_\phi = 0$, and $d\hat{n} = \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta$.

Collecting it all together we have

$$
\begin{aligned}
d\hat{e}_\theta &= -\frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\phi - \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{n} \\
d\hat{e}_\phi &= \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R\tan\theta}\hat{e}_\theta - \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{n} \\
d\hat{n} &= \frac{(d\vec{r}\cdot\hat{e}_\theta)}{R}\hat{e}_\theta + \frac{(d\vec{r}\cdot\hat{e}_\phi)}{R}\hat{e}_\phi
\end{aligned}
$$

Notice, that these are purely geometrical formulas.

Now let's consider a pendulum. In our local system of coordinates it's radius vector is

$$\vec{\xi} = x\hat{e}_\theta + y\hat{e}_\phi = \xi\cos\psi\hat{e}_\theta + \xi\sin\psi\hat{e}_\phi.$$

The velocity is then

$$\dot{\vec{\xi}} = \dot{\xi}(\cos\psi\hat{e}_\theta + \sin\psi\hat{e}_\phi) + \xi\dot{\psi}(-\sin\psi\hat{e}_\theta + \cos\psi\hat{e}_\phi) + \xi(\cos\psi\frac{\partial\hat{e}_\theta}{\partial\vec{r}} + \sin\psi\frac{\partial\hat{e}_\phi}{d\vec{r}})\frac{d\vec{r}}{dt}.$$

When we calculate $\dot{\vec{\xi}}^2$ we only keep terms no more than first order in $d\vec{r}/dt$

$$\dot{\vec{\xi}}^2 \approx \dot{\xi}^2 + \xi^2\dot{\psi}^2 + 2\xi^2\dot{\psi}\frac{\hat{e}_\phi\cdot\partial\hat{e}_\theta}{\partial\vec{r}}\frac{d\vec{r}}{dt} = \dot{\xi}^2 + \xi^2\dot{\psi}^2 + 2\xi^2\dot{\psi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt}$$

The potential energy does not depend on ψ, so the Lagrange equation for ψ is simply $\frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}} = 0$. Moreover, as ξ is fast when we take the derivative $\frac{d}{dt}$ we differentiate only ξ. Then

$$4\xi\dot{\xi}\dot{\psi} + 4\xi\dot{\xi}\frac{1}{R\tan\theta}\frac{\hat{e}_\phi\cdot d\vec{r}}{dt} = 0$$

so

$$\dot{\psi} = -\frac{1}{R\tan\theta}\frac{R\sin\theta d\phi}{dt} = -\cos\theta\frac{d\phi}{dt}$$

Finally,

$$d\psi = -\cos\theta d\phi.$$

LECTURE 14
Oscillations with parameters depending on time. Parametric resonance.

14.1. Physics Festival.

- Kapitza pendulum
- Non-newtonian fluid
- Fire Piston
- Pendulum Cart
- Chaldni plate

14.2. Generalities

Now we consider a situation when the parameters of the oscillator depend on time and the frequency of this dependence is comparable to the frequency of the oscillator. We start from the equation

$$\ddot{x} = -\omega^2(t)x,$$

where $\omega(t)$ is a periodic function of time. The interesting case is when $\omega(t)$ is almost a constant ω_0 with a small correction which is periodic in time with period T. Then the case which we are interested in is when $2\pi/T$ is of the same order as ω_0. We are going to find the resonance conditions. Such resonance is called "parametric resonance".

First we notice, that if the initial conditions are such that $x(t=0)=0$, and $\dot{x}(t=0)=0$, then $x(t)=0$ is the solution and no resonance happens. This is very different from the case of the usual resonance.

Let's assume, that we found two linearly independent solutions $x_1(t)$ and $x_2(t)$ of the equation. All the solutions are just linear combinations of $x_1(t)$ and $x_2(t)$.

If a function $x_1(t)$ is a solution, then function $x_1(t+T)$ must also be a solution, as T is a period of $\omega(t)$. It means, that the function $x_1(t+T)$ is a linear combination of functions $x_1(t)$ and $x_2(t)$. The same is true for the function $x_2(t+T)$. So we have

$$\begin{pmatrix} x_1(t+T) \\ x_2(t+T) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

We can always choose such $x_1(t)$ and $x_2(t)$ that the matrix is diagonal. In this case

$$x_1(t+T) = \mu_1 x_1(t), \qquad x_2(t+T) = \mu_2 x_2(t)$$

so the functions are multiplied by constants under the translation on one period. The most general functions that have this property are

$$x_1(t) = \mu_1^{t/T} X_1(t), \qquad x_2(t) = \mu_2^{t/T} X_2(t),$$

where $X_1(t)$, and $X_2(t)$ are periodic functions of time.

The numbers μ_1 and μ_2 cannot be arbitrary. The functions x_1 and x_2 satisfy the Wronskian equation

$$W(t) = \dot{x}_1 x_2 - \dot{x}_2 x_1 = \text{const}$$

So on one hand $W(t+T) = \mu_1\mu_2 W(t)$, on the other hand $W(t)$ must be constant. So

$$\mu_1\mu_2 = 1.$$

Now, if x_1 is a solution so must be x_1^*. It means that either both μ_1 and μ_2 are real, or $\mu_1^* = \mu_2$. In the later case we have $|\mu_1| = |\mu_2| = 1$ and no resonance happens. In the former case we have $\mu_2 = 1/\mu_1$ (either both are positive, or both are negative). Then we have

$$x_1(t) = \mu^{t/T} X_1(t), \qquad x_2(t) = \mu^{-t/T} X_2(t).$$

We see, that one of the solutions is unstable, it increases exponentially with time. This means, that a small initial deviation from the equilibrium will exponentially grow with time. This is the parametric resonance.

14.3. Resonance.

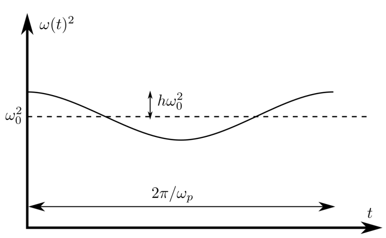


Figure 1

Let's now consider the following dependence of ω on time

$$\omega^2 = \omega_0^2(1 + h\cos\omega_p t)$$

where $h \ll 1$.

- The most interesting case is when $\omega_p \sim 2\omega_0$. Explain.

So I will take $\omega_p = 2\omega_0 + \epsilon$, where $\epsilon \ll \omega_0$. The equation of motion is

$$\ddot{x} + \omega_0^2 x[1 + h\cos(2\omega_0 + \epsilon)t] = 0$$

(Mathieu's equation)

We seek the solution in the form

$$x = a(t)\cos(\omega_0 + \epsilon/2)t + b(t)\sin(\omega_0 + \epsilon/2)t$$

and retain only the terms first order in ϵ assuming that $\dot{a} \sim \epsilon a$ and $\dot{b} \sim \epsilon b$. We then substitute this solution into the equation use the identity

$$\cos(\omega_0 + \epsilon/2)t\cos(2\omega_0 + \epsilon)t = \frac{1}{2}\cos 3(\omega_0 + \epsilon/2)t + \frac{1}{2}\cos(\omega_0 + \epsilon/2)t$$

and neglect the terms with frequency $\sim 3\omega_0$ as they are off the resonance. The result is

$$-\omega_0(2\dot{a} + b\epsilon + \frac{1}{2}h\omega_0 b)\sin(\omega + \epsilon/2)t + \omega_0(2\dot{b} - a\epsilon + \frac{1}{2}h\omega_0 a)\cos(\omega + \epsilon/2)t = 0$$

So we have a pair of equations

$$2\dot{a} + b\epsilon + \frac{1}{2}h\omega_0 b = 0$$
$$2\dot{b} - a\epsilon + \frac{1}{2}h\omega_0 a = 0$$

We look for the solution in the form $a, b \sim a_0, b_0 e^{st}$, then

$$2sa_0 + b_0\epsilon + \frac{1}{2}h\omega_0 b = 0, \qquad 2sb_0 - a_0\epsilon + \frac{1}{2}h\omega_0 a_0 = 0.$$

The compatibility condition gives

$$s^2 = \frac{1}{4}\left[(h\omega_0/2)^2 - \epsilon^2\right].$$

Notice, that e^s is what we called μ before. The condition for the resonance is that s is real. It means that the resonance happens for

$$-\frac{1}{2}h\omega_0 < \epsilon < \frac{1}{2}h\omega_0$$

- The range of frequencies for the resonance depends on the amplitude h.
- The amplification s, also depends on the amplitude h.
- In case of dissipation the solution acquires a decaying factor $e^{-\lambda t}$, so s should be substituted by $s - \lambda$. Then in order for the instability to occur we must have $s > \lambda$ so the range of instability is given by $\frac{1}{4}\left[(h\omega_0/2)^2 - \epsilon^2\right] > \lambda^2$:

$$-\sqrt{(h\omega_0/2)^2 - 4\lambda^2} < \epsilon < \sqrt{(h\omega_0/2)^2 - 4\lambda^2}$$

- At finite dissipation the parametric resonance requires finite amplitude $h = 4\lambda/\omega_0$.
- Other resonances occur $\omega_0/\omega_p = n/2$.

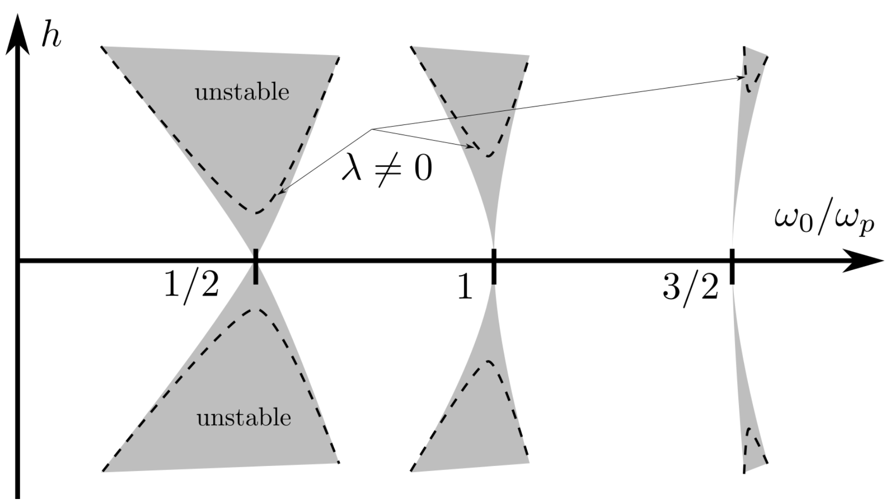


Figure 2

LECTURE 15
Oscillations of an infinite series of springs. Oscillations of a rope. Phonons.

15.1. Physics Festival.

- Kapitza pendulum
- Non-newtonian fluid
- Fire Piston
- Pendulum Cart
- Chaldni plate

15.2. Series of springs.

Consider one dimension string of N masses m connected with identical springs of spring constants k. The first and the last masses are connected by the same springs to a wall. The question is what are the normal modes of such system?

- The difference between the infinite number of masses and finite, but large — zero mode.

This system has N degrees of freedom, so we must find N modes. We call x_i the displacement of the ith mass from its equilibrium position. The Lagrangian is:

$$L = \sum_{i=1}^{N} \frac{m\dot{x}_i^2}{2} - \frac{k}{2}\sum_{i=0}^{N+1}(x_i - x_{i+1})^2, \qquad x_0 = x_{N+1} = 0.$$

15.2.1. First solution

The matrix $-\omega^2 k_{ij} + u_{ij}$ is

$$-\omega^2 k_{ij} + u_{ij} = \begin{pmatrix} -m\omega^2+2k & -k & 0 & \dots & \dots \\ -k & -m\omega^2+2k & -k & 0 & \dots \\ 0 & -k & -m\omega^2+2k & -k & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

This is $N \times N$ matrix. Let's call its determinant D_N. We then see

$$D_N = (-m\omega^2+2k)D_{N-1} - k^2 D_{N-2}, \qquad D_1 = -m\omega^2+2k, \qquad D_2 = (-m\omega^2+2k)^2 - k^2$$

This is a linear difference equation with constant coefficients. The solution should be of the form $D_N = a^N$. Then we have

$$a^2 = (-m\omega^2 + 2k)a - k^2, \qquad a = \frac{-m\omega^2 + 2k \pm i\sqrt{m\omega^2(4k - m\omega^2)}}{2}.$$

So the general solution and initial conditions are

$$D_N = Aa^{N-1} + \bar{A}\bar{a}^{N-1}, \qquad A + \bar{A} = -m\omega^2 + 2k, \qquad Aa + \bar{A}\bar{a} = (-m\omega^2 + 2k)^2 - k^2.$$

The solution is $A = \frac{a^2}{a-\bar{a}}$. Now in order to find the normal frequencies we need to solve the following equation for ω.

$$D_N = \frac{a^2}{a - \bar{a}} a^{N-1} - \frac{\bar{a}^2}{a - \bar{a}} \bar{a}^{N-1} = 0, \qquad \text{or} \qquad \left(\frac{a}{\bar{a}}\right)^{N+1} = 1.$$

We now say that $a = ke^{i\phi}$, ($|a|^2 = k^2$) where $\cos\phi = \frac{-m\omega^2 - 2k}{2k}$ then

$$e^{2i\phi(N+1)} = 1, \qquad 2\phi(N+1) = 2\pi n, \qquad \text{where} \quad n = 1 \dots N.$$

So we have

$$\cos\phi = \cos\frac{\pi n}{N+1} = \frac{-m\omega^2 - 2k}{2k}, \qquad \omega_n^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N+1)}.$$

15.2.2. Second solution.

From the Lagrangian we find the equations of motion

$$\ddot{x}_j = -\frac{k}{m}(2x_j - x_{j+1} - x_{j-1}), \qquad x_0 = x_{N+1} = 0.$$

We look for the solution in the form

$$x_j = \sin(\beta j)\sin(\omega t), \qquad \sin\beta(N+1) = 0.$$

Substituting this guess into the equation we get

$$\begin{aligned}
-\omega^2\sin(\beta j) &= -\frac{k}{m}\left(2\sin(\beta j) - \sin\beta(j+1) - \sin\beta(j-1)\right) \\
&= -\frac{k}{m}\Im\left(2e^{ij\beta} - e^{i(j+1)\beta} - e^{i(j-1)\beta}\right) = -\frac{k}{m}\Im e^{ij\beta}\left(2 - e^{i\beta} - e^{-i\beta}\right) = \frac{k}{m}\Im e^{ij\beta}\left(e^{i\beta/2} - e^{-i\beta/2}\right)^2 \\
&= -4\frac{k}{m}\Im e^{ij\beta}\sin^2(\beta/2) = -4\frac{k}{m}\sin(j\beta)\sin^2(\beta/2).
\end{aligned}$$

So we have

$$\omega^2 = 4\frac{k}{m}\sin^2(\beta/2),$$

but β must be such that $\sin\beta(N+1) = 0$, so $\beta = \frac{\pi n}{N+1}$, and we have

$$\omega^2 = 4\frac{k}{m}\sin^2\frac{\pi n}{2(N+1)}, \qquad n = 1, \dots, N$$

15.3. A rope.

The potential energy of a (2D) rope of shape $y(x)$ is $T\int_0^L \sqrt{1+y'^2}dx \approx \frac{T}{2}\int_0^L y'^2 dx$. The kinetic energy is $\int_0^L \frac{\rho}{2}\dot{y}^2 dx$, so the Lagrangian is

$$L = \int_0^L \left(\frac{\rho}{2}\dot{y}^2 - \frac{T}{2}y'^2\right) dx, \qquad y(0) = y(L) = 0.$$

In order to find the normal modes we need to decide on the coordinates in our space of functions $y(x,t)$. We will use a standard Fourier basis $\sin kx$ and write any function as

$$y(x,t) = \sum_k A_k(t)\sin kx, \qquad \sin kL = 0$$

The constants $A_k(t)$ are the coordinates in the Fourier basis. We then have

$$L = \frac{L}{2}\sum_k \left(\frac{\rho}{2}\dot{A}_k^2 - \frac{T}{2}k^2 A_k^2\right)$$

We see, that it is just a set of decoupled harmonic oscillators and k just enumerates them. The normal frequencies are

$$\omega_k^2 = \frac{T}{\rho}k^2, \qquad \omega = \sqrt{\frac{T}{\rho}}k.$$

- We also see, that the wavelength $\lambda = 2\pi/k$. So using $\omega = 2\pi f$ we find that $\lambda f = \sqrt{T/\rho}$. So the speed of these waves is

$$c^2 = T/\rho.$$

LECTURE 16
Motion of a rigid body. Kinematics. Kinetic energy. Momentum. Tensor of inertia.

16.1. Kinematics.

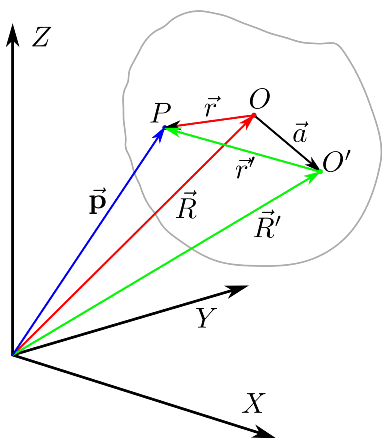


Figure 1

We will use two different system of coordinates XYZ — fixed, or external inertial system of coordinates, and xyz the moving, or internal system of coordinates which is attached to the body itself and moves with it.

Let's $\vec{R}$ be radius vector of the center of mass O of a body with respect to the external frame of reference, $\vec{r}$ be the radius vector of any point P of the body with respect to the center of mass O, and $\vec{\mathbf{p}}$ the radius vector of the point P with respect to the external frame of reference: $\vec{\mathbf{p}} = \vec{R}+\vec{r}$. For any infinitesimal displacement $d\vec{\mathbf{p}}$ of the point P we have

$$d\vec{\mathbf{p}} = d\vec{R} + d\vec{r} = d\vec{R} + d\vec{\phi} \times \vec{r}.$$

Or dividing by dt we find the velocity of the point P as

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}, \qquad \vec{v} = \frac{d\vec{\mathbf{p}}}{dt}, \quad \vec{V} = \frac{d\vec{R}}{dt}, \quad \vec{\Omega} = \frac{d\vec{\phi}}{dt}.$$

- Notice, that ϕ is not a vector, while $d\vec{\phi}$ is.

In the previous calculation the fact that O is a center of mass has not been used, so for any point O' with a radius vector $\vec{R}' = \vec{R} + \vec{a}$ we find the radius vector of the point P to be $\vec{r}' = \vec{r} - \vec{a}$, and we must have $\vec{v} = \vec{V}' + \vec{\Omega}' \times \vec{r}'$. Now substituting $\vec{r} = \vec{r}' + \vec{a}$ into $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$ we get $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \vec{r}'$. So we conclude that

$$\vec{V}' = \vec{V} + \vec{\Omega} \times \vec{a}, \qquad \vec{\Omega}' = \vec{\Omega}.$$

The last equation shows, that the vector of angular velocity is the same and does not depend on the particular moving system of coordinates. So $\vec{\Omega}$ can be called the angular velocity of the body.

If at some instant the vectors $\vec{V}$ and $\vec{\Omega}$ are perpendicular for some choice of O, then they will be perpendicular for any other O': $\vec{\Omega} \cdot \vec{V} = \vec{\Omega} \cdot \vec{V}'$. Then it is possible to solve the equation for $\vec{a}$

$$\vec{V}+\vec{\Omega}\times\vec{a}=0.$$

These are three inhomogeneous linear equations for the components of the vector $\vec{a}$.

If we take the dot product of the above equation with $\vec{\Omega}$, we find that $\vec{\Omega}\cdot\vec{V}=0$, this is the requirement that the above equation has a solution (rhs must be orthogonal to the zero modes).

If we multiply the above equation by $\vec{V}$ "vectorly" we find

$$0=\vec{\Omega}(\vec{V}\cdot\vec{a})-\vec{a}(\vec{V}\cdot\vec{\Omega})=\vec{\Omega}(\vec{V}\cdot\vec{a}),$$

so $\vec{V}\cdot\vec{a}=0$. Finally, multiplying the main equation by $\vec{\Omega}$ "vectorly" we find

$$0=\vec{\Omega}\times\vec{V}+\vec{\Omega}(\vec{\Omega}\cdot\vec{a})-\vec{a}\Omega^2=\vec{\Omega}\times\vec{V}-\vec{a}\Omega^2.$$

So

$$\vec{a}=\frac{\vec{\Omega}\times\vec{V}}{\Omega^2}.$$

So in this case there exist a point (it may be outside of the body) with respect to which the whole motion is just a rotation. The line parallel to $\vec{\Omega}$ which goes through this point is called "instantaneous axis of rotation". (In the general case the instantaneous axis can be made parallel to $\vec{V}$.)

- In general both the magnitude and the direction of $\vec{\Omega}$ are changing with time, so is the "instantaneous axis of rotation".

16.2. Kinetic energy.

The total kinetic energy of a body is the sum of the kinetic energies of its parts. Lets take the origin of the moving system of coordinates to be in the center of mass. Then

$$K=\frac{1}{2}\sum m_\alpha v_\alpha^2=\frac{1}{2}\sum m_\alpha\left(\vec{V}+\vec{\Omega}\times\vec{r}_\alpha\right)^2=\frac{1}{2}\sum m_\alpha\vec{V}^2+\sum m_\alpha\vec{V}\cdot\vec{\Omega}\times\vec{r}_\alpha+\frac{1}{2}\sum m_\alpha\left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$
$$=\frac{MV^2}{2}+\vec{V}\cdot\vec{\Omega}\times\sum m_\alpha\vec{r}_\alpha+\frac{1}{2}\sum m_\alpha\left[\vec{\Omega}\times\vec{r}_\alpha\right]^2$$

For the center of mass $\sum m_\alpha\vec{r}_\alpha=0$ and we have

$$K=\frac{MV^2}{2}+\frac{1}{2}\sum m_\alpha\left(\vec{\Omega}^2\vec{r}_\alpha^2-(\vec{\Omega}\cdot\vec{r}_\alpha)^2\right)=\frac{MV^2}{2}+\frac{I_{ij}\Omega^i\Omega^j}{2},$$

where

$$I_{ij}=\sum m_\alpha\left(\delta_{ij}\vec{r}_\alpha^2-r_\alpha^i r_\alpha^j\right).$$

I_{ij} is the tensor of inertia. This tensor is symmetric and positive definite. The diagonal components of the tensor are called moments of inertia.

16.3. Angular momentum

The origin is at the center of mass. So we have

$$\vec{M} = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum m_\alpha \vec{r}_\alpha \times (\vec{\Omega} \times \vec{r}_\alpha) = \sum m_\alpha \left(r_\alpha^2 \vec{\Omega} - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\Omega}) \right)$$

Writing this in components we have

$$M_i = \sum m_\alpha \left(\delta_{ij} \vec{r}_\alpha^2 - r_\alpha^i r_\alpha^j \right) \Omega^j$$

or

$$M_i = I_{ij} \Omega^j.$$

- In general the direction of angular momentum $\vec{M}$ and the direction of the angular velocity $\vec{\Omega}$ do not coincide.

16.4. Tensor of inertia.

Tensor of inertia is a symmetric tensor of rank two. As any such tensor it can be reduced to a diagonal form by an appropriate choice of the moving axes. Such axes are called the principal axes of inertia. The diagonal components I_1, I_2, and I_3 are called the principal moments of inertia.

- Notice, that these axes are "attached" to the body and thus rotate with the body.

In this axes the kinetic energy is simply

$$K = \frac{I_1 \Omega_1^2}{2} + \frac{I_2 \Omega_2^2}{2} + \frac{I_3 \Omega_3^2}{2}.$$

(a) If all three principal moments of inertia are different, then the body is called "asymmetrical top".
(b) If two of the moments coincide and the third is different, then it is called "symmetrical top".
(c) If all three coincide, then it is "spherical top".

For any plane figure if z is perpendicular to the plane, then $I_1 = \sum m_\alpha y_\alpha^2$, $I_2 = \sum m_\alpha x_\alpha^2$, and $I_3 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = I_1 + I_2$. If symmetry demands that $I_1 = I_2$, then $\frac{1}{2} I_3 = I_1$. Example: a disk, a square.

If the body is a line, then (if z is along the line) $I_1 = I_2$, and $I_3 = 0$. Such system is called "rotator".

LECTURE 17
Motion of a rigid body. Rotation of a symmetric top. Euler angles.

Reminders:

- Physics Festival.
- Tensor of inertia.
- Angular momentum.
- Principal axes, principal moments.

Kinematics:

- Spherical top.
- Arbitrary top rotating around one of its principal axes.
- Symmetric top.

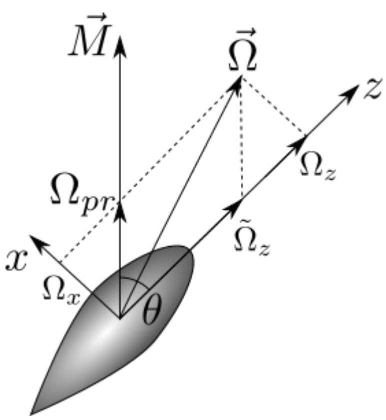


Figure 1

Consider a free rotation of a symmetric top $I_x = I_y \neq I_z$, where x, y, and z are the principal axes. The direction of the angular momentum does not coincide with the direction of any principle axes. Let's say, that the angle between $\vec{M}$ and the moving axes z at some instant is θ. We chose as the axis x the one that is in plane with the two vectors $\vec{M}$ and $\hat{z}$.

During the motion the total angular momentum is conserved.

The whole motion can be thought as two rotations one the rotation of the body around the axes z and the other, called precession, is the rotation of the axis z around the direction of the vector $\vec{M}$.

We can think of vector $\vec{\Omega}$ in two different ways

$$\vec{\Omega} = \hat{z}\Omega_z + \hat{x}\Omega_x \tag{17.1}$$

$$\vec{\Omega} = \frac{\vec{M}}{M}\Omega_{pr} + \hat{z}\tilde{\Omega}_z \tag{17.2}$$

and angular momentum

$$\vec{M} = \Omega_x I_x \hat{x} + \Omega_z I_z \hat{z}, \tag{17.3}$$

Multiplying (17.3) by $\hat{z}$ (at this instant of time) we get

$$\Omega_z = \frac{M}{I_z}\cos\theta.$$

In order to find the angular velocity of precession we multiply (17.2) and (17.3) by $\hat{x}$ and get

$$\Omega_x = \frac{\hat{x}\cdot\vec{M}}{M}\Omega_{pr} \qquad \text{and} \qquad \hat{x}\cdot\vec{M} = I_x\Omega_x$$

or

$$\Omega_x = \frac{I_x}{M}\Omega_x\Omega_{pr}.$$

This equation has two solutions $\Omega_x = 0$ – which corresponds to $\vec{M} \parallel \hat{z}$, or if $\Omega_x \neq 0$

$$\Omega_{pr} = \frac{M}{I_x}.$$

17.1. Euler's angles

The total rotation of a rigid body is described by three angles. There are different ways to parametrize rotations. Here we consider what is called Euler's angles.

The fixed coordinates are XYZ, the moving coordinates xyz. The plane xy intersects the plane XY along the line ON called the line of nodes.

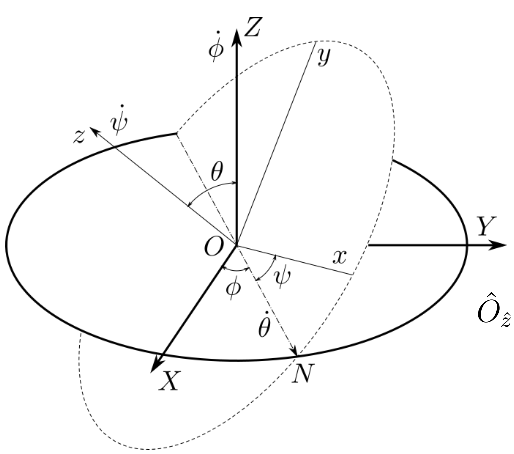


Figure 2

The angle θ is the angle between the Z and z axes. The angle ϕ is the angle between the X axes and the line of nodes, and the angle ψ is the angle between the x axes and the line of nodes.

Initially the axes XYZ and xyz coincide. Let's denote $\hat{O}_{\hat{\xi}}(\alpha)$ a rotation around a unit vector $\hat{\xi}$ on the angle α. Then in order to get the orientation on the picture we need to perform three separate rotations

$$\hat{O}_{\hat{z}}(\psi) \circ \hat{O}_{\hat{Z}}(\phi) \circ \hat{O}_{\hat{X}}(\theta)$$

The angle θ is from 0 to π, the ϕ and ψ angles are from 0 to 2π.

I need to find the components of the angular velocity $\vec{\Omega}$ of in the moving frame and the time derivative of the angles $\dot\theta$, $\dot\phi$, and $\dot\psi$.

(a) The **vector** $\vec{\dot\theta}$ is along the line of nodes, so its components along x, y, and z are $\dot\theta_x = \dot\theta\cos\psi$, $\dot\theta_y = -\dot\theta\sin\psi$, and $\dot\theta_z = 0$.

(b) The **vector** $\vec{\dot\phi}$ is along the Z direction, so its component along z is $\dot\phi_z = \dot\phi\cos\theta$. Its components along x and y are $\dot\phi_y = \dot\phi\sin\theta\cos\psi$, and $\dot\phi_x = \dot\phi\sin\theta\sin\psi$.

(c) The **vector** $\vec{\dot\psi}$ is along the z direction, so $\dot\psi_z = \dot\psi$, and $\dot\psi_x = \dot\psi_y = 0$.

We now collect all angular velocities along each axis as $\Omega_x = \dot\theta_x + \dot\phi_x + \dot\psi_x$ etc. and find

$$\begin{aligned}
\Omega_x &= \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi \\
\Omega_y &= -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi \\
\Omega_z &= \dot\phi\cos\theta + \dot\psi
\end{aligned}$$

These equations allow us to first solve problem in the moving system of coordinates, find Ω_x, Ω_y, and Ω_z, and then calculate $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$.

- Symmetric top again.

Consider the symmetric top again $I_y = I_x$. We take Z to be the direction of the angular momentum. We can take the axis x coincide with the line of nodes. Then $\psi = 0$ (but $\dot{\psi} \neq 0!$), and we have $\Omega_x = \dot{\theta}$, $\Omega_y = \dot{\phi}\sin\theta$, and $\Omega_z = \dot{\phi}\cos\theta + \dot{\psi}$.

The components of the angular momentum are $M_x = I_x\Omega_x = I_x\dot{\theta}$, $M_y = I_y\Omega_y = I_x\dot{\phi}\sin\theta$, and $M_z = I_z\Omega_z$. On the other hand $M_z = M\cos\theta$, $M_x = 0$, and $M_y = M\sin\theta$. Comparing those we find

$$\dot{\theta} = 0, \qquad \Omega_{pr} = \dot{\phi} = \frac{M}{I_x}, \qquad \Omega_z = \frac{M}{I_z}\cos\theta.$$

LECTURE 18
Symmetric top in gravitational field.

The angles are unconstrained and change $0 < \theta < \pi$, $0 < \psi, \phi < 2\pi$.

We want to consider the motion of the symmetric top ($I_x = I_y$) whose lowest point is fixed. We call this point O. The line ON is the line of nodes.

- Line of nodes is an intersection between XY and xy planes.

The Euler angles θ, ϕ, and ψ fully describe the orientation of the top.

Instead of defining the tensor of inertia with respect to the center of mass, we will define it with respect to the point O. The principal axes with trough this point are parallel to the ones through the center of mass. The principal moment I_z does not change under such shift, the principal moment with respect to the axes x and y become by $I = I_x + ml^2$, where l is the distance from the point O to the center of mass.

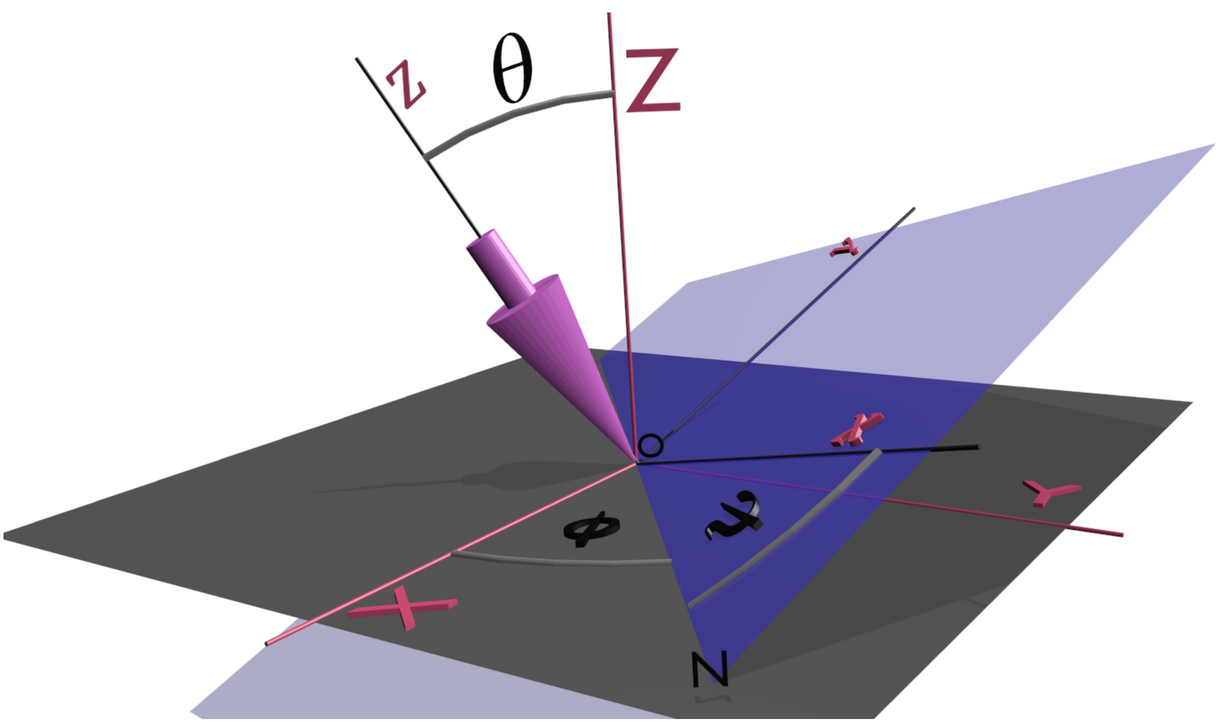


Figure 1

$$\Omega_x = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$
$$\Omega_y = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$
$$\Omega_z = \dot{\phi}\cos\theta + \dot{\psi}$$

The kinetic energy of the symmetric top is

$$K = \frac{I_z}{2}\Omega_z^2 + \frac{I}{2}\left(\Omega_x^2 + \Omega_y^2\right) = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

The potential energy is simply $mgl\cos\theta$, so the Lagrangian is

$$L = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta$$

We see that the Lagrangian does not depend on ϕ and ψ – this is only correct for the symmetric top. The corresponding momenta $M_Z = \frac{\partial L}{\partial \dot{\phi}}$ and $M_3 = \frac{\partial L}{\partial \dot{\psi}}$ are conserved.

$$M_3 = I_z(\dot{\psi} + \dot{\phi}\cos\theta), \qquad M_Z = (I\sin^2\theta + I_z\cos^2\theta)\dot{\phi} + I_z\dot{\psi}\cos\theta.$$

The energy is also conserved

$$E = \frac{I_z}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta.$$

The values of M_Z, M_3, and E are given by the initial conditions.

So we have three unknown functions $\theta(t)$, $\phi(t)$, and $\psi(t)$ and three conserved quantities. The conservation laws then completely determine the whole motion.

From equations for M_Z and M_3 we have

$$\dot{\phi} = \frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$
$$\dot{\psi} = \frac{M_3}{I_3} - \cos\theta\frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$$

We then substitute the values of the $\dot{\phi}$ and $\dot{\psi}$ into the expression for the energy and find

$$E' = \frac{1}{2}I\dot{\theta}^2 + U_{eff}(\theta),$$

where

$$E' = E - \frac{M_3^2}{2I_z} - mgl, \qquad U_{eff}(\theta) = \frac{(M_Z - M_3\cos\theta)^2}{2I\sin^2\theta} - mgl(1 - \cos\theta).$$

This is an equation of motion for a $1D$ motion, so we get

$$t = \sqrt{\frac{I}{2}}\int\frac{d\theta}{\sqrt{E' - U_{eff}(\theta)}}.$$

This is an elliptic integral.

The effective potential energy goes to infinity when $\theta \to 0, \pi$. The function θ oscillates between θ_{min} and θ_{max} which are the solutions of the equation $E' = U_{eff}(\theta)$. These oscillations are called *nutations*. As $\dot{\phi} = \frac{M_Z - M_3\cos\theta}{I\sin^2\theta}$ the motion depends on weather $M_Z - M_3\cos\theta$ changes sign in between θ_{min} and θ_{max}.

We can find a condition for the stable rotation about the Z axes. For such rotation $M_3 = M_Z$, so the effective potential energy is

$$U_{eff} = \frac{M_3^2}{2I}\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} - 2mgl\sin^2(\theta/2) \approx \left(\frac{M_3^2}{8I} - \frac{1}{2}mgl\right)\theta^2,$$

where the last is correct for small θ. We see, that the rotation is stable if $M_3^2 > 4Imgl$, or $\Omega_z^2 > \frac{4Imgl}{I_z^2}$.

LECTURE 19

Rolling coin. An example of the rigid body dynamics.

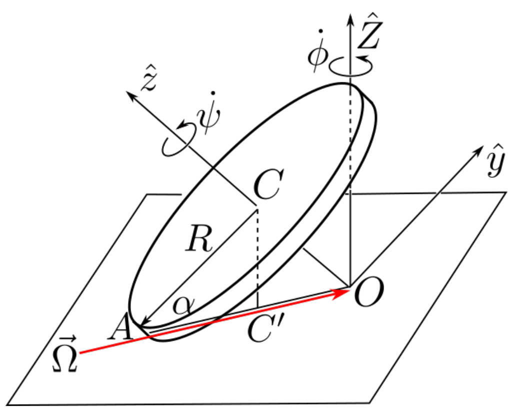


Figure 1

In this lecture we consider the dynamics of the following object:

A uniform thin disc (a coin) of mass M rolls without slipping on a horizontal plane. The disc makes an angle α with the plane, the center of the disc C moves around the horizontal circle with a constant speed v. The axis of symmetry of the disc CO intersects the horizontal plane in the point O. The point O is stationary.

- Intuitively clear, that this problem is overdetermined. We want to see why.

First, some geometrical facts

$$|OC| = R\tan\alpha, \qquad |OA| = \frac{R}{\cos\alpha}, \qquad |OC'| = \frac{R}{\cos\alpha} - R\cos\alpha = R\frac{\sin^2\alpha}{\cos\alpha}.$$

We chose the internal system of coordinates xyz as shown on the figure. In this system the principal moments of inertia are

$$I_z = \frac{1}{2}MR^2, \qquad I_y = I_x = \frac{1}{4}MR^2 + M|OC|^2 = MR^2\left(\frac{1}{4} + \tan^2\alpha\right)$$

19.1. Kinematics.

Simple way. According to the problem statement the points O and A are stationary at this moment. So the are on the *instantaneous* axis of rotation. It means that the vector $\vec{\Omega}$ is along this axis.

The point C has a velocity v. For any point $\vec{r}$ of a rotating body the velocity is $\vec{v} = \vec{\Omega}\times\vec{r}$. So we see, that $v = \Omega|OC|\cos\alpha$, or

$$\Omega = \frac{v}{R}\frac{1}{\sin\alpha}.$$

So we know both the direction and the magnitude of the vector $\vec{\Omega}$.

In the internal system of coordinates xyz we then have

$$\Omega_z = -\Omega\sin\alpha = -\frac{v}{R}, \qquad \Omega_y = \Omega\cos\alpha = \frac{v}{R}\frac{\cos\alpha}{\sin\alpha}.$$

Euler angles. We can find the same result from the Euler angles. As this is symmetric top, we can set $\psi = 0$, but we need to know θ, $\dot\theta$, $\dot\phi$, and $\dot\psi$. According to the figure the Euler angle $\theta = \alpha$, so $\dot\theta = 0$. As the velocity of the point C is v, we can write $\dot\phi|OC'| = v$, so
$$\dot\phi = \frac{v}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$
The Euler $\dot\psi$ must be found from non-slipping condition. This condition means that the velocity of the point A is zero. So we can write $\dot\phi|OA| + \dot\psi R = 0$. So
$$\dot\psi = -\frac{\dot\phi}{\cos\alpha} = -\frac{v}{R}\frac{1}{\sin^2\alpha}.$$
(the $-$ sign is important here!)

Now we use our relations:
$$\begin{aligned}
\Omega_x = \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi = &\quad 0\\
\Omega_y = -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi = &\quad \frac{v}{R}\frac{\cos\alpha}{\sin\alpha}\\
\Omega_z = \dot\phi\cos\theta + \dot\psi = \tfrac{v}{R}\tfrac{\cos^2\alpha}{\sin^2\alpha} - \tfrac{v}{R}\tfrac{1}{\sin^2\alpha} = &\quad -\frac{v}{R}
\end{aligned}$$
These are exactly the results we got earlier.

19.2. Dynamics.

The main dynamic equations are $\frac{d\vec M}{dt} = \vec\tau$ and $\vec F = M\vec a$. Let's start with the angular momentum.

In the internal system of coordinates the angular momentum is a constant vector
$$\vec M = I_z\Omega_z\hat z + I_y\Omega_y\hat y.$$
However, the whole internal system of coordinates is rotating around $\hat Z$ with frequency $\dot\phi$, so $\dot{\hat z} = \dot\phi\hat Z\times\hat z$ and $\dot{\hat y} = \dot\phi\hat Z\times\hat y$. Using $\hat Z = \hat y\sin\alpha + \hat z\cos\alpha$ and $\hat y\times\hat z = \hat x$ we get
$$\dot{\vec M} = I_z\Omega_z\dot{\hat z} + I_y\Omega_y\dot{\hat y} = \dot\phi\left(I_z\Omega_z\sin\alpha - I_y\Omega_y\cos\alpha\right)\hat x = -\frac{v^2}{R^2}\frac{\cos\alpha}{\sin^3\alpha}\left(I_z\sin^2\alpha + I_y\cos^2\alpha\right)\hat x.$$

There are three forces that act on the coin: the gravity Mg applied to the point C, pointing down; the normal force N applied to the point A and pointing up; and the friction force F applied to the point A and pointing towards point O. As the center of mass does not move in the Z direction, the normal force and the gravity must compensate each other, so $N = Mg$ (N is up). We want to compute the total torque with respect to point O acting on the coin. The torque of a force $\vec F$ applied at point $\vec r$ is $\vec\tau = \vec r\times\vec F$. So the torque of the friction force is zero. The torque of the gravity is $\vec\tau_g = -\vec{OC}\times Mg\hat Z = -RMg\tan\alpha\hat z\times\hat Z = \hat x RMg\tan\alpha\sin\alpha$. The torque of the normal force is $\vec\tau_N = -Mg|OA|\hat x$. So the total torque is
$$\vec\tau = RMg\left(\frac{\sin^2\alpha}{\cos\alpha} - \frac{1}{\cos\alpha}\right)\hat x = -MgR\cos\alpha\hat x.$$
Notice, that this result would be much easier to obtain if we simply computed the torques with respect to point A, but this is not a trivial statement, as point A is not inertial.

Thus we have
$$\frac{v^2}{R^2}\frac{\cos\alpha}{\sin^3\alpha}\left(I_z\sin^2\alpha + I_y\cos^2\alpha\right) = MgR\cos\alpha$$

Substituting here The values of I_y and I_z we get

$$\frac{1}{4}\frac{v^2}{R}\frac{\cos\alpha}{\sin^3\alpha}\left(1+5\sin^2\alpha\right)=g\cos\alpha.$$

- Notice, that $\alpha=\pi/2$ (or $\cos\alpha=0$) is a solution for any v and R – as expected.

For $\cos\alpha\neq 0$ we get

$$\frac{1}{4}\frac{v^2}{R}\left(1+5\sin^2\alpha\right)=g\sin^3\alpha.$$

- So v, R, and α cannot be arbitrary!!!

19.3. Friction force.

The center of mass of the coin moves around the circle of radius $|OC'|$ with velocity v, so its acceleration is $\frac{v^2}{|OC'|}=\frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}$ The force that provides this acceleration is the friction force, so

$$F=M\frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

However, this force cannot be larger than $\mu N=\mu Mg$, so we have

$$\mu g>\frac{v^2}{R}\frac{\cos\alpha}{\sin^2\alpha}.$$

Using the previous result $\frac{v^2}{Rg}=\frac{4\sin^3\alpha}{1+5\sin^2\alpha}$ we get

$$\frac{4\cos\alpha\sin\alpha}{1+5\sin^2\alpha}<\mu$$

19.4. Lagrangian dynamics.

In order to study the full Lagrangian dynamics of the coin we first must assume, that the non-slipping condition is valid at all times – otherwise we will have dissipation and the Lagrangian method would not work. Second we should not assume that the angle α is constant in time, so the kinetic energy term will have $\dot{\alpha}$.

But there is more subtle issue. Our coordinate are α, ϕ, ψ, X, and Y, where X and Y are the position of the coin center of mass (its Z coordinate is $R\cos\alpha$ – not an independent variable.) The Lagrangian will not explicitly depend on X and Y, it only depends on the velocity v^2. However, the velocity is also not independent and is given by $\dot{\phi}$. So there are only three variables – the Euler angles.

LECTURE 20
Motion of a rigid body. Euler equations. Stability of asymmetric top.

20.1. Euler equations.

Let's write the vector $\vec{M}$ in the following form

$$\vec{M} = I_x\Omega_x\hat{x} + I_y\Omega_y\hat{y} + I_z\Omega_z\hat{z}.$$

I want to use the fact that the angular momentum is conserved $\dot{\vec{M}} = 0$. In order to differentiate the above equation I need to use $\dot{\hat{x}} = \vec{\Omega} \times \hat{x}$ etc, then

$$0 = \dot{\vec{M}} = I_x\dot{\Omega}_x\hat{x} + I_y\dot{\Omega}_y\hat{y} + I_z\dot{\Omega}_z\hat{z} + I_x\Omega_x\vec{\Omega} \times \hat{x} + I_y\Omega_y\vec{\Omega} \times \hat{y} + I_z\Omega_z\vec{\Omega} \times \hat{z}.$$

Multiplying the above equation by $\hat{x}$, will find

$$0 = I_x\dot{\Omega}_x + I_y\Omega_y\vec{\Omega} \cdot [\hat{y} \times \hat{x}] + I_z\Omega_z\vec{\Omega} \cdot [\hat{z} \times \hat{x}],$$

or

$$I_x\dot{\Omega}_x = (I_y - I_z)\Omega_y\Omega_z.$$

Analogously for $\hat{y}$ and $\hat{z}$, and we get the Euler equations:

$$\begin{aligned} I_x\dot{\Omega}_x &= (I_y - I_z)\Omega_y\Omega_z \\ I_y\dot{\Omega}_y &= (I_z - I_x)\Omega_z\Omega_x \\ I_z\dot{\Omega}_z &= (I_x - I_y)\Omega_x\Omega_y \end{aligned}$$

One can immediately see, that the energy is conserved.

For a symmetric top $I_y = I_x$ we find that $\Omega_z = \text{const.}$, then denoting $\omega = \Omega_z\frac{I_z - I_x}{I_x}$ we get

$$\begin{aligned} \dot{\Omega}_x &= -\omega\Omega_y \\ \dot{\Omega}_y &= \omega\Omega_x \end{aligned}$$

The solution is

$$\Omega_x = A\cos\omega t, \qquad \Omega_y = A\sin\omega t.$$

So the vector $\vec{\Omega}$ rotates around the z axis with the frequency ω. So does the vector $\vec{M}$ – this is the picture in the moving frame of reference. It is the same as the one before.

Let's check it. Using Euler angles we can write with respect to some external system of coordinates.

$$\begin{aligned}
\Omega_x &= \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi = A\cos\omega t \\
\Omega_y &= -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi = A\sin\omega t \\
\Omega_z &= \dot\phi\cos\theta + \dot\psi = \Omega_z
\end{aligned}$$

Multiplying the first equation by $\cos\psi$, the second by $\sin\psi$ and subtracting one from another we get

$$\dot\theta = A\cos(\psi + \omega t).$$

The external system is not well defined and we have a freedom to chose one which is the most convenient. So let's chose the one where $\dot\theta = 0$. Such system does not necessarily exist, so we must check that this guess is consistent with the rest of full set of equations.

The requirement $\dot\theta = 0$ means

$$\psi = \pi/2 - \omega t.$$

The first two equations then give the same relation

$$\dot\phi\sin\theta = A$$

and the third one gives

$$\dot\phi\cos\theta = \Omega_z + \omega = \Omega_z\frac{I_z}{I_x},$$

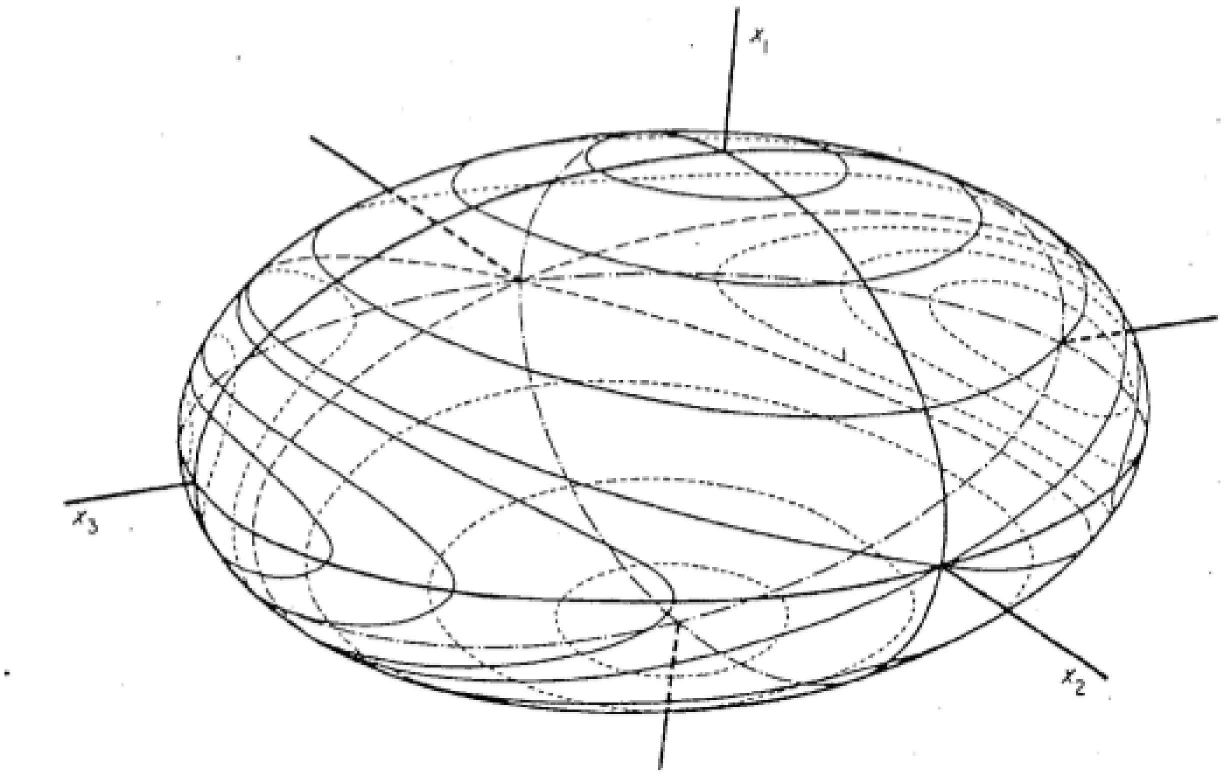
so we see, that from these two equations $\tan\theta = \frac{AI_x}{\Omega_z I_z}$ is indeed a constant, so $\dot\theta = 0$ is consistent. Moreover, as $A = \sqrt{\Omega_x^2 + \Omega_y^2} = \Omega_\perp$ we see, that $\tan\theta = \frac{I_x\Omega_\perp}{I_z\Omega_z} = \frac{M_\perp}{M_z}$ as it should be, because $\vec M$ is constant.

We also see, that

$$\Omega_{pr} = \dot\phi = \frac{\Omega_z I_z}{\cos\theta}\frac{1}{I_x} = \frac{M_z}{\cos\theta}\frac{1}{I_x} = \frac{M}{I_x}$$

20.2. Stability of the free rotation of a asymmetric top.

- Different meaning of stability. Static stability and dynamic stability.



Conservation of energy and the magnitude of the total angular momentum read

$$\frac{I_x\Omega_x^2}{2} + \frac{I_y\Omega_y^2}{2} + \frac{I_z\Omega_z^2}{2} = E$$
$$I_x^2\Omega_x^2 + I_x^2\Omega_x^2 + I_x^2\Omega_x^2 = M^2$$

In terms of the components of the angular momentum these equations read

$$\frac{M_x^2}{2I_x} + \frac{M_y^2}{2I_y} + \frac{M_z^2}{2I_z} = E$$
$$M_x^2 + M_y^2 + M_z^2 = M^2$$

The first equation describes an ellipsoid with the semiaxes $\sqrt{2I_xE}$, $\sqrt{2I_yE}$, and $\sqrt{2I_zE}$. The second equation describes a sphere of a radius M. The initial conditions give us E and M, the true solution must satisfy the conservation lows at all times. So the vector $\vec{M}$ will lie on the lines of intersection of the ellipsoid, and sphere. Notice, how different these lines

LECTURE 21
Statics.

Static conditions:

- Sum of all forces is zero. $\sum \vec{F}_i = 0$.
- Sum of all torques is zero: $\sum \vec{r}_i \times \vec{F}_i = 0$.

If the sum of all forces is zero, then the torque condition is independent of where the coordinate origin is.

$$\sum(\vec{r}_i + \vec{a}) \times \vec{F}_i = \sum \vec{r}_i \times \vec{F}_i + \vec{a} \times \sum \vec{F}_i$$

Examples

- A bar on two supports.
- A block with two legs moving on the floor with μ_1 and μ_2 coefficients of friction.
- A ladder in a corner.

A problem for students in class:

- A bar on three supports.

Elastic deformations:

- Continuous media. Scales.
- Small, only linear terms.
- No nonelastic effects.
- Static.
- Isothermal.

Definition of derivatives.

LECTURE 22
Strain and Stress.

22.1. Strain

Let the unstrained lattice be given positions x_i and the strained lattice be given positions $x'_i = x_i + u_i$. The distance dl between two points in the unstrained lattice is given by $dl^2 = dx_i^2$. The distance dl'^2 between two points in the strained lattice is given by

$$\begin{aligned} dl'^2 = dx_i'^2 = (dx_i + du_i)^2 &= dx_i^2 + 2dx_i du_i + du_i^2 \\ &= dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k} dx_j dx_k \\ &= dl^2 + 2u_{ik} dx_i dx_k, \end{aligned} \tag{22.1}$$

where

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right) \tag{22.2}$$

Normally we will take only the case of small strains, for which

$$u_{ik} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \tag{22.3}$$

Can diagonalize the real symmetric u_{ik}, and get orthogonal basis set. In that local frame $(1,2,3)$ have $dx'_1 = dx_1(1 + u_{11})$, etc. Hence the new volume is given by

$$\begin{aligned} dV' = dx'_1 dx'_2 dx'_3 &\approx dx_1 dx_2 dx_3 (1 + u_{11} + u_{22} + u_{33}) \\ &= dV(1 + u_{ii}), \end{aligned} \tag{22.4}$$

where the trace u_{ii} is invariant to the coordinate system used. Hence the fractional change in the volume is given by

$$\frac{\delta(dV)}{dV} = u_{ii}. \tag{22.5}$$

We see also, that in the *linerar* approximation (22.3) if we write

$$dx'_i = (\delta_{ij} + u_{ij})\, dx_j$$

and compute dl'^2 we will reproduce (22.1), so we have

$$du_i = dx'_i - dx_i = u_{ij} dx_j,$$

or $u_i = \int_\Gamma u_{ij} dx_j$.

22.2. Stress

The forces are considered to be short range.

Consider a volume V that is acted on by internal stresses. The force on it due to the internal stresses is given by

$$\mathcal{F}_i = \int \frac{d\mathcal{F}_i}{dV} dV = \int F_i dV. \tag{22.6}$$

However, because the forces are short-range it should also be possible to write them as an integral over the surface element $dS_i = n_i dS$, where $\hat{n}$ is the outward normal (L&L use df_i for the surface element). Thus we expect that

$$\mathcal{F}_i = \int \sigma_{ij} dS_j \tag{22.7}$$

for some σ_{ij}. Thinking of it as a set of three vectors (labeled by i) with vector index j, we can apply Gauss's Theorem to rewrite this as

$$\mathcal{F}_i = \int \frac{\partial \sigma_{ij}}{\partial x_j} dV, \tag{22.8}$$

so comparison of the two volume integrals gives

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{22.9}$$

Because there are no self-forces (by Newton's Third Law), these forces must come from material that is outside V.

In equilibrium when only the internal stresses act we have $F_i = \frac{\partial \sigma_{ij}}{\partial x_j} = 0$. If there is a long-range force, such as gravity acting, with force $F_i^g = \rho g_i$, where ρ is the mass density and g_i is the gravitational field, then in equilibrium $F_i + F_i^g = 0$. This latter case is important for objects with relatively small elastic constant per unit mass, because then they must distort significantly in order to support their weight.

When no surface force is applied, the stress at the surface is zero. When there is a surface force P_i per unit area, this determines the stress force $\sigma_{ij}\hat{n}_j$, so

$$P_i = \sigma_{ij}\hat{n}_j \tag{22.10}$$

If the surface force is a pressure, then $P_i = -P\hat{n}_i = \sigma_{ij}\hat{n}_j$. The only way this can be true for any $\hat{n}$ is if

$$\sigma_{ij} = -P\delta_{ij}. \tag{22.11}$$

Just as the force due to the internal stresses should be written as a surface integral, so should the torque. Each of the three torques is an antisymmetric tensor, so we consider

$$\begin{aligned} M_{ik} &= \int (F_i x_k - F_k x_i) dV = \int \left(\frac{\partial \sigma_{ij}}{\partial x_j} x_k - \frac{\partial \sigma_{kj}}{\partial x_j} x_i \right) dV \\ &= \int \left(\frac{\partial(\sigma_{ij} x_k)}{\partial x_j} - \frac{\partial(\sigma_{kj} x_i)}{\partial x_j} - (\sigma_{ik} - \sigma_{ki}) \right) dV \\ &= \int (\sigma_{ij} x_k - \sigma_{kj} x_i) dS_j - \int (\sigma_{ik} - \sigma_{ki}) dV. \end{aligned} \tag{22.12}$$

To eliminate the volume term we require that

$$\sigma_{ik} = \sigma_{ki}. \tag{22.13}$$

LECTURE 23
Work, Stress, and Strain.

This lecture we want to find a connection between the stress and the strain tensors. We are working in a linear and local approximation, so the connection must be linear and have a form

$$u^{ij} = D^{ijkl}\sigma^{kl},$$

where D^{ijkl} is a material dependent local tensor of the fourth rank. Not all elements of this tensor are independent especially if the material is uniform and isotropic. Moreover, the requirement of stability of the equilibrium (when $u^{ij} = 0$) must lead to some inequalities between the elements of the tensor D. As it is a tensor of the fourth rank it is very tedious to analyze. So instead we will work around it.

23.1. Work against Internal Stresses

Let's imagine an experiment when we want to slightly change the field $\vec{u}(\vec{r})$ while keeping the shape of the object intact. In order to do that we have to do work against the internal stresses. The force we need to apply to a piece of volume dV is $-F_i$ (where F_i is the force due to the internal stresses) So we need to do the work $\delta R_{our} = -F_i\delta u_i$. The internal forces then do the work $\delta R = F_i\delta u_i$. Hence the total work done by the internal stresses is given by

$$\begin{aligned}\delta W &= \int \delta R dV = \int F_i \delta u_i dV = \int \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i dV \\ &= \int \frac{\partial(\sigma_{ij}\delta u_i)}{\partial x_j} dV - \int \sigma_{ij}\frac{\partial(\delta u_i)}{\partial x_j} dV. \qquad (23.1)\end{aligned}$$

If we transform the first integral to a surface integral, by Gauss's Theorem, and take $\delta u_i = 0$ on the surface — we fix the boundary, we do not change the shape of the object, then we eliminate the first term. If we use the symmetry of σ_{ik} and the small-amplitude form of the strain, then the last term can be rewritten so that we deduce that

$$\delta R = -\sigma_{ik}\delta u_{ki}. \qquad (23.2)$$

23.1.1. Thermodynamics

We now assume the system to be in thermodynamic equilibrium. Using the energy density $d\epsilon$ and the entropy density s, the first law of thermodynamics gives

$$d\epsilon = T ds - dR = T ds + \sigma_{ik} du_{ki}. \qquad (23.3)$$

Defining the free energy density $F = \epsilon - Ts$ we have

$$dF = -sdT + \sigma_{ik}du_{ki}. \tag{23.4}$$

In the next section we consider the form of the free energy density as a function of T and u_{ik}.

23.2. Elastic Energy

- An example with a spring (or compressed stick). $F = -kx$, considering doubling of the spring in width and then in length we find $F = \frac{SE}{L}x$, so $\frac{F}{S} = E\frac{x}{L}$. $F/S \to$stress, $x/L \to$ strain. On the other hand, from symmetry $x \to -x$ we see, that the energy $\mathcal{E} \sim x^2$, or $\mathcal{E} = \frac{kx^2}{2}$ and then the force by the spring $F = -\frac{d\mathcal{E}}{dx} = -kx$.

The elastic equations must be linear, as this is the accuracy which we work with. The energy density then must be quadratic in the strain tensor. We thus need to construct a scalar out of the strain tensor in the second order. If we assume that the body is isotropic, then the only way to do that is:

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ik}^2. \tag{23.5}$$

- Notice, that in this approach instead of working with the tensor of the fourth rank we are working with a scalar – free energy.

Here λ and μ are the only parameters (in the isotropic case). They are called *Lamé coefficients,* and in particular μ is called the *shear modulus* or *modulus of rigidity.* Note that u_{ii} is associated with a volume change, by (22.5). The quantity

$$\tilde{u}_{ik} = u_{ik} - \frac{1}{3}\delta_{ik}u_{jj} \tag{23.6}$$

satisfies $\tilde{u}_{ii} = 0$, and is said to describe a pure shear.

With this definition we have

$$u_{ik} =\tilde{u}_{ik} + \frac{1}{3}\delta_{ik}u_{jj} \tag{23.7}$$

$$u_{ik}^2 =\tilde{u}_{ik}^2 + \frac{2}{3}\tilde{u}_{ii}u_{kk} + \frac{1}{3}u_{jj}^2 = \tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2. \tag{23.8}$$

Hence (23.5) becomes

$$F =F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu(\tilde{u}_{ik}^2 + \frac{1}{3}u_{jj}^2) = F_0 + \frac{1}{2}Ku_{ii}^2 + \mu\tilde{u}_{ik}^2. \qquad (K \equiv \lambda + \frac{2}{3}\mu) \tag{23.9}$$

In this form the two elastic terms are independent of one another. For the elastic energy to correspond to a stable system, each of them must be positive, so $K > 0$ and $\mu > 0$.

23.2.1. Stress

On varying u_{ik} at fixed T the free energy of (23.9) changes by

$$\begin{aligned} dF =& Ku_{ii}du_{kk} + 2\mu\tilde{u}_{ik}d\tilde{u}_{ik} = Ku_{ii}du_{kk} + 2\mu\tilde{u}_{ik}(du_{ik} - \frac{1}{3}\delta_{ik}du_{jj}) \\ =& Ku_{ii}du_{kk} + 2\mu\tilde{u}_{ik}du_{ik} = Ku_{jj}\delta_{ik}du_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}\right)du_{ik}, \end{aligned} \tag{23.10}$$

so comparison with (23.4) gives

$$\sigma_{ik} = K u_{jj} \delta_{ik} + 2\mu (u_{ik} - \frac{1}{3}\delta_{ik} u_{jj}). \tag{23.11}$$

Note that $\sigma_{jj} = 3K u_{jj}$, so that

$$u_{jj} = \frac{\sigma_{jj}}{3K}.$$

We now solve (24.8) for u_{ik}:

$$\begin{aligned} u_{ik} =& \frac{1}{3}\delta_{ik} u_{jj} + \frac{\sigma_{ik} - K u_{jj}\delta_{ik}}{2\mu} \\ =& \frac{\sigma_{ik}}{2\mu} + \delta_{ik}\left(\frac{1}{3} - \frac{K}{2\mu}\right)\frac{\sigma_{jj}}{3K} \\ =& \delta_{ik}\frac{\sigma_{jj}}{9K} + \frac{1}{2\mu}\left(\sigma_{ik} - \frac{1}{3}\sigma_{jj}\delta_{ik}\right). \end{aligned}$$

In the above the first term has a finite trace and the second term has zero trace.

LECTURE 24
Elastic Modulus'.

Results of last lecture:

$$\sigma_{ik} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}). \tag{24.1}$$

$$u_{ik} = \delta_{ik}\frac{\sigma_{jj}}{9K} + \frac{1}{2\mu}\left(\sigma_{ik} - \frac{1}{3}\sigma_{jj}\delta_{ik}\right). \tag{24.2}$$

Taking the trace of either equation we get

$$u_{jj} = \frac{\sigma_{jj}}{3K}. \tag{24.3}$$

This lecture is about the physical meaning of the elastic constants.

24.1. Bulk Modulus and Young's Modulus

24.1.1. Hydrostatic compression.

For hydrostatic compression the force on a small tile is always perpendicular to that tile, so the vector of force and the vector area of the tile have the exactly opposite directions. It means that $\sigma_{ik} = -P\delta_{ik}$, so (24.3) gives

$$u_{jj} = -\frac{P}{K}. \qquad \text{(hydrostatic compression)} \tag{24.4}$$

We can think of this as being a δu_{jj} that gives a $\delta V/V$, by (22.5), due to $P = \delta P$, so

$$\frac{1}{K} = -\frac{\delta u_{jj}}{\delta P} = -\left.\frac{1}{V}\frac{\partial V}{\partial P}\right|_T. \tag{24.5}$$

So K is inverse isothermal compressibility $\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ as defined in thermodynamics.

24.1.2. Uni-direction compression.

Now let there be a compressive for per unit area P along z axis for a system with normal along z, so that $\sigma_{zz} = -P$, but $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{ii} = -P$. By (24.2) we have $u_{ik} = 0$ for $i \neq k$, and

$$u_{xx} = u_{yy} = \frac{P}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right), \tag{24.6}$$

$$u_{zz} = -\frac{P}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right) = -\frac{P}{E}, \qquad E \equiv \frac{9K\mu}{3K+\mu}. \tag{24.7}$$

Notice, that for positive pressure (compression) u_{zz} is always negative, as both $K > 0$ and $\mu > 0$, and hence $E > 0$.

The coefficient of P is called the *coefficient of extension.* Its inverse E is called *Young's modulus,* or the *modulus of extension.*

In particular a spring constant can be found by

$$\Delta z = u_{zz}L = -\frac{PL}{E} = -\frac{L}{AE}F, \qquad k = \frac{AE}{L}$$

24.1.3. Poisson's ratio.

For the previous experiment we can define *Poisson's ratio* σ via

$$u_{xx} = -\sigma u_{zz}. \tag{24.8}$$

Then we find that

$$\sigma = -\frac{u_{xx}}{u_{zz}} = \frac{\left(\frac{1}{2\mu} - \frac{1}{3K}\right)}{\left(\frac{1}{3K} + \frac{1}{\mu}\right)} = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}. \tag{24.9}$$

Since K and μ are positive, the maximum value for σ is $\frac{1}{2}$ and the minimum value is -1. All materials in Nature (except some) have $\sigma > 0$.

Notice, that the volume is changing by $\frac{\delta dV}{dV} = u_{ii} = u_{zz}(1-2\sigma)$, so if $\sigma = 1/2$ the volume does not change – incompressible liquid. The requirements that when we compress the volume cannot increase is the requirement that $\sigma < 1/2$.

Often one uses E and σ instead of K and μ. We leave it to the reader to show that

$$\lambda = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}, \tag{24.10}$$

$$\mu = \frac{E}{2(1+\sigma)}, \tag{24.11}$$

$$K = \frac{E}{3(1-2\sigma)}. \tag{24.12}$$

24.2. Twisted rod.

Let's take a circular rod of radius a and length L and twist its end by a small angle θ. We want to calculate the torque required for that.

- We first guess the right solution.

Two cross-section a distance dz from each other are twisted by the angle $\frac{\theta}{L}dz$ with respect to each other. So a point at distance r from the center on the cross-section at $z + dz$ is shifted by the vector $d\vec{u} = r\frac{\theta}{L}dz\vec{e}_\phi$ in comparison to that point in the cross-section at z. We thus see that the strain tensor is

$$u_{z\phi} = u_{\phi z} = \frac{1}{2}\frac{du_\phi}{dz} = \frac{1}{2}r\frac{\theta}{L}$$

and all other elements are zero.

The relation between u_{ij} and σ_{ij} is local, so we can write them in any local system of coordinates. So as the strain tensor is trace-less

$$\sigma_{z\phi} = \sigma_{\phi z} = \mu r \frac{\theta}{L}.$$

and all other elements are zeros.

- Notice, that for that stress tensor $\frac{\partial \sigma_{z\phi}}{\partial z} = \frac{\partial \sigma_{z\phi}}{\partial \phi} = 0$, so the condition of equilibrium is satisfied and our guess was right.

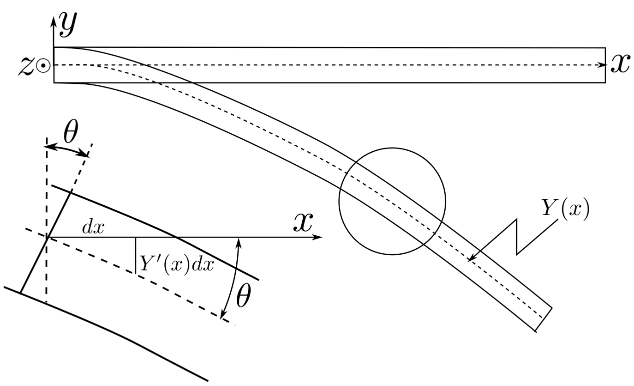
Now we calculate the torque on we need to apply to the end. To a small area ds at a point at distance r from the end we need to apply a force $dF_\phi \vec{e}_\phi = \sigma_{\phi z} dS \vec{e}_\phi$. The torque of this force with respect to the center is along z direction and is given by $d\tau = rF_\phi = r\sigma_{\phi z} dS$. So the total torque is

$$\tau = \int r\sigma_{\phi z} dS = \int_0^a r\mu r \frac{\theta}{L} r dr d\phi = \mu \frac{\theta}{L} \int_0^a r^3 dr d\phi = \frac{\pi}{2} \frac{\mu}{L} a^4 \theta.$$

So we can measure μ in this experiment by the following way

(a) Prepare rods of different radii and lengths.
(b) For each rod measure torque τ as a function of angle θ.
(c) For each rod plot τ as a function of θ. Verify, that for small enough angle τ/θ does not depend on θ and is just a constant. This constant is a slope of each graph at small θ.
(d) Plot this constant as a function of $\frac{\pi a^4}{2L}$. Verify, that the points are on a straight line for small $\frac{\pi a^4}{2L}$. The slope of this line at small $\frac{\pi a^4}{2L}$ is the sheer modulus μ.
(e) One can also measure $\frac{\pi a^4}{2L}$ by measuring the frequency of oscillations of a disk on known moment of inertia hanged on a thread.

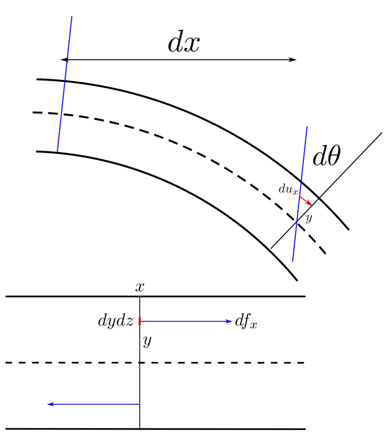
LECTURE 25
Small deformation of a beam.



Let's consider a small deformation of a (narrow) beam with rectangular cross-section under gravity.

- x coordinate is along undeformed beam, y is perpendicular to it, pointing up.
- Nothing depends on z. $\hat{z}$ points towards us.
- Part of the beam is compressed, part is stretched.
- *Neutral surface.* The coordinates of the neutral surface is $Y(x)$.
- Deformation is small, $|Y'(x)| \ll 1$.

Under these conditions the angle $\theta(x) \approx Y'(x)$. So the change of the angle $\theta(x)$ between two near points is $d\theta = Y''(x)dx$.



The neutral surface is neither stretched, nor compressed. The line which is a distance y from this surface is stretched (compressed) in x direction by $du_x = y d\theta = yY''dx$, so we have

$$u_{xx} = \frac{\partial u_x}{\partial x} = y\frac{\partial^2 Y(x)}{\partial x^2}.$$

- The stretching (compression) proportional to the second derivative, as the first derivative describes the uniform rotation of the beam.

There is no confining in the y or z directions, so we find that

$$\sigma_{xx} = -Eu_{xx} = -Ey\frac{\partial^2 Y(x)}{\partial x^2}.$$

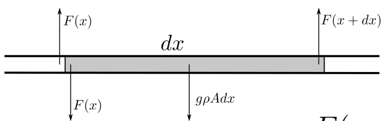
Consider a cross-section of the beam at point x. The force in the x direction of the $dydz$ element of the beam is $df_x = \sigma_{xx}dzdy$. The torque which acts from the **right** part on the

left is

$$\tau(x)=\int y\sigma_{xx}dydz=-E\frac{\partial^2Y(x)}{\partial x^2}\int y^2dzdy=-IAE\frac{\partial^2Y(x)}{\partial x^2},\qquad I=\frac{\int y^2dydz}{\int dydz}.$$

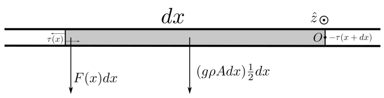
(A is the cross-section area.) This torque is in the negative $\hat{z}$ direction.

The beam is at equilibrium. So if we take a small portion of it, between x and $x+dx$, the total force and torque on it must be zero. Let's consider these two conditions one by one:



Force: Let's say that the y component of the force on the cross-section at x with which the **right** side is acting on the **left** is $F(x)$.

$$-F(x+dx)+F(x)+\rho gAdx=0,\qquad \frac{\partial F}{\partial x}=\rho gA.$$



Torque: The total torque (with respect to the point $x+dx$, positive is counterclockwise) acting on this portion is

$$-\tau(x+dx)+\tau(x)+F(x)dx+\frac{1}{2}m\rho gA(dx)^2=0,\qquad \frac{\partial\tau}{\partial x}=F(x).$$

From these equations we find

$$\frac{\partial^2\tau}{\partial x^2}=\frac{\partial F}{\partial x}=\rho gA,\qquad IAE\frac{\partial^4Y(x)}{\partial x^4}=-\rho gA.$$

The general solution of this equation is simply

$$Y(x)=-\frac{\rho g}{24IE}x^4+\frac{C_3}{6}x^3+\frac{C_2}{2}x^2+C_1x+C_0.$$

$$\tau(x)=-IAE\frac{\partial^2Y(x)}{\partial x^2},\qquad \text{along } -\hat{z} \text{ direction}$$

$$F(x)=-IAE\frac{\partial^3Y(x)}{\partial x^3},\qquad \text{along } +\hat{y} \text{ direction} \tag{25.1}$$

Both the force and the torque is from the **right** on the **left** side of a cross-section at x.

- The constants must be found from the boundary conditions.

25.1. A beam with free end. A diving board.

We need to determine four unknown constants. C_0, C_1, C_2, and C_3.

We take $y=0$ at $x=0$ — fixing the position of one end — which gives $C_0=0$. Another condition is that at $x=0$ the board is horizontal – **the end is clamped** ,

$$Y'(x=0)=0$$

This determines $C_1=0$.

At the other end (distance L) both the force and the torque are zero — it is a free end, so we get the conditions

$$F(x=L)=\left.\frac{\partial^3Y(x)}{\partial x^3}\right|_{x=L}=0,\qquad \tau(x=L)=\left.\frac{\partial^2Y(x)}{\partial x^2}\right|_{x=L}=0.$$

These two conditions will define $C_3=\frac{\rho g}{IE}L$ and $C_2=-\frac{\rho g}{2IE}L^2$.

$$Y(x)=-\frac{\rho g}{24IE}x^2\left(x^2-4xL+6L^2\right).$$

In particular,

$$Y(x = L) = -\frac{\rho g}{8IE} L^4.$$

Notice the proportionality to the fourth power.

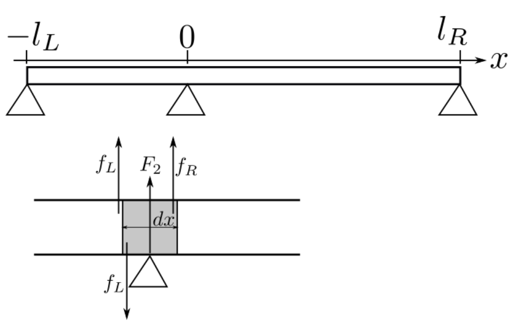
Different modes for the boundary conditions.

- Clamped.
- Supported.
- Free.

LECTURE 26
A rigid beam on three supports.

26.1. The force on the middle.

Consider an absolutely rigid $E = \infty$ horizontal beam with its ends fixed. Let's see how the force on the central support changes as a function of height h of this support. For $h < 0$ the force is zero. For $h > 0$ the force is infinite and $h \to 0_-$ and $h \to 0_+$ are very different. So the situation is unphysical. It means that the order of limits first $E \to \infty$ and then $h \to 0$ is wrong. We need to take the limits in the opposite order: first take $h = 0$ and then $E \to \infty$. In this order the limits are well defined. So we need to solve the static horizontal beam on three supports for large, but finite E and then take the limit $E \to \infty$ at the very end, when we already know the solution. Luckily we know how to solve this problem for large E!



The beam is of length L. The central support has a coordinate $x = 0$ and is at the distance l_L from the left end and at the distance l_R from the right end ($l_R + l_L = L$).

The central support exerts a force F_2 on the beam. This force is at a single point.

- It means that there is a jump in the internal elastic forces at $x = 0$.

We then need to consider the shape of the beam to be given by two functions: $Y_L(x)$ and $Y_R(x)$. As all supports are at the same height we must have $Y_L(x = 0) = Y_L(x = -l_L) = Y_R(x = 0) = Y_R(x = l_R) = 0$, so

$$Y_L = -\tfrac{\rho g}{24IE}x(x + l_L)\left(x^2 + C_1^L x + C_0^L\right) \qquad \text{for } -l_L < x < 0$$
$$Y_R = -\tfrac{\rho g}{24IE}x(x - l_R)\left(x^2 + C_1^R x + C_0^R\right) \qquad \text{for } 0 < x < l_R$$

In order to find the force *from the middle support on the beam* F_2, lets consider a small (length dx) element right on top of the middle support. The sum of all forces must be zero, so we get

$$F_2 + f_R - f_L - \rho A g dx = 0.$$

Taking the limit $dx \to 0$ we find

$$F_2 = f_L - f_R = IAE\left(\left.\frac{d^3Y_R}{dx^3}\right|_{x=0} - \left.\frac{d^3Y_L}{dx^3}\right|_{x=0}\right) = -\frac{\rho g A}{4}\left(C_1^R - C_1^L - l_R - l_L\right).$$

Check the units.

The boundary conditions are

- The beam is smooth at $x=0$: $\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0}$.
- The torques on both ends are zero, $\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=-l_L} = \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=l_R} = 0$.
- The torque at $x=0$ is continuous: $\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=0}$.

We thus have four conditions and four unknowns.

We now see what the boundary conditions give one by one:

- $$l_L C_0^L = -l_R C_0^R.$$
- $$3l_R^2 + 2C_1^R l_R + C_0^R = 0, \qquad 3l_L^2 - 2C_1^L l_L + C_0^L = 0.$$
- $$C_0^L + l_L C_1^L = C_0^R - l_R C_1^R.$$

These are four linear equation for four unknowns. We only need a combination $C_1^R - C_1^L$ from them. Solving the equations we find

$$C_1^R - C_1^L = -\frac{1}{2}(l_R + l_L)\frac{l_R^2 + l_R l_L + l_L^2}{l_R l_L}.$$

and hence the force is

$$F_2 = \frac{\rho g A}{8}(l_R + l_L)\left(1 + \frac{(l_R + l_L)^2}{l_R l_L}\right) = \frac{Mg}{8}\left(1 + \frac{L^2}{l(L-l)}\right).$$

where l is the distance between the left end and the central support.

After this we find that

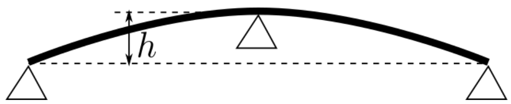
$$F_L = \frac{Mg}{8}\left(3 + \frac{l}{L} - \frac{L}{l}\right), \qquad F_L = \frac{Mg}{8}\left(3 + \frac{L-l}{L} - \frac{L}{L-l}\right).$$

In particular

- The answer does not depend on E! So the limit $E \to \infty$ is well defined!
- If $l = L/2$, we have $F_2 = \frac{5}{8}Mg$, $F_L = F_R = \frac{3}{16}Mg$. The guy at the center carries more than half of the total weight!
- If $l \to 0$ ($l \to L$), then F_2 and F_L (F_R) diverges. Why?

26.2. The force as a function of h.

Now let's finish this problem and compute how the force F_2 depends on the height h of the middle support. We simplify the problem by considering the middle support to be in the center.



We expect that the result for the force on the center support will be linear in h as for a spring.

- This is different from the situation of two unstretched springs. The difference is the torques that appears at bending.

So the result should have the form $F(h) = -\frac{5}{8}Mg - kh$. The spring constant k will depend on the Young modulus E. It is also clear, that if we fix the position of the ends (this is what we do for the solution) the spring constant will not depend on g, as it will be the same even without gravity. The force is always proportional to the combination EIA.

- The dimensional analysis then gives $k \sim \frac{EIA}{l^2}\frac{h}{l}$.

The prefactor should be just a number.

Again we have two functions $Y_L(x)$ and $Y_R(x)$ and the following boundary conditions

- The ends are at $Y = 0$, so

$$Y_L(x=-l) = 0, \qquad Y_R(x=l) = 0.$$

- At $x = 0$ we must have $Y = h$, so

$$Y_L(x=0) = h, \qquad Y_R(x=0) = h.$$

- The beam is smooth at the center

$$\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0},$$

- The torques at the ends are zero.

$$\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=-l} = 0, \qquad \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=l} = 0,$$

- The torque is continuous at the center

$$\left.\frac{\partial^2 Y_L}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 Y_R}{\partial x^2}\right|_{x=0}.$$

The force *on the support from the beam* is given by (different sign then before)

$$F = -IAE\left(\left.\frac{d^3Y_R}{dx^3}\right|_{x=0} - \left.\frac{d^3Y_L}{dx^3}\right|_{x=0}\right).$$

The first two conditions are satisfied by the functions of the form

$$Y_L(x) = -\frac{\rho g}{24IE}(x+l)\left(x^3 + C_2^L x^2 + C_1^L x - \frac{24IE}{\rho g l}h\right)$$
$$Y_R(x) = -\frac{\rho g}{24IE}(x-l)\left(x^3 + C_2^R x^2 + C_1^R x + \frac{24IE}{\rho g l}h\right)$$

The rest four conditions are enough to determine four unknown constants. As the result we have for the force *on the support*

$$F(h) = -\frac{5}{8}Mg - 6\frac{EIA}{l^2}\frac{h}{l}.$$

It has the expected form. One can see, that

$$F(h=0) = -\frac{5}{8}Mg$$
$$F = 0, \quad \text{for } h = -l\tfrac{5}{48}\tfrac{Mgl^2}{EIA}$$
$$F = -Mg, \quad \text{for } h = l\tfrac{3}{48}\tfrac{Mgl^2}{EIA}.$$

LECTURE 27
Hydrodynamics of Ideal Fluid: Mass conservation and Euler equation.

27.1. Hydrostatics.

For the statics of liquid we can use the elastic theory. The main difference between the solid body and the liquid is that the liquid has zero sheer coefficient. In this case the equation

$$\sigma_{ik} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj}).$$

tells us that the stress tensor is diagonal and we can use $\sigma_{ij} = -P\delta_{ij}$. The constant P is called pressure. We then have

$$\frac{\delta V}{V} = u_{ii} = \frac{\sigma_{ii}}{3K} = -\frac{P}{K}.$$

The constant K is then given by the equation of state for the liquid.

The equilibrium condition

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho g_i,$$

gives

$$\frac{\partial P}{\partial x_i} = -\rho g_i, \qquad \vec{\nabla} P = -\rho \vec{g}$$

So $P = \rho g h$.

Consider a small volume dV. The force which acts on it is the weight $\rho\vec{g}dV$ and the force of the hydrostatic pressure. We see, that the force of the hydrostatic pressure is $d\vec{f} = -dV\vec{\nabla}P$.

27.2. Hydrodynamics

- Separation of scales.
- Separation of time scales.
- Universality.

Ideal fluid means that there is no viscosity.

27.2.1. Mass conservation.

The liquid is now moving. Mass current: amount of mass dM through an area dS during time dt is $dM = Idt$, I is proportional to dS and depends on the orientation, so $I = \vec{j} \cdot d\vec{S}$. $\vec{j}$ is the current density and is

$$\vec{j} = \rho\vec{v}.$$

Mass conservation, consider a small volume dV

- during time dt the amount of mass in the volume changes by $\delta m = -dt \oint \vec{j} \cdot d\vec{S} = -dt \int \vec{\nabla} \cdot \vec{j} dV$.
- The change of mass is $dt \int \dot{\rho} dV$.
- As it is correct for any volume we have

$$\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0.$$

This is called continuity equation. It represents the fact that the mass cannot appear or disappear. It will also be correct for any conserved quantity with the correct definition of "current".

27.2.2. Euler equation.

We can describe the flow of liquid in two different ways:

- Describe the position and the velocity of the "liquid particles" as the function of time.
- Introduce the fields $\rho(\vec{r}, t)$ and $\vec{v}(\vec{r}, t)$ of density and velocity and describe the dynamics of these fields.

Describe the two point of views.

Consider a small volume dV... Derivation of the equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\nabla\vec{P}$$

In case there is gravity

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P + \vec{g}.$$

This equation together with the continuity equation and the equation of state are a full set of equations which must be supplied with the boundary conditions.

LECTURE 28
Hydrodynamics of Ideal Fluid: Incompressible fluid, potential flow.

28.1. Incompressible liquid.

In case of incompressible liquid the equation of state is particularly simple: the density is constant. So we have

$$\vec{\nabla}\cdot\vec{v} = 0$$
$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\left(\frac{P}{\rho} + \Phi_g\right).$$

Using the formula

$$\vec{v}\times\mathrm{curl}\vec{v} = \frac{1}{2}\vec{\nabla}v^2 - (\vec{v}\cdot\vec{\nabla})\vec{v}$$

we can rewrite the Euler equation as

$$\frac{\partial\vec{v}}{\partial t} - \vec{v}\times\mathrm{curl}\vec{v} = -\vec{\nabla}\left(\frac{P}{\rho} + \Phi_g + \frac{1}{2}v^2\right).$$

If we now take curl of both sides we'll get

$$\frac{\partial}{\partial t}\mathrm{curl}\vec{v} - \mathrm{curl}(\vec{v}\times\mathrm{curl}\vec{v}) = 0$$

Notice, that this equation is identically satisfied if $\mathrm{curl}\vec{v} = 0$. Which in turn identically satisfied by $\vec{v} = \vec{\nabla}\phi$ for some function ϕ. The continuity equation then gives

$$\Delta\phi = 0.$$

This is so called potential flow. We need to supplement this equation with the boundary conditions. The simplest one is that on each boundary the component of the fluid velocity perpendicular to the boundary equals to the component of the boundary velocity perpendicular to the boundary.

Now substituting $\vec{v} = \vec{\nabla}\phi$ into the Euler equation we find

$$\vec{\nabla}\left(\frac{\partial\phi}{\partial t} + \frac{P}{\rho} + \Phi_g + \frac{1}{2}v^2\right) = 0, \qquad \frac{\partial\phi}{\partial t} + \frac{P}{\rho} + \Phi_g + \frac{1}{2}v^2 = f(t)$$

28.2. Potential flow around a moving sphere

Consider a sphere of radius R moving with the velocity $\vec{u}$ in the ideal incompressible fluid. The flow of the fluid around the sphere is potential, so we need to solve the equation

$$\Delta\phi = 0, \qquad \vec{n}\cdot\vec{v}|_{\text{on sphere}} = \vec{n}\cdot\vec{u}, \qquad \vec{v}|_{r\to\infty} \to 0.$$

where the boundary conditions demand that the normal component of the fluid on the sphere equals the normal component of the element of the sphere.

The function ϕ is the scalar. It must linearly depend on the velocity $\vec{u}$ as both the Laplace equation and the boundary conditions are linear. This is analogous to the dipole field in the electrostatics, so the solution must be of the form

$$\phi = a\vec{u}\cdot\vec{\nabla}\frac{1}{r}.$$

This is the field produced by the dipole $\vec{d} = a\vec{u}$, so the velocity (electric field) is

$$\vec{v} = \frac{a}{r^3}\left[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}\right].$$

So on the sphere surface we have

$$\vec{v}\cdot\vec{n}|_{r=R} = \frac{2a}{R^3}(\vec{u}\cdot\vec{n})$$

and we see, that $a = \frac{R^3}{2}$ and

$$\phi = -\frac{R^3}{2r^2}\vec{u}\cdot\vec{n}, \qquad \vec{v} = \frac{R^3}{2r^3}\left[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}\right].$$

In order to calculate the pressure use $\frac{\partial\phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}v^2 = \frac{P_0}{\rho}$. We then need to calculate $\frac{\partial\phi}{\partial t}$. In order to do we must remember, that the sphere is moving, so

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\vec{u}}\cdot\dot{\vec{u}} - \vec{u}\cdot\nabla\phi.$$

We then find

$$P = P_0 + \frac{1}{8}\rho u^2(9\cos^2\theta - 5) + \frac{1}{2}\rho R\vec{n}\cdot\frac{d\vec{u}}{dt}.$$

We can calculate the total force acting on the sphere

$$\vec{F} = \oint Pd\vec{S}.$$

The integration of the first two terms in P gives zero. For the last term we find

$$F_i = \frac{1}{2}\rho R\frac{du_j}{dt}4\pi R^2\overline{n_j n_i} = \frac{1}{2}\frac{4\pi}{3}\rho R^3\frac{du_j}{dt}.$$

(it is clear that $\overline{n_j n_i}$ must be diagonal. Also $\overline{n_x n_x} = \overline{n_y n_y} = \overline{n_z n_z}$ and $\overline{n_i n_i} = 1$) So we find that

$$\vec{F} = \frac{1}{2}\rho R\frac{du_j}{dt}4\pi R^2\overline{n_j n_i} = \frac{2\pi}{3}\rho R^3\frac{d\vec{u}}{dt}.$$

Notice:

- Without the viscosity the force is zero if the velocity of the sphere does not change.

- The liquid just effectively changes the mass of the sphere by the value

$$\frac{1}{2}\frac{4}{3}\pi R^3\rho.$$

  Half of the expelled mass.