Modern Physics. Phys 222

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Contents

LECTURE 1
Introduction. Geometry.

- Contact info.
- Zoom. Use space bar. Pin video.
- Office hours.
- eCampus.
- Homework submissions. PDF SINGLE FILE.
- Homeworks (deadlines, collaborations!!!!! make study groups, mistakes, etc.)
- Homeworks on Friday lectures.
- Extra problems.
- Lecture, feedback. Going too fast, etc.
- Book.
- Grading.
- Exams.
- Language.
- Course content and philosophy.
- Questions: profound vs. stupid.

What do we know?

- Calculus (derivatives, integrals, partial derivatives, Taylor expansion, integration over a path, Fourier transformation.)
- Linear algebra (vectors, matrices, eigen values, eigen vectors.)
- Complex variables.
- Mechanics.
- Electrodynamics.
- Geometry.

Geometry

- What is the sum of all angles in a triangle? Why?
- What is distance?
- Metric tensor.
- A story of an ant on a sphere. Sum of the angles in a triangle. The number π.

What is a straight line?

- Length of a curve as a functional.
- Functional, variations, Extremum.
- Straight line in Euclidean space in Cartesian coordinates.

LECTURE 2
Mechanics.

- Home work solutions
- Calculus.
- Physical world and its description.
  - System of coordinates.
  - Straight line in Euclidean space in Cartesian coordinates: $y = ax + b$.
  - Straight line in Euclidean space in Polar coordinates $r = \frac{a}{\cos(\phi - \phi_0)}$.

LECTURE 3
Geometry. Topology.

Geometry

- What is a straight line?
- Metric tensor in Cartesian coordinate system: $(dl)^2 = (dx)^2 + (dy)^2$.
- Metric tensor in polar coordinates $(dl)^2 = (dr)^2 + r^2(d\phi)^2$.
- Change of variables in the integral: under the change $x = r(\phi)\cos(\phi)$, $y = r(\phi)\sin(\phi)$ we have
$$L = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{r^2 + (r')^2} d\phi.$$
- Metric tensor on a sphere $(dl)^2 = R^2(d\theta)^2 + R^2(d\phi)^2 \sin^2\theta$.
- "Straight" line on a sphere.
- What is our space?

Topology.

- Number of vertices V, edges E, and faces F.
- Compute $V + F - E$ for several polyhedral.
- $V + F - E$ as invariant.
- A face must have no holes.
- Continuum limit.
- $V + F - E$ for torus.
- $V + F - E = 2 - 2g$
- A story of an ant.
- What does it have to do with physics?

LECTURE 4
Mechanical world. Galilean invariance. Newton laws.

Mechanics

- A body (simplification – point like object of a certain mass) in an empty space. Process is independent of observer. No universal frame of reference.
- Galilean invariance.
- Galilean invariance in increments.

$$dx' = dx + V\,dt$$
$$dt' = dt.$$

- Time reversal. No universal clock.
- Interactions. What is force?
- Newton laws. Differential equations.
  - Motion with constant acceleration in $1D$.

  $$v = v_0 + at$$
  $$x = x_0 + v_0 t + \frac{at^2}{2}$$

  These are correct ONLY(!!!) for the case of constant acceleration.
  - Oscillator.
  - Gravity.

LECTURE 5
Homework. Work. Conservation laws.

- Homeworks.
- Calculus of many variables.
  - Differential
$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \equiv \vec{\nabla} U \cdot d\vec{r} \equiv \frac{\partial U}{\partial \vec{r}} \cdot d\vec{r} \equiv \mathrm{grad} U \cdot d\vec{r}$$
  - When is a 1-form
$$f_x(x,y)dx + f_y(x,y)dy$$
    a differential of some function? What conditions do the functions $f_x(x,y)$ and $f_y(x,y)$ need to satisfy in order for the above 1-form to be a full differential?
$$\frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x}$$
  - Examples:
    — Full differential: $ydx + xdy$
$$f_x(x,y) = y, \quad f_y(x,y) = x, \qquad \frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x} = 1, \qquad U(x,y) = xy.$$
    — Not a differential: $ydx - xdy$
$$f_x(x,y) = y, \quad f_y(x,y) = -x, \qquad \frac{\partial f_x}{\partial y} = 1 \neq \frac{\partial f_y}{\partial x} = -1$$
    There is no function U!
- $$\vec{F} = m\vec{a}, \qquad \vec{a} = \frac{d\vec{v}}{dt}, \qquad \vec{v} = \frac{d\vec{r}}{dt}$$
- Work. Work as a path integral in a force field.
$$W = \int_\Gamma \vec{F} \cdot d\vec{r} = \int_\Gamma m\frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt}dt = \int_\Gamma m\frac{d\vec{v}}{dt} \cdot \vec{v}dt = \int_\Gamma \frac{d\frac{m\vec{v}^2}{2}}{dt}dt = \frac{m\vec{v}_f^2}{2} - \frac{m\vec{v}_i^2}{2} = \Delta K$$
  Work depends on the path.
- Conservative forces. Work does not depend on path! It depends only on initial and final points!
$$\oint \vec{F} \cdot d\vec{r} = 0.$$

- Consider a small loop.

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \Rightarrow \quad \vec{F} = -\frac{\partial U}{\partial \vec{r}}.$$

- Large loop as a sum small of loops.

LECTURE 6
Newton's law. Energy conservation. Motion in $1D$.

So far:

- Newton's law
$$\vec{F} = m\vec{a}, \qquad \vec{a} = \frac{d\vec{v}}{dt}, \qquad \vec{v} = \frac{d\vec{r}}{dt}$$
- Newton's law as a differential equation
$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}(\vec{r}, t), \qquad \vec{r}(t = t_0) = \vec{r}_0, \qquad \vec{v}(t = t_0) = \vec{v}_0$$
- Example:
  - $\vec{F}(\vec{r}, t)$ does not depend on $\vec{r}$ and t. Say $\vec{F} = -mg\hat{y}$, where g is an arbitrary constant and $\hat{y}$ is a unit vector in the y direction. Then the equation of motion is
$$\frac{d^2\vec{r}}{dt^2} = -g\hat{y}, \quad \text{or} \quad \frac{d^2x}{dt^2} = 0, \quad \frac{d^ty}{dt^2} = -g$$
    This equation(s) must be supplied with initial conditions, say at $t = 0$
$$x(t = 0) = x_0, \quad v_x(t = 0) = v_{x0}, \quad y(t = 0) = y_0, \quad v_y(t = 0) = v_{y0}.$$
  - The solution of the equation of motion with the given initial conditions is
$$x(t) = x_0 + v_{x0}t, \qquad y(t) = y_0 + v_{y0}t - \frac{gt^2}{2}.$$

6.1. Energy conservation.

- Work. Work as a path integral in a force field.
$$W = \int_\Gamma \vec{F} \cdot d\vec{r}$$
  Work depends on the path.
- If the motion is due to a force field $\vec{F}$, then
$$W = \int_\Gamma \vec{F} \cdot d\vec{r} = K_f - K_i \equiv \Delta K, \qquad K = \frac{m\vec{v}^2}{2}.$$
  where Γ is the trajectory.

- Conservative forces. Work does not depend on path! It depends only on initial and final points! Then if initial and final points are the same, the work must be zero. So for any closed loop
$$\oint \vec{F} \cdot d\vec{r} = 0.$$
- It means, that a force filed $\vec{F}(\vec{r})$ is conservative if, and only if there is a function $U(\vec{r})$ such that
$$\vec{F} = -\frac{\partial U}{\partial \vec{r}}$$

Consequences:

- Non-uniqueness of U.
- Voltage. Kirchhoff's law.
- Potential difference. Why do you need ground.
- Energy. For a conservative force:
$$\Delta K = K_f - K_i = \int_\Gamma \vec{F} \cdot d\vec{r} = -\int_\Gamma \frac{\partial U}{\partial \vec{r}} \cdot d\vec{r} = -\int_\Gamma dU = -U_f + U_i$$
so
$$K_i + U_i = K_f + U_f, \qquad E = K + U, \qquad E_i = E_f$$
Full energy is conserved! $E = \frac{m\vec{v}^2}{2} + U(\vec{r})$.
- Time translation invariance. Energy conservation.
- Translation invariance. Momentum conservation.

6.2. Motion in $1D$.

- Conservative forces in $1D$. In $1D$ every force which depends only on coordinate $F(x)$ is a conservative force. We can always construct the potential energy
$$U(x) = -\int_{x_0}^{x} F(x')dx', \qquad F(x) = -\frac{\partial U(x)}{\partial x}.$$
- Energy conservation. Motion in $1D$. $E = \frac{mv^2}{2} + U(x)$ so $U(x) < E$. Let's initial conditions be $x(t_0) = x_0$ and $v(t_0) = v_0$, the $E = \frac{mv_0^2}{2} + U(x_0)$. Then we can write for any moment of tile
$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + U(x) = E$$
$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$
$$dt = \pm\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{E - U(x)}}$$
$$t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x(t)} \frac{dx}{\sqrt{E - U(x)}}$$
- Oscillator.
  - Let's assume that our initial conditions are
$$x(t = 0) = 0, \qquad v(t = 0) = v_0.$$

- The force is $F = -kx$, the corresponding potential energy

$$U(x) = \frac{kx^2}{2}, \qquad F(x) = -\frac{\partial U}{\partial x} = -kx$$

- The energy is conserved, so we compute it at $t = 0$

$$E = \frac{mv^2(t=0)}{2} + \frac{kx^2(t=0)}{2} = \frac{mv_0^2}{2}.$$

- Now we have

$$t = \pm\sqrt{\frac{m}{2}} \int_0^{x(t)} \frac{dx}{\sqrt{E - \frac{kx^2}{2}}}.$$

- Taking this integral we find

$$t = \sqrt{\frac{m}{k}} \arcsin\left(\sqrt{\frac{k}{2E}} x(t)\right)$$

- Inverting this equation

$$x(t) = \sqrt{\frac{2E}{m}} \sin\left(\sqrt{\frac{k}{m}} t\right)$$

- Using the value of E and usual frequency $\omega = \sqrt{k/m}$ we get

$$x(t) = \frac{v_0}{\omega} \sin(\omega t).$$

- Motion in 1D in arbitrary potential $U(x)$.
- Let's draw the function $U(x)$ – see figure 1.
- We also draw a line E – it is conserved, it is a constant.
- As kinetic energy K is always positive, we must have

$$U(x) < E.$$

- So the shaded regions on the figure are inaccessible/prohibited for a particle of energy E.
- At the points of intersection of the lines $U(x)$ and E, the kinetic energy is zero, so the velocity is zero. These points are called turning points.
- If a particle is in between two of such points it must go back and force in between them. If the particle has only one of such points, as of the far left and far right on the figure, then the particle will go to infinity. These are the only two possibilities.
- If the particle is in between two turning points, say x_1 and x_2 on the figure. The period of its motion can be computed by

$$T = 2\sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}$$

  The extra factor of 2 is because the particle should go back and forth.

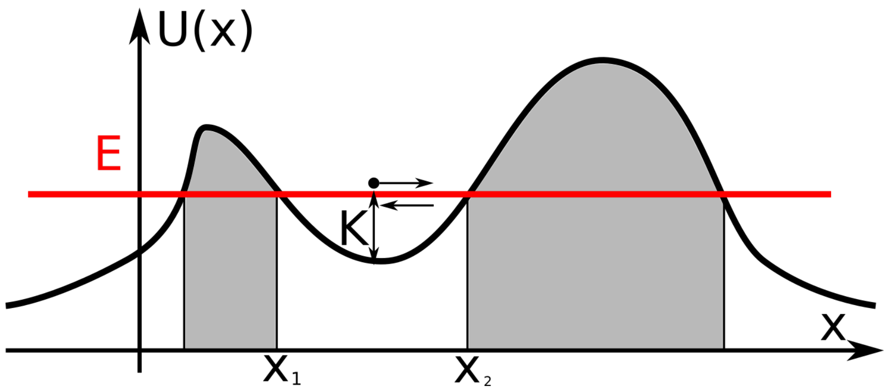


Figure 1. $1D$ potential $U(x)$ – black line.

LECTURE 7
Hamiltonian.

Why do we need a new formulation? Conservative force is given by a single scalar function $U(\vec{r})$. Newtonian formulation demands that we work with forces – a vector function. It is an overkill. There must be a way to avoid dealing with vectors.

Symmetries are hard to notice in vector formulation.

7.1. Hamiltonian formulation.

- Energy conservation.
  - Consider a particle moving in a potential field
$$m\frac{d^2\vec{r}}{dt^2} = -\frac{\partial U}{\partial \vec{r}}, \qquad +\text{initial conditions.}$$
  - Consider energy as a function of velocity and coordinates: $E(\vec{v},\vec{r}) = \frac{m\vec{v}^2}{2} + U(\vec{r})$. At this point we want to think of $\vec{v}$ and $\vec{r}$ as independent variables.
  - Full vs. partial derivatives.
  - The particle moves according to the equation of motion. If we solve it, we will know $\vec{r}(t)$ and $\vec{v}(t)$. We then can stick these functions into our function $E(\vec{v},\vec{r})$ and get $E(\vec{v}(t),\vec{r}(t))$ a function of t.
  - Let's compute, how this energy $E(\vec{v}(t),\vec{r}(t))$ changes with time
$$\frac{dE}{dt} = \frac{\partial E}{\partial \vec{v}}\cdot\frac{d\vec{v}}{dt} + \frac{\partial E}{\partial \vec{r}}\cdot\frac{d\vec{r}}{dt} = m\vec{v}\cdot\frac{d^2\vec{r}}{dt^2} + \frac{\partial U}{\partial \vec{r}}\cdot\vec{v} = \vec{v}\cdot\left(m\frac{d^2\vec{r}}{dt^2} + \frac{\partial U}{\partial \vec{r}}\right) = 0.$$

    In other words energy is conserved on the trajectory.
- Momentum. Dispersion relation
$$\dot{p} = -\frac{\partial U}{\partial \vec{r}}, \qquad K = \frac{\vec{p}^2}{2m}$$
- Consider a function $H(\vec{p},\vec{r}) = \frac{\vec{p}^2}{2m} + U(\vec{r})$. We take $\vec{p}$ and $\vec{r}$ as independent variables in this function.
- We then see, that our equations of motion are
$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}$$
$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}$$

- These equations are called Hamiltonian equations. The function H is called Hamiltonian.

Hamiltonian is a function of coordinates and momenta.

- Consider the value of the Hamiltonian on a trajectory: $\vec{p}(t)$, $\vec{r}(t)$: $H(\vec{p}(t), \vec{r}(t))$,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \vec{p}} \cdot \dot{\vec{p}} + \frac{\partial H}{\partial \vec{r}} \cdot \dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{p}} - \frac{\partial H}{\partial \vec{r}} \cdot \frac{\partial H}{\partial \vec{r}} = 0$$

- Energy as a value of a Hamiltonian on a trajectory.
- Phase space.
- Hamiltonian is an arbitrary (specific for a given problem) function on a phase space.
- Second Hamiltonian equation gives the velocity for general dispersion relation.
- Phase space trajectories.

7.2. Functionals.

- Definition of functionals.
  - Correspondence/map "number to number" is called a function.
  - Correspondence/map "function to number" is called a functional.
- Examples.

LECTURE 8
Lagrangian.

8.1. Lagrangian formulation.

- Homework.
- Definition of functionals.
  - Correspondence/map "number to number" is called a function.
  - Correspondence/map "function to number" is called a functional.
- Examples.
- For functions which satisfy the boundary conditions $f(x_A) = f_A$ and $f(x_B) = f_B$ in many cases the functional can be represented by

  $$\mathcal{S}[f(x)] = \int_{x_A}^{x_B} L(f'(x), f(x))dx$$

  The function of two variables $L(\ , \ )$ defines the functional.
- We ask the following question: given a functional $\mathcal{S}[f(x)]$, what function $f(x)$ (which satisfies the boundary condition) gives us the minimal value of the functional?
  - We consider only the functions which satisfy the boundary conditions.
  - Let's assume that we found the function $f_0(x)$ that solves our problem.
  - Let's change this function a little and see how the value of the functional will change. So we consider a function $f(x)$

    $$f(x) = f_0(x) + \delta f(x), \qquad \delta f(x_A) = 0, \quad \delta f(x_B) = 0$$

    The last two equalities are due to the fact, that the function $f(x)$ must satisfy the same boundary conditions.
  - The new value of the functional is

    $$\mathcal{S}[f(x)] = \int_{x_A}^{x_B} L(f_0'(x) + \delta f'(x), f_0(x) + \delta f(x))dx$$

    Remember L is just a function of two variables, so we can write

    $$L(f_0'(x) + \delta f'(x), f(x) + \delta f(x)) = L(f_0'(x), f_0(x)) + \frac{\partial L}{\partial f'}\delta f' + \frac{\partial L}{\partial f}\delta f,$$

    So we have

    $$\mathcal{S}[f(x)] = \mathcal{S}[f_0(x)] + \int_{x_A}^{x_B} \left(\frac{\partial L}{\partial f_0'} \frac{d\delta f}{dx} + \frac{\partial L}{\partial f_0}\delta f \right) dx,$$

Taking the integral by parts in the first term we get

$$\mathcal{S}[f(x)] = \mathcal{S}[f_0(x)] + \int_{x_A}^{x_B} \left(-\frac{d}{dx}\frac{\partial L}{\partial f_0'} + \frac{\partial L}{\partial f_0} \right) \delta f(x) dx$$

Now we see, that depending on $\delta f(x)$ the integral can be either positive, or negative. But it must never be negative, because $\mathcal{S}[f_0(x)]$ is the minimum! It means that the expression in the brackets must be zero! We then found, that the function $f_0(x)$ must be such as to satisfy the following equation

$$\frac{d}{dx}\frac{\partial L}{\partial f_0'} - \frac{\partial L}{\partial f_0} = 0$$

- Hamilton principle. Action. Minimal action.
- Lagrangian.
  - Lagrangian.
  $$L(\dot{r}, r) = K - U.$$
  - Action
  $$\mathcal{A}[\vec{r}(t)] = \int_{t_i}^{t_f} L(\dot{\vec{r}}, \vec{r}) dt$$

Lagrangian is a function of coordinates and velocities.

- Euler-Lagrange equation.
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial L}{\partial \vec{r}}$$
- Examples:
  - 1D motion in a potential $U(x)$
  $$L(\dot{x}, x) = \frac{m\dot{x}^2}{2} - U(x).$$
  Euler-Lagrange equation – left hand side:
  $$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = m\ddot{x}$$
  Euler-Lagrange equation – right hand side:
  $$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$$
  Euler-Lagrange equation:
  $$m\ddot{x} = -\frac{\partial U}{\partial x}$$
- One can use any set of numbers as coordinates.
  - Examples.
  - Oscillator.
  - Pendulum.
  - Pendulum with a spring.
  - Pendulum with a spring on a wedge.
  - Double pendulum. etc.

LECTURE 9
Oscillations with dissipation (friction).

9.1. Euler formula

$$e^{i\phi} = \cos(\phi) + i\sin(\phi).$$

which also mean

$$\cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2}, \qquad \sin(\phi) = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

and

$$e^{i\pi} = -1.$$

9.2. Oscillators.

- Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$$

- Oscillator, pendulum, electric resonator (LC-circuit)

$$m\ddot{x} = -kx, \qquad ml\ddot{\phi} = -mg\sin\phi \approx -mg\phi, \qquad -L\ddot{Q} = \frac{Q}{C},$$

  All of these equation have the same form

$$\ddot{x} = -\omega_0^2 x, \qquad \omega_0^2 = \begin{cases} k/m \\ g/l \\ 1/LC \end{cases}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- Notice the minus sign! This is a very important minus sign!!! It quarantines that the oscillator returns back – oscillates, instead of running away.
- The solution

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) = |C|\cos(\omega_0 t - \phi), \qquad B = x_0, \qquad \omega_0 A = v_0.$$

- $A = |C|\sin(\phi)$, $B = |C|\cos(\phi)$, or $|C| = \sqrt{A^2 + B^2}$ — amplitude; $\phi = \tan^{-1}(A/B)$ — phase.
- Oscillates forever. Frequency is ω_0. The frequency can be read straight from the equation.

- The solution can be written as
$$x(t) = \Re C e^{i\omega t}, \qquad C = |C| e^{i\omega t}$$

9.3. Oscillations with dissipation (friction).

- Oscillations with friction:
$$m\ddot{x} = -kx - 2\alpha\dot{x}, \qquad -L\ddot{Q} = \frac{Q}{C} + R\dot{Q},$$
- The sign of α.
  - The mechanical energy of an oscillator is $E = \frac{mv^2}{2} + \frac{kx^2}{2}$.
  - Let's compute, how it changes with time
$$\frac{dE}{dt} = mv\dot{v} + kxv = -vkx - 2\alpha v^2 + kxv = -2\alpha v^2.$$
  - Under the dissipation the mechanical energy must decrease at all times. Notice, that this requires, that
$$\alpha > 0.$$
  - The case $\alpha < 0$ would correspond to pumping of energy into the system.
- Consider
$$\ddot{x} = -\omega_0^2 x - 2\gamma\dot{x}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$
This is a linear equation with constant real coefficients. We look for the solution in the form $x = \Re C e^{i\omega t}$, where ω and C are complex constants.
$$\omega^2 - 2i\gamma\omega - \omega_0^2 = 0, \qquad \omega = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$
- Two solutions, two independent constants.
- Two cases: $\gamma < \omega_0$ and $\gamma > \omega_0$.
- In the first case (underdamping):
$$x = e^{-\gamma t}\Re\left[C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right] = C e^{-\gamma t}\sin\left(\Omega t + \phi\right), \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}$$
Decaying oscillations. Shifted frequency.
- In the second case (overdamping):
$$x = A e^{-\Gamma_- t} + B e^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}, \qquad \Gamma_+ > \Gamma_- > 0$$
- For the initial conditions $x(t=0) = x_0$ and $v(t=0) = 0$ we find $A = x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}$, $B = -x_0\frac{\Gamma_-}{\Gamma_+ - \Gamma_-}$. For $t \to \infty$ the B term can be dropped as $\Gamma_+ > \Gamma_-$, then $x(t) \approx x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-} e^{-\Gamma_- t}$.
- At $\gamma \to \infty$, $\Gamma_- \to \frac{\omega_0^2}{2\gamma} \to 0$. The motion is arrested. The example is an oscillator in honey.

LECTURE 10
Oscillations with external force. Resonance.

- Homework.

10.1. Comments on dissipation.

- Time reversibility. A need for a large subsystem.
- Locality in time.

10.2. Resonance

- Let's add an external force:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- The full solution is the sum of the solution of the homogeneous equation with any solution of the inhomogeneous one. This full solution will depend on two arbitrary constants. These constants are determined by the initial conditions.
- Let's assume, that $f(t)$ is not decaying with time. The solution of the inhomogeneous equation also will not decay in time, while any solution of the homogeneous equation will decay. So in a long time $t \gg 1/\gamma$ The solution of the homogeneous equation can be neglected. In particular this means that the asymptotic of the solution does not depend on the initial conditions.
- Let's now assume that the force $f(t)$ is periodic. with some period. It then can be represented by a Fourier series. As the equation is linear the solution will also be a series, where each term corresponds to a force with a single frequency. So we need to solve

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$

where f is the force's amplitude.
- From the equation (linear) it is obvious, the amplitude of $x(t)$ will be proportional to the force amplitude f.
- Let's look at the solution in the form $x = f\Im Ce^{i\Omega_f t}$, and use $\sin(\Omega_f t) = \Im e^{i\Omega_f t}$. We then get

$$C = \frac{1}{\omega_0^2 - \Omega_f^2 + 2i\gamma\Omega_f} = |C|e^{-i\phi},$$

$$|C| = \frac{1}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}}, \quad \tan\phi = \frac{2\gamma\Omega_f}{\omega_0^2 - \Omega_f^2}$$

$$x(t) = f\Im|C|e^{i\Omega_f t - i\phi} = f|C|\sin\left(\Omega_f t - \phi\right),$$

- Resonance frequency:
$$\Omega_f^r = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\Omega^2 - \gamma^2},$$
where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ is the frequency of the damped oscillator.
- Phase changes sign at $\Omega_f^\phi = \omega_0 > \Omega_f^r$. Importance of the phase – phase shift.
- To analyze resonant response we analyze $|C|^2$.
- The most interesting case $\gamma \ll \omega_0$, then the response $|C|^2$ has a very sharp peak at $\Omega_f \approx \omega_0$:
$$|C|^2 = \frac{1}{(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2} \approx \frac{1}{4\omega_0^2}\frac{1}{(\Omega_f - \omega_0)^2 + \gamma^2},$$
so that the peak is very symmetric.
- $|C|^2_{\max} \approx \frac{1}{4\gamma^2\omega_0^2}$.
- to find HWHM we need to solve $(\Omega_f - \omega_0)^2 + \gamma^2 = 2\gamma^2$, so HWHM $= \gamma$, and FWHM $= 2\gamma$.
- Q factor (quality factor). The good measure of the quality of an oscillator is $Q = \omega_0/\mathrm{FWHM} = \omega_0/2\gamma$. (decay time) $= 1/\gamma$, period $= 2\pi/\omega_0$, so $Q = \pi\frac{\text{decay time}}{\text{period}}$.
- For a grandfather's wall clock $Q \approx 100$, for the quartz watch $Q \sim 10^4$.

10.3. Response.

- Response. The main quantity of interest. What is "property"?
- The equation
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t).$$
The LHS is time translation invariant!
- Multiply by $e^{i\omega t}$ and integrate over time. Denote
$$x_\omega = \int_{-\infty}^{\infty} x(t)e^{i\omega t}dt.$$
Then we have
$$\left(-\omega^2 - 2i\gamma\omega + \omega_0^2\right)x_\omega = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt, \qquad x_\omega = -\frac{\int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt'}{\omega^2 + 2i\gamma\omega - \omega_0^2}$$
- The inverse Fourier transform gives
$$x(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}x_\omega = -\int_{-\infty}^{\infty} f(t')dt'\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \int_{-\infty}^{\infty}\chi(t-t')f(t')dt'.$$
- Where the response function is $(\gamma < \omega_0)$
$$\chi(t) = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \begin{cases} e^{-\gamma t}\frac{\sin(t\sqrt{\omega_0^2-\gamma^2})}{\sqrt{\omega_0^2-\gamma^2}} & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases}, \qquad \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

- Causality principle. Poles in the lower half of the complex ω plane. True for any (linear) response function. The importance of $\gamma > 0$ condition.

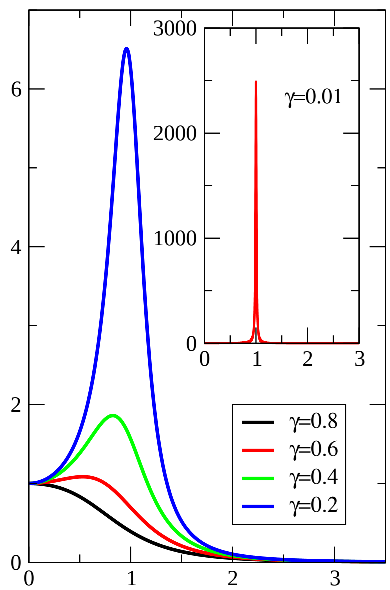


Figure 1. Resonant response. For insert $Q = 50$.

LECTURE 11
Spontaneous symmetry braking.

- Dissipation:
  - coupling to a large system.
  - Locality in time – not very important.
  - Decreases the energy.
- Noise:
  - Any system with dissipation will have noise.
  - Kicks a system out of unstable equilibriums. Or does not allow a system to freeze in a wrong extremum.
- No matter how small the dissipation and noise are they together ensure that the system finds the minimum of the potential energy.
- Close to a minimum every function can be described as $\frac{kx^2}{2}$ – a harmonic oscillator.

General procedure: If we know the potential energy function $U(\vec{r})$

- First, find the position of the minimums.
- Find which minimum has the lowest energy.
- Use Taylor expansion of the Potential energy function around the minimum to the second order.
- Use it as k to find the oscillation/resonance frequency.

11.1. Spontaneous symmetry braking.

The mystery of a broken symmetry.

- The fundamental laws are translationally invariant, but the world around us is not.
- A magnet below transition picks a particular direction on the magnetization.
- And so on.

The symmetry of a solution does not have to have full symmetry of the equation.

11.1.1. Example.

A bead on a vertical rotating hoop.

- Potential energy:

$$U(\theta) = mgR(1 - \cos\theta).$$

- Kinetic energy:
$$K = \frac{m}{2}R^2\dot\theta^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta.$$
- The Lagrangian.
$$L = \frac{m}{2}R^2\dot\theta^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta - mgR(1-\cos\theta).$$
- The Lagrangian can be written as
$$L = \frac{m}{2}R^2\dot\theta^2 - U_{eff}(\theta),$$
where the "effective" potential energy is
$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1-\cos\theta).$$
- Equation of motion.
$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta}.$$
or
$$R\ddot\theta = -\frac{1}{mR}\frac{\partial U_{eff}(\theta)}{\partial\theta} = (\Omega^2 R\cos\theta - g)\sin\theta.$$
There are four equilibrium points: $\frac{\partial U_{eff}}{\partial\theta} = 0$
$$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R}$$
- Critical Ω_c. The second two equilibriums are possible only if
$$\frac{g}{\Omega^2 R} < 1, \qquad \Omega > \Omega_c = \sqrt{g/R}.$$
- Effective potential energy for $\Omega \sim \Omega_c$. Assuming $\Omega \sim \Omega_c$ we are interested only in small θ. So
$$U_{eff}(\theta) \approx \frac{1}{2}mR^2(\Omega_c^2 - \Omega^2)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4$$
$$U_{eff}(\theta) \approx mR^2\Omega_c(\Omega_c - \Omega)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4$$
- Spontaneous symmetry breaking. Plot the function $U_{eff}(\theta)$ for $\Omega < \Omega_c$, $\Omega = \Omega_c$, and $\Omega > \Omega_c$. Discuss universality.
- Small oscillations around $\theta = 0$, $\Omega < \Omega_c$
$$mR^2\ddot\theta = -mR^2(\Omega_c^2 - \Omega^2)\theta, \qquad \omega = \sqrt{\Omega_c^2 - \Omega^2} \approx \sqrt{2\Omega_c(\Omega_c - \Omega)}.$$
- Small oscillations around θ_0, $\Omega > \Omega_c$.
$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mrR(1-\cos\theta),$$

$$\frac{\partial U_{eff}}{\partial\theta} = -mR(\Omega^2 R\cos\theta - g)\sin\theta, \qquad \frac{\partial^2 U_{eff}}{\partial\theta^2} = mR^2\Omega^2\sin^2\theta - mR\cos\theta(\Omega^2 R\cos\theta - g)$$
$$\left.\frac{\partial U_{eff}}{\partial\theta}\right|_{\theta=\theta_0} = 0, \qquad \left.\frac{\partial^2 U_{eff}}{\partial\theta^2}\right|_{\theta=\theta_0} = mR^2(\Omega^2 - \Omega_c^4/\Omega^2) \approx 2mR^2(\Omega^2 - \Omega_c^2) \approx 4mR^2\Omega_c(\Omega - \Omega_c)$$

So the Tylor expansion gives
$$U_{eff}(\theta \sim \theta_0) \approx \text{const} + \frac{1}{2}4\Omega_c mR^2(\Omega - \Omega_c)(\theta - \theta_0)^2$$

The frequency of small oscillations then is

$$\omega = 2\sqrt{\Omega_c(\Omega - \Omega_c)}.$$

- The effective potential energy for small θ and $|\Omega - \Omega_c|$

$$U_{eff}(\theta) = \frac{1}{2}a(\Omega_c - \Omega)\theta^2 + \frac{1}{4}b\theta^4.$$

- θ_0 for the stable equilibrium is given by $\partial U_{eff}/\partial\theta = 0$

$$\theta_0 = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \sqrt{\frac{a}{b}(\Omega - \Omega_c)} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

Plot $\theta_0(\Omega)$. Non-analytic behavior at Ω_c.

- Response: how θ_0 responses to a small change in Ω.

$$\frac{\partial\theta_0}{\partial\Omega} = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \frac{1}{2}\sqrt{\frac{a}{b}}\frac{1}{\sqrt{(\Omega-\Omega_c)}} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

Plot $\frac{\partial\theta_0}{\partial\Omega}$ vs Ω. The response *diverges* at Ω_c.

LECTURE 12
Oscillations with time dependent parameters.

12.1. Oscillations with time dependent parameters.

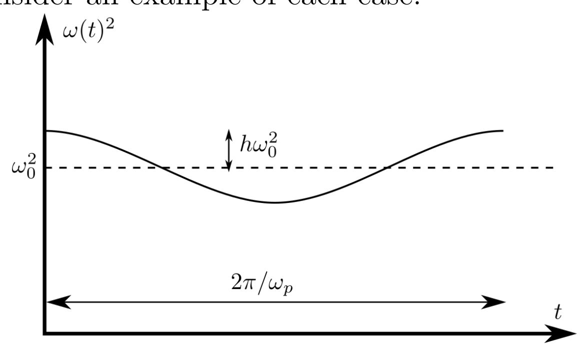
Let's consider the following problem

- The parameters of the oscillator (either k, or l for a pendulum, or C and L in circuit, etc) depend on time.
- There is no external force acting on the oscillator.

It is most interesting when the dependence of parameters on time is periodic, say with a period $T_p = 2\pi/\omega_p$. It is also most interesting, when the parameters do not change by much, so that we have almost intact oscillator with its own frequency close to ω_0.

After the time T_p the whole system returns back where it was, but the state of the system does not have to be the same.

We will distinguish between three different cases: $\omega_p \ll \omega_0$, $\omega_p \gg \omega_0$, $\omega_p \sim \omega_0$. Below we consider an example of each case.



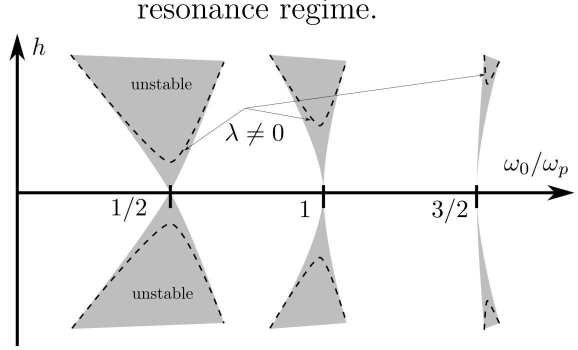
- $\omega_p \ll \omega_0$ — Foucault pendulum as an example of slow change of the parameter $\Delta\phi$ =solid angle of the path. (quantum: Berry phase 1984; classical: Hannay angle 1985.)
  - Topological in nature.
  - Universal.
- $\omega_p \gg \omega_0$ — Kapitza pendulum. (demo) Criteria: $\overline{\left(\dot{\xi}\right)^2} > gl$.
  - Importance of the time scale separation.
  - Averaging out fast processes – a natural thing to do.
  - Importance of non-linearity.

  – Universal mechanism – averaging over fast degrees of freedom leads to the change of the dynamics of the slow degree of freedom through non-linearity.

- $\omega_p \sim \omega_0$ — parametric resonance ($\omega_p = \frac{2}{n}\omega_0$)

$$\ddot{x} = -\omega^2(t)x, \qquad \omega^2(t) = \omega_0^2(1 + h\cos(\omega_p t)), \quad h \ll 1$$

  Different from the usual resonance:
  – If the initial conditions $x(t = 0) = 0$, $\dot{x}(t = 0) = 0$, then $x(t) = 0$.
  – Frequency ω_p is a fraction of ω_0.
  – At finite dissipation one must have a finite amplitude h in order to get to the resonance regime.



LECTURE 13
Waves.

13.1. Waves.

- Homework.
- Waves. Ripples. Sound waves. Light waves.
- More is different. Waves as collective excitations.
- Amplitude, phase.
- Linearity. Superposition.
- Acoustic beat `https://en.wikipedia.org/wiki/Beat_(acoustics)`
$$\sin(\omega_1 t) + \sin(\omega_2 t) = 2\cos\left(\frac{\omega_1 - \omega_2}{2} t\right)\sin\left(\frac{\omega_1 + \omega_2}{2} t\right)$$
- Interference.
- Wave front. Rays.
- Snell's Law.
- Green's picture. Huygens's principle. `https://en.wikipedia.org/wiki/Huygens%E2%80%93Fresnel_principle`.
- Diffraction. `https://en.wikipedia.org/wiki/Diffraction`
- Resonator.
- Wave in a loop.
- Difference between waves and particles (particle's stream).
- Acoustic Doppler effect `https://en.wikipedia.org/wiki/Doppler_effect`. The source of frequency f_s is moving towards the stationary observer with velocity v_s. The observer hears the frequency f_o:
$$f_o = f_s \frac{c}{c - v_s}.$$
Discuss the role of the medium.
- Anderson localization.

LECTURE 14
Currents

- Current. Mass current. General current.
- Current density: vector.
- Charge/mass conservation:

$$\dot{\rho} + \nabla \vec{j} = 0$$

- Voltage. Current.
- Capacitor. Inductance.
- Resistor. Ohm's law.

$$V = IR, \qquad \vec{j} = \sigma \vec{E}.$$

- Kirchhoff's law.
- Phasor diagrams.

$$V_L = i\omega L I_L, \qquad V_C = -i\frac{1}{\omega C} I_C, \qquad V_R = R I_R$$

LECTURE 15
Gauss theorem. Lorenz force.

15.1. Gauss theorem.

Notations:

- A boundary of a surface is a line.
- A boundary of a piece of volume is a surface.
- Boundary of a surface or volume Ω is denoted $\partial\Omega$.
- A boundary has no boundary $\partial\partial\Omega = 0$.

Consider an arbitrary vector field $\vec{E}(\vec{r})$. Consider an arbitrary surface Σ. The flux of the vector field $\vec{E}$ over the through the surface Σ is defined as

$$\int_\Sigma \vec{E} \cdot d\vec{S}.$$

There is no "correct" way to define which direction $d\vec{S}$ is pointing to.

Consider a piece of volume Ω with the boundary $\partial\Omega$. We can compute the flux of a vector field $\vec{E}$ through the surface $\partial\Omega$

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S}.$$

We can now define that $d\vec{S}$ is pointing outside.

The Gauss theorem states that for any (smooth) vector field $\vec{E}$ and for arbitrary volume Ω:

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \int_\Omega \nabla \cdot \vec{E} dV.$$

Am example of the Gauss theorem in $1D$ is the familiar formula

$$\int_a^b \frac{df(x)}{dx} dx = f(b) - f(a)$$

On the left hand side we have $1D$ bulk integral of $1D$ "divergence" $\frac{df}{dx}$ over $1D$ "volume" — the interval $[a, b]$. On the right hand side we have the "boundary integral" the sum (with the correct signs) of the function value in the points a and b.
Example of the use of the Gauss theorem:

- Current as a flux of current density field.

- By the definition of the current density and by **the charge conservation law** the total charge inside a volume Ω changes according to
$$\dot{Q} = -\oint_{\partial\Omega} \vec{j} \cdot d\vec{S} = -\int_{\Omega} \nabla \cdot \vec{j} dV.$$
The last equality is the Gauss theorem.
- By the definition of the charge density $\rho(\vec{r})$ we have
$$Q = \int_{\Omega} \rho dV, \qquad \text{and} \qquad \dot{Q} = \int_{\Omega} \dot{\rho} dV$$
- So we have
$$\int_{\Omega} \dot{\rho} dV = -\int_{\Omega} \nabla \cdot \vec{j} dV, \qquad \text{or} \qquad \int_{\Omega} \left(\dot{\rho} + \nabla \cdot \vec{j}\right) dV = 0.$$
- As this is correct for any Ω we must have
$$\dot{\rho} + \nabla \cdot \vec{j} = 0.$$
The continuity equation!

15.2. Current density.

Computing current density through local quantities.

- Current density. Charge density ρ, collective velocity $\vec{v}$:
$$\vec{j} = \rho\vec{v}.$$
If charges are electrons and the density of electrons is n, then $\rho = en$, where e is the charge of an electron.
- Continuity equation
$$\dot{\rho} + \nabla \cdot (\rho\vec{v}) = 0.$$

15.3. Lorenz force.

If we know position and velocities of all charges, we can find $\vec{j}$ etc. We just need to solve the Newton equations: $\vec{F} = m\vec{a}$. But what is $\vec{F}$? Force on a charge q.

- Lorenz force.
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}.$$
- Problem with Lorenz force.
- Examples:
  - Cyclotron radius, cyclotron frequency.
  - Force on a piece of wire.

Next question: Where $\vec{E}$ and $\vec{B}$ come from?

LECTURE 16
Gauss law. Vector field circulation.

- Homework.
- Exam announcement.

16.1. Gauss Law for electric field.

Do not confuse it with the Gauss theorem. The Gauss theorem states that for any (smooth) vector field $\vec{E}$ and for arbitrary volume Ω:

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \int_{\Omega} \nabla \cdot \vec{E} dV.$$

Gauss theorem is a very general theorem it has no physics content.

- We can define the flux of a Electric field $\vec{E}(\vec{r})$ through a surface Σ.

$$\Phi_E = \int_{\Sigma} \vec{E} \cdot d\vec{S}.$$

- If we have the some charge distribution with arbitrary charge density $\rho(\vec{r})$, then for any/arbitrary volume of space Ω we have
- Gauss's Law:

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_{\Omega} \rho dV.$$

- This law has a very clear physical content: the charges are the sources of the electric field flux.
- This law can be thought of as another form of the Coulomb law.
- In fact if we take the Coulomb law as the established physical law we can prove the Gauss law, and vice verse.
- Examples for very symmetric charge distributions:
  - Charged sphere.
  - Charged plane.
  - Electric field of a charged wire.
- Local form of the Gauss's Law
  - Using Gauss theorem we write

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \int_{\Omega} \nabla \cdot \vec{E} dV$$

- So the Gauss law can be written as

$$\int_\Omega \nabla\cdot\vec{E}dV = \frac{1}{\epsilon_0}\int_\Omega \rho dV, \qquad \text{or} \qquad \int_\Omega \left(\nabla\cdot\vec{E} - \frac{1}{\epsilon_0}\rho\right) dV = 0$$

- As it must be correct for any Ω we have

- the local form of the Gauss's Law

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}.$$

This can be viewed as yet another form of the Coulomb law.

16.2. Gauss Law for magnetic field.

- There are no magnetic charges.
- For the flux of the magnetic field $\vec{B}(\vec{r})$ through any closed surface $\partial\Omega$ is zero

$$\oint_{\partial\Omega} \vec{B}\cdot d\vec{S} = 0.$$

- It's local version (using Gauss theorem)

$$\nabla\cdot\vec{B} = 0.$$

16.3. Circulation of a vector field.

Another construction for the vector fields

- Circulation of a vector field.
- For any vector field $\vec{A}(\vec{r})$ and any **oriented** path Γ we can compute

$$\int_\Gamma \vec{A}\cdot d\vec{r}.$$

- If the path Γ is closed, then such an integral

$$\oint_\Gamma \vec{A}\cdot d\vec{r}.$$

is called circulation.
- Examples.

LECTURE 17
Maxwell Equations.

- Notation

$$\mathrm{curl}\vec{A} \equiv \nabla \times \vec{A}, \quad \mathrm{div}\vec{A} \equiv \nabla \cdot \vec{A} \quad \mathrm{grad}U \equiv \nabla U, \quad \Delta f \equiv \nabla \cdot \nabla f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- So far we know

$$\text{Gauss's law:} \qquad \oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \tfrac{1}{\epsilon_0} \int_\Omega \rho dV, \qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{Gauss's law magnetic:} \qquad \oint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0, \qquad \nabla \cdot \vec{B} = 0$$

This is not enough: first, we have only two scalar equation for 6 components of the fields 3 for electric field and 3 for magnetic; second, there is no time derivatives – no dynamics of the fields.

17.1. Circulation of a vector field.

Another construction for the vector fields

- Circulation of a vector field.
- For any vector field $\vec{A}(\vec{r})$ and any **oriented** path Γ we can compute

$$\int_\Gamma \vec{A} \cdot d\vec{r}.$$

- If the path Γ is closed and **oriented**, then such an integral

$$\oint_\Gamma \vec{A} \cdot d\vec{r}.$$

  is called circulation.
- Examples.

"Gauss" theorem for circulation. Stokes' theorem.

Let's take an arbitrary vector field $\vec{A}(\vec{r})$ and an arbitrary piece of surface Σ with the boundary $\partial\Sigma$. The boundary is a closed path. If we **chose** an orientation of $\partial\Sigma$, we can define the circulation of $\vec{A}$ over $\partial\Sigma$

$$\oint_{\partial\Sigma} \vec{A} \cdot d\vec{r}.$$

- Circulation of a vector field.

$$\oint_{\partial\Sigma} \vec{A} \cdot d\vec{r} = \int_\Sigma \nabla \times \vec{A} \cdot d\vec{S}.$$

- Orientation of Σ is induced by the orientation of $\partial\Sigma$ by the right hand rule.
- Independence of $\int_\Sigma \nabla \times \vec{A} \cdot d\vec{s}$ of Σ. Consider Σ_1 and Σ_2 with the common boundary $\partial\Sigma$. The orientation of both Σ_1 and Σ_2 is induced by the orientation of $\partial\Sigma$. The flux of the vector field $\nabla \times \vec{A}$ through $\Sigma_1 \cup \Sigma_2$ is $\Phi_{\Sigma_1\cup\Sigma_2} = \Phi_{\Sigma_2} - \Phi_{\Sigma_1}$

$$\Phi_{\Sigma_2} - \Phi_{\Sigma_1} = \Phi_{\Sigma_1\cup\Sigma_2} = \int_{\Sigma_1\cup\Sigma_2} \nabla \times \vec{A} \cdot d\vec{S} = \int_\Omega \nabla \cdot \nabla \times \vec{A} dV = 0.$$

- Example of a circulation: work of a force vector field over a closed path.

$$\mathcal{W} = \int_{\partial\Sigma} \vec{F} \cdot d\vec{r} = \int_\Sigma \nabla \times \vec{F} \cdot d\vec{s}.$$

if the force is a potential force, then $\vec{F} = \nabla U$ and

$$\mathcal{W} = \int_\Sigma \nabla \times \nabla U \cdot d\vec{s} = 0.$$

17.2. Faraday's Law.

You are familiar with this law in the form

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

- Faraday's Law, Circulation of Electric field. (zero in statics)

$$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int_\Sigma \vec{B} \cdot d\vec{S}.$$

- Faraday's Law is independent of Σ — it only depends on $\partial\Sigma$.

$$\Phi_{\Sigma_1} - \Phi_{\Sigma_2} = \Phi_{\Sigma_1\cup\Sigma_2} = \int_{\Sigma_1\cup\Sigma_2} \vec{B} \cdot d\vec{S} = \int_\Omega \nabla \cdot \vec{B} dV = 0.$$

- Local version of the Faraday's law. Using

$$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{r} = \int_\Sigma \nabla \times \vec{E} \cdot d\vec{s}$$

we get

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Now this equation has a time derivative of the magnetic field. We also need an equation which has a time derivative of the electric field.

17.3. Ampere's Law.

- Ampere's Law, Circulation of Magnetic field.

$$\oint_{\partial\Sigma} \vec{B} \cdot d\vec{r} = \mu_0 \int_\Sigma \vec{j} \cdot d\vec{S}$$

- Problem with the Ampere's Law. As written it depends on Σ. Consider

$$\int_{\Sigma_1} \vec{j} \cdot d\vec{S} - \int_{\Sigma_2} \vec{j} \cdot d\vec{S} = \int_{\Sigma_1\cup\Sigma_2} \vec{j} \cdot d\vec{S} = \int_\Omega \nabla \cdot \vec{j} dV = -\frac{d}{dt}\int_\Omega \rho dV = -\epsilon_0 \frac{d}{dt}\int_\Omega \nabla \cdot \vec{E} dV =$$
$$-\epsilon_0 \frac{d}{dt}\int_{\Sigma_1\cup\Sigma_2} \vec{E} \cdot d\vec{S} = -\epsilon_0 \frac{d}{dt}\int_{\Sigma_1} \vec{E} \cdot d\vec{S} + \epsilon_0 \frac{d}{dt}\int_{\Sigma_2} \vec{E} \cdot d\vec{S}$$

We see, that

$$\int_{\Sigma_1} \vec{j} \cdot d\vec{S} + \epsilon_0 \frac{d}{dt} \int_{\Sigma_1} \vec{E} \cdot d\vec{S} = \int_{\Sigma_2} \vec{j} \cdot d\vec{S} + \epsilon_0 \frac{d}{dt} \int_{\Sigma_2} \vec{E} \cdot d\vec{S}$$

So that the combination $\int_\Sigma \vec{j} \cdot d\vec{S} + \epsilon_0 \frac{d}{dt} \int_\Sigma \vec{E} \cdot d\vec{S}$ is independent of Σ. If there is no electric field, then it is the same as just $\int_\Sigma \vec{j} \cdot d\vec{S}$. So we should write

- Ampere's law, corrected.

$$\oint_{\partial\Sigma} \vec{B} \cdot d\vec{r} = \mu_0 \int_\Sigma \vec{j} \cdot d\vec{S} + \mu_0 \epsilon_0 \frac{d}{dt} \int_\Sigma \vec{E} \cdot d\vec{S}.$$

- Local form of the Ampere's law

$$\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}.$$

17.4. Full set of Maxwell equations

Gauss's law:	$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_\Omega \rho dV,$	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Gauss's law magnetic:	$\oint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0,$	$\nabla \cdot \vec{B} = 0$
Faraday's law:	$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_\Sigma \vec{B} \cdot d\vec{S},$	$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
Ampere's law:	$\oint_{\partial\Sigma} \vec{B} \cdot d\vec{r} = \mu_0 \int_\Sigma \vec{j} \cdot d\vec{S} + \mu_0 \epsilon_0 \frac{d}{dt} \int_\Sigma \vec{E} \cdot d\vec{S},$	$\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$

In addition we should supply

- Initial conditions.
- Boundary conditions.
- "Material law". Plasmons.

Consequences:

- Coulomb law.
- Charge conservation – Gauss's and Ampere's laws.

LECTURE 18
Maxwell equations. Gauge invariance.

- Exam. Homework.

18.1. Maxwell equations.

Full set of Maxwell equations:

Gauss's law:	$\oint_{\partial\Omega} \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_\Omega \rho dV,$	$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$
Gauss's law magnetic:	$\oint_{\partial\Omega} \vec{B}\cdot d\vec{S} = 0,$	$\nabla\cdot\vec{B} = 0$
Faraday's law:	$\oint_{\partial\Sigma} \vec{E}\cdot d\vec{r} + \frac{d}{dt}\int_\Sigma \vec{B}\cdot d\vec{S} = 0,$	$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$
Ampere's law:	$\oint_{\partial\Sigma} \vec{B}\cdot d\vec{r} - \mu_0\epsilon_0\frac{d}{dt}\int_\Sigma \vec{E}\cdot d\vec{S} = \mu_0\int_\Sigma \vec{j}\cdot d\vec{S},$	$\nabla\times\vec{B} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j}$

In addition we should supply

- Initial conditions.
- Boundary conditions.
- "Material law". Plasmons.

Consequences:

- Coulomb law. The non-trivial spherically symmetric solution of the static Gauss and Faraday's equations with the boundary condition $\vec{E}(\vec{r}\to\infty)\to 0$ gives the Coulomb law.
- Charge conservation – Gauss's and Ampere's laws.
  - The time derivative of the Gauss law gives
$$\nabla\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial\rho}{\partial t}.$$
  - The div of the Ampere's law gives (we use $\nabla\cdot\nabla\times\vec{B} = 0$ for any $\vec{B}$)
$$-\mu_0\epsilon_0\nabla\cdot\frac{\partial\vec{E}}{\partial t} = \mu_0\nabla\cdot\vec{j}.$$
  - Comparing these two equations we get
$$\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0.$$
    The charge conservation law (the continuity equation).

Analysis of the equations.

- Units. From Faraday's law $\frac{[E]}{[l]} = \frac{[B]}{[t]}$, or $[E] = \frac{[l]}{[t]}[B]$. From Ampere's law $\frac{[B]}{[l]} = [\mu_0\epsilon_0]\frac{[E]}{[t]}$. So $\frac{1}{[\mu_0\epsilon_0]} = \frac{[l]^2}{[t]^2}$ – units of the square of the velocity.
- We have 8 equations for only 6 unknown functions $\vec{E}$ and $\vec{B}$.
- The equations impose two constraints on their right hand sides.
- The first constraint is that the charge is conserved $\nabla \cdot \vec{j} + \partial\rho/\partial t = 0$. It comes from Gauss's and Ampere's laws.
- The second one is trivial and comes from Gauss's magnetic and Faraday's laws. (If we had magnetic charges, this constraint would give us the conservation of magnetic charge.)

18.2. Gauge fields.

- Solve magnetic Gauss's and Faraday's laws (both equations have zeros on the right hand sides.)
$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}.$$
- The fields ϕ and $\vec{A}$ are called potential and vector potential respectively.

If we express $\vec{E}$ and $\vec{B}$ through the gauge fields $\vec{A}$ and ϕ the magnetic Gauss's law and the Faraday's law are automatically satisfied (notice, that these the laws that have zeros on RHS) The other two laws can be written as ($\Delta \equiv \nabla^2$.)

$$-\Delta\phi - \frac{\partial \nabla \cdot \vec{A}}{\partial t} = \frac{\rho}{\epsilon_0}$$
$$-\Delta\vec{A} + \vec{\nabla}(\nabla \cdot \vec{A}) + \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

- Notice, that now we have four equations and four unknowns ϕ and $\vec{A}$. But we still have one constraint on the functions ρ and $\vec{j}$ in the right hand sides. So effectively we have only three equations. It means that we have a freedom to chose a "gauge" for our fields ϕ and $\vec{A}$.
- This freedom, however, must not change the physical fields $\vec{E}$ and $\vec{B}$.

18.3. Gauge invariance.

- Gauge transformation, for any $f(\vec{r}, t)$ the transformation
$$\vec{A} \to \vec{A} + \nabla f, \qquad \phi \to \phi - \frac{\partial f}{\partial t}$$
does not change $\vec{E}$ and $\vec{B}$. But $\vec{E}$ and $\vec{B}$ are the only physically observable fields. So no matter what physical property we comute the result must be invariant under these gauge transformations.

Gauge symmetry (gauge freedom) allows us to chose any gauge we want. This choice is done by imposing an additional constraint on the fields ϕ and $\vec{A}$.
There are many particularly useful gauges. I give here two examples:

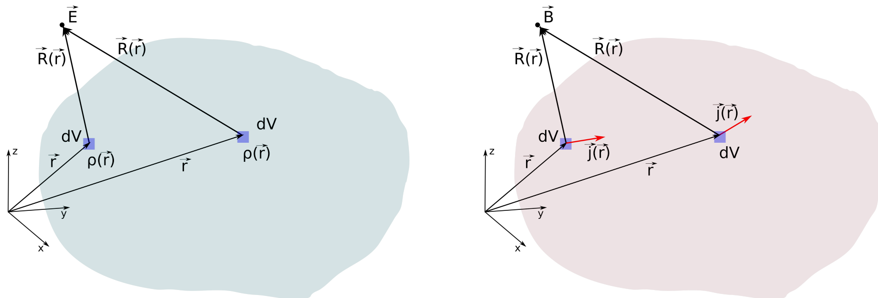


Figure 1. Illustration for the Coulomb and Biot-Savart laws.

Coulomb gauge. This gauge is given by the following gauge fixing condition

$$\nabla \cdot \vec{A} = 0.$$

The Maxwell equations then become

$$-\Delta\phi = \frac{\rho}{\epsilon_0}$$

$$-\Delta\vec{A} + \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

Lorenz gauge. This gauge is given by the following gauge fixing condition

$$\nabla \cdot \vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} = 0.$$

The Maxwell equations then become

$$-\Delta\phi + \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\epsilon_0}$$

$$-\Delta\vec{A} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

Notice, that both equations in this gauge can be written as

$$\left(-\Delta + \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\phi \\ \vec{A}\end{pmatrix} = \begin{pmatrix}\rho/\epsilon_0 \\ \mu_0\vec{j}\end{pmatrix}.$$

Also notice, that the combination $1/\sqrt{\epsilon_0\mu_0}$ has units of velocity.

18.4. Biot-Savart law.

In particular, if we are looking for the static solutions, meaning that neither ρ nor $\vec{j}$ depend on time and there is no EM waves around then neither ϕ nor $\vec{A}$ will depend on time (more precisely we can find a solution when neither ϕ nor $\vec{A}$ depend on time) Both Coulomb Lorenz gauges then give ($\partial_t\phi = 0$ and $\partial_t\vec{A} = 0$).

$$-\Delta\phi = \frac{\rho}{\epsilon_0}$$

$$-\Delta\vec{A} = \mu_0\vec{j}$$

Notice, that the equations look exactly the same. We know that the solution of the first equation for a point like charge is given by the Coulomb potential

$$d\phi = \frac{1}{4\pi\epsilon_0}\frac{\rho dV}{R}$$

So the solution of the second equation (for the "point like" current) must be

$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{j}dV}{R}$$

So for any static distribution of charges and currents we can find the electric and magnetic fields taking the gradient of $d\phi$ and the curl of $d\vec{A}$.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\rho dV\vec{R}}{R^3}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{dV\vec{j}\times\vec{R}}{R^3}$$

So for any static distribution of charges and currents we can find the electric and magnetic fields using the Coulomb and Biot-Savart laws.

The familiar form of the Biot-Savart law

$$d\vec{B} = -\frac{\mu_0}{4\pi}\frac{I\vec{R}\times d\vec{l}}{R^3}.$$

is obtained by assuming the current density is inside the small piece of wire of length dl and cross-section dS, then $\vec{j}dV = \vec{j}dSdl = Id\vec{l}$.

18.5. Light.

- Maxwell equations in vacuum — no static solutions.
- Wave equation.
- General solution of the wave equation.
- Speed of light.

LECTURE 19
Let there be light! Electromagnetic waves. Speed of light.

We saw that the Maxwell equation contain everything that we know about electric and magnetic fields. Here we will find out what else they have.

$$\begin{aligned}
\text{Gauss's law:} \qquad & \nabla \cdot \vec{E} = 0 \\
\text{Gauss's law magnetic:} \qquad & \nabla \cdot \vec{B} = 0 \\
\text{Faraday's law:} \qquad & \nabla \times \vec{E} + \tfrac{\partial \vec{B}}{\partial t} = 0 \\
\text{Ampere's law:} \qquad & \nabla \times \vec{B} - \mu_0 \epsilon_0 \tfrac{\partial \vec{E}}{\partial t} = 0
\end{aligned}$$

- Maxwell equations show the dynamics of the fields $\vec{E}$ and $\vec{B}$ themselves, independent of the dynamics of the sources/charges/currents.
- Consider Maxwell equations in vacuum — there are no static solutions.
- However, there are dynamical solutions.

  Acting by $\nabla\times$ on Faraday's law and using $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \Delta \vec{E}$ and the Gauss and Ampere's laws we get

$$\Delta \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

- Wave equation, $1D$.

$$\frac{\partial^2 \vec{E}}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

  - Boundary condition: $\vec{E}(t, x \to \pm\infty) \to 0$.
  - Initial condition: $\vec{E}(t = 0, x) = \vec{E}_0(x)$ – it can be thought as a boundary condition in time.
- General solution of the wave equation.

$$\vec{E}(x, t) = \vec{E}_0(x \pm ct).$$

  (According to the Gauss magnetic and Ampere's laws magnetic field will also be generated.)
- Speed of light.

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

- Problem with the speed of light.

  In Galilean/Newtonian mechanics any velocity depends on the frame of reference the observer is in. However, the Maxwell equations are valid in any frame of reference, so in any frame of reference the e-m wave propagates with the velocity c. This is a clear contradiction.
- Both Maxwell equations/theory as well as Galilean/Newtonian mechanics were thoroughly tested in many many experiments.
- Idea of Aether — a special universal frame of reference. Michelson-Morley experiment. `https://en.wikipedia.org/wiki/Michelson%E2%80%93Morley_experiment`.

What are the space-time transformations that leave the Maxwell equations invariant?

- Galilean transformation. Transformations that leave the Newton' equation invariant:
$$dx = dx' + Vdt', \qquad dt = dt'$$
- Lorenz transformation. Transformations that leave the wave equation invariant. Look for the transformation in the form
$$dx = Adt' + Bdx', \qquad dt = Cdt' + Ddx'$$
  then considering the transformation of variables $x'(x,t)$ and $t'(x,t)$ and using the chain rule and the definition of differential we get
$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = B\frac{\partial}{\partial x} + D\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = A\frac{\partial}{\partial x} + C\frac{\partial}{\partial t}$$
  So that
$$\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \left(B^2 - \frac{1}{c^2}A^2\right)\frac{\partial^2}{\partial x^2} + \left(D^2 - \frac{1}{c^2}C^2\right)\frac{\partial^2}{\partial t^2} + 2\left(BD - \frac{1}{c^2}AC\right)\frac{\partial^2}{\partial x\partial t}$$
  In order for the wave equation not to change its form we must have
$$B^2 - \frac{1}{c^2}A^2 = 1, \qquad D^2 - \frac{1}{c^2}C^2 = -\frac{1}{c^2}, \qquad BD - \frac{1}{c^2}AC = 0$$
  We have three equation with four unknowns. The solution depends on one parameter γ and can be written as
$$dx = \frac{\gamma c dt'}{\sqrt{1-\gamma^2}} + \frac{dx'}{\sqrt{1-\gamma^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-\gamma^2}} + \frac{\gamma dx'}{\sqrt{1-\gamma^2}},$$
  This is called Lorentz transformation.
- At this stage γ is an arbitrary parameter. These transformation rules do not have any physical content. It is so far just a mathematical statement that Lorenz transformation with arbitrary γ will leave the wave (in fact Maxwell) equations invariant.
- In order to understand the physical meaning of these transformations we need to figure out what is γ.
- Comparing the Lorenz transformation to the Galileo transformation we find that $\gamma = V/c$
$$dx = \frac{Vdt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{Vdx'/c}{\sqrt{1-V^2/c^2}},$$

- Now we understand that the Lorenz transformation tells us how to go from one frame of references to another!
- These transformations also tell us that our space-time has a very different structure than what was thought before.

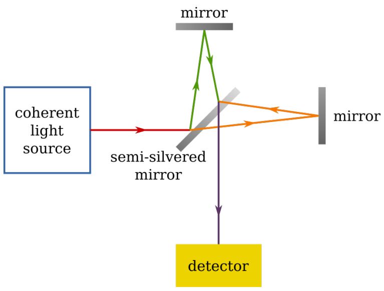


Figure 1. A Michelson interferometer uses the same principle as the original experiment. But it uses a laser for a light source.

LECTURE 20
Special theory of relativity.

What we have done so far:

- Intuition: Translation and time translation invariance, Galilean inviriance $\longrightarrow$ Newtonian mechanics. Experiments to check the validity.
- Experiments with magnetic and electric fields: Lorenz force, Gauss laws (both), Faraday's law, Ampere's law + writing it all in the form that makes sense $\longrightarrow$ Maxwell equations. Experiments to check the validity.
- Comparing the Newton's dynamics and Maxwell equations $\longrightarrow$ conundrum $\longrightarrow$ Lorenz transformation.
- Galilean inviriance is only approximate $\longrightarrow$ Newtonian mechanics is only approximate, it works only if speeds are much less then the speed of light (whether we can use the Newtonian mechanics or not depends on the problem and on the accuracy we need. The Newtonian mechanics will always have the corrections of the order of $(v/c)^2$. In many cases these corrections are beyond the resolution of our experimental devices.)

Lorenz transformation:

- Lorenz transformation. If V is along the x direction, then

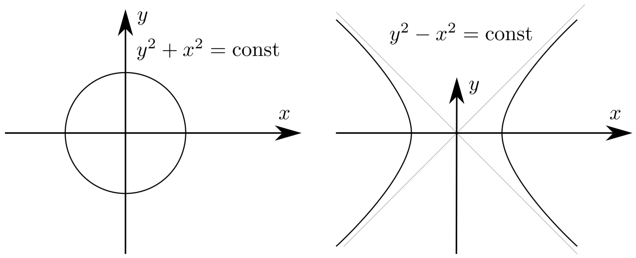
$$cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{Vdx'/c}{\sqrt{1-V^2/c^2}}, \qquad dx = \frac{Vdt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}, \qquad dy = dy', \qquad dz = dz'.$$

- The inverse of the Lorenz transformation has the same form:

$$dx' = -\frac{Vdt}{\sqrt{1-V^2/c^2}} + \frac{dx}{\sqrt{1-V^2/c^2}}, \qquad cdt = \frac{cdt}{\sqrt{1-V^2/c^2}} - \frac{Vdx/c}{\sqrt{1-V^2/c^2}},$$

  with $V \to -V$, as expected.
- These transformations tell us that our space-time has a very different structure than what was thought before.
- Lorenz transformation is the transformation that leaves the interval $ds^2 = c^2dt^2 - dx^2$ invariant.
- $ds^2 = c^2dt^2 - dx^2$ — metric of space-time!
- Event is a point of a space-time. Interval ds is the "distance" between the Events.
- This provides a true metric for the space-time. So the full space-time has geometry!

- A space (space-time) with such metric is called Minkowskii space. The metric is called Minkowskii metric.
- Lorenz transformation is a "rotation" of the space-time.
- GPS, LHC.



Consequences:

- Events that are simultaneous in one frame of reference are not necessarily simultaneous in another (In contrast to Galilean transformation.)
- Velocity transformation: $v' = dx'/dt'$, $v = dx/dt$.

$$v = \frac{V + v'}{1 + \frac{Vv'}{c^2}}.$$

  - If both $v', V \ll c$, then $v = V + v'$ – our usual Galilean result!
  - If $v' = c$, then $v = c$! The e.-m. wave indeed travels with the same speed in all frames of references!
- Time change. The experiment is the following: A person in the moving (primed) frame is staying put, so $dx' = 0$, and measures the time interval dt' so the time interval dt in the frame of reference at rest is:

$$dt = \frac{dt'}{\sqrt{1 - V^2/c^2}}$$

- Twin's paradox.
- Length change. The experiment is the following: a stick in the moving frame of reference is measured by a person in the same (moving) frame of reference (so the stick is not moving with respect to this person) The result is dx'. The length of this stick is now measured in the frame of references at rest. In order to do that the researcher must note the positions of the ends of the stick at the same moment of time in his frame! so for his measurement $dt = 0$. It then means that $cdt' = -\frac{V}{c}dx'$, and

$$dx = \frac{-V^2 dx'/c^2 + 1}{\sqrt{1 - V^2/c^2}} dx' = dx'\sqrt{1 - V^2/c^2}.$$

- Doppler effect. The speed of light is the same for every observer. However, different observers see this light differently.

  The light source S' moves with respect to the observer S with velocity V directly away. In the frame S' the distance between two wave fronts is $dx' = c/f'$, the time between them is just one period $dt' = T' = 1/f'$. In the frame S we then have

$$dx = \frac{V/f'}{\sqrt{1 - V^2/c^2}} + \frac{c/f'}{\sqrt{1 - V^2/c^2}}, \qquad cdt = \frac{c/f'}{\sqrt{1 - V^2/c^2}} + \frac{V/f'}{\sqrt{1 - V^2/c^2}}.$$

First we notice, that $cdt = dx$ as it must be – the speed of light is the same for both observers.(It also means that the Minkowskii interval $(ds)^2 = (cdt)^2 - (dx)^2 = 0$ for light. So the light is the straight line in Minkowskii space.) Second, we notice, that

$$f = \frac{1}{dt} = \sqrt{\frac{c-v}{c+v}} f'.$$

This is Doppler effect.

There are special mathematical notations that make it much easier to work in Minkowskii, (or any other) space.

LECTURE 21
Special theory of relativity. General theory of relativity.

- Homework.

Doppler effect.

- Red shift.
- Blue shift.
- Velocity of the stars in the galaxy
- Hable constant.
- Universe expansion.
- Distance to the stars.
- Light year, parsec (3.3 light years) `https://en.wikipedia.org/wiki/Parsec`
- Astronomical distances: `https://en.wikipedia.org/wiki/List_of_nearest_galaxies`
  - Distance to the Sun ~ 8 light minutes.
  - Distance to the closest other star, Alpha Centauri: 4.367 light years.
  - The diameter of the Milky Way Galaxy: $\sim 100 - 180$ thousand light years. Milky way galaxy has a lot of satellite galaxies.
  - The distance to the next large galaxy, Andromeda: 2.5 million light years.
- Look into the past. Microwave background radiation `https://en.wikipedia.org/wiki/Cosmic_microwave_background`.

Dynamics.

- Energy and momentum.
$$dE = F dx, \quad dp = F dt, \quad ds^2 = c^2 dt^2 - dx^2 = (c^2 dp^2 - dE^2)/F^2$$
so $E^2 - c^2p^2 = \text{const}$ must be invariant under the Lorenz transformation. For small p using Taylor expansion and comparing to $E = p^2/2m_0$ (remember, energy is defined up to a constant) we find
$$E^2 = c^2p^2 + m_0^2c^4,$$
where m_0 – mass at rest. In particular for $p = 0$ we have $E = m_0c^2$ – energy at rest.
- Momentum and velocity.

  Energy as a function of momentum is Hamiltonian, so we can write the Hamiltonian equations of motion:
$$\dot{x} = \frac{\partial E(p)}{\partial p}, \qquad \dot{p} = -\frac{\partial E(p)}{\partial x}$$

The first equation gives $v = \dot{x}$:

$$v = \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}}, \quad \text{or} \quad p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}.$$

One can also say, that

$$p = mv, \qquad m = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

- Energy and velocity.
  Using p in $E(p)$ we find
  $$E = c^2p^2 + m_0^2c^4 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} = mc^2.$$
- Example. Nuclear binding energy energy.
  - A nucleus consists of N neutrons and P protons.
  - We know the mass of each neutron m_N and each proton m_P.
  - We measure the mass of the nucleus M.
  - The binding energy is
  $$E = (Nm_n + Pm_P - M)c^2.$$

21.1. A bit of general theory of relativity.

- Inertial and gravitational masses.
  - Compare the Newton's third law and Newton's gravity
  $$\vec{F} = m\vec{a}, \qquad F = \frac{GmM}{R^2}.$$
  These two laws assume very different experiments: In the first, one applies a force and measures the acceleration; in the second one keeps two masses stationary at distance R to each other and measures the force.
  - In the first experiment we measure the response of a free object to an applied force. In the second we measure the gravity force between two objects.
  - How come the masses in the two laws are the same? More precisely, how come if we double the mass m in the first experiment we will also double the force in the second experiment?
  - It does not happen to any other force. The Coulomb force, for example, will not double.
- Non-inertial frame of references.
  - Imagine, that you are staying in a closed box (big enough, but you cannot see the outside) Consider two situations: in one the box is accelerating with a constant acceleration; in the second a planet is moved close to the box.
  - Is there any way for you do distinguish between these?
  - Because the inertial (third law) and gravitational (Newton's gravity) masses are the same, there is no way to tell if your box is accelerating, or you are in the gravitational field.
  - In fact the only way to do that is to look at infinity. The gravity decay's at infinity, but the "fake" force in the accelerated frame does not.

  - This suggests, that the gravity is equivalent to the local non-inertial frame. This translates to the local curvature of the space-time.
- Space-time metric.
- Black holes `https://en.wikipedia.org/wiki/Black_hole`.
- Gravitational lensing `https://en.wikipedia.org/wiki/Gravitational_lens`.
- Gravitation waves: The gravitational field has its own dynamics $\longrightarrow$ the waves of gravity can propagate! `https://en.wikipedia.org/wiki/Gravitational_wave`

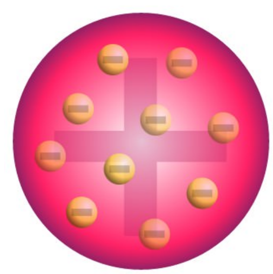
LECTURE 22
Problems with classical theory.

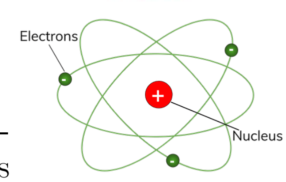
22.1. Waves vs stream of particles

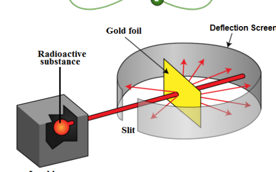
- Common features:
  - Energy flux. Amount of energy which crosses a unit area per unit time.
  - Momentum flux. Amount of momentum which crosses a unit area per unit time
- Difference
  - Waves: diffraction and interference, or, in one word, phase.
  - Particles: number of particles.

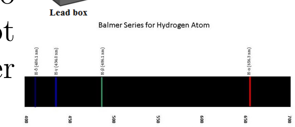
22.2. Particles are waves.

- Atom stability. In the classical theory if an electron orbits the positively charged center the electron, as it moves with acceleration, will emit e.-m. waves. So it will lose the energy. As the electron loses the energy its orbit must shrink until it reaches the central nucleus and the atom collapses. One can compute the time it takes for an atom to collapse: $t \approx 2 \times 10^{-11}$s, `http://www.physics.princeton.edu/~mcdonald/examples/orbitdecay.pdf`
  - Plum Pudding model.
  - Rutherford experiment.
- Atomic spectra. In the classical theory an electron orbiting a nucleus should emit e.-m. waves of all frequencies. In other words the spectrum of emitted light should be continuous. However in the experiment the spectrum is discrete – consists of several sharp spectral lines.
- It looks as if there is some sort of diffraction for electrons. So that only few states are available for an electron – so it cannot collapse. And when electron transition from one state to another it emits light of a very specific frequency.



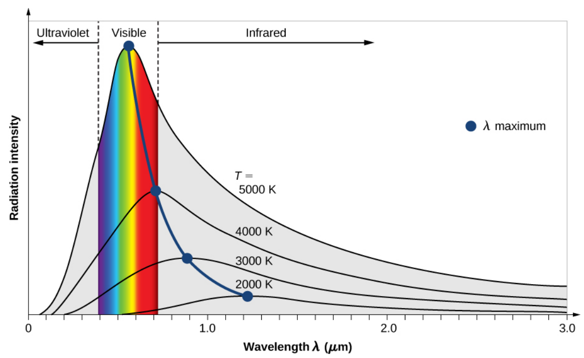






22.3. Waves are particles.

- Black body radiation `https://en.wikipedia.org/wiki/Black_body`.



Observations:
  - Goes to zero when $\lambda \to 0$ (or $f \to \infty$).
  - Has a maximum! Nothing in the Maxwell equations can give a scale!

$$\lambda_{\text{max}} T = 2.898 \times 10^{-3} m \cdot K.$$

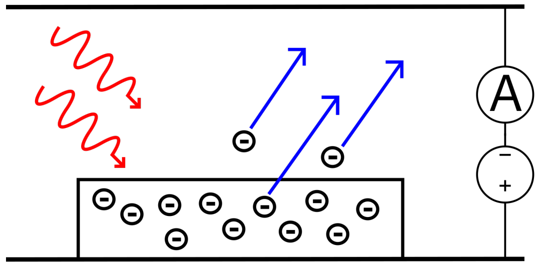
Planck's formula:

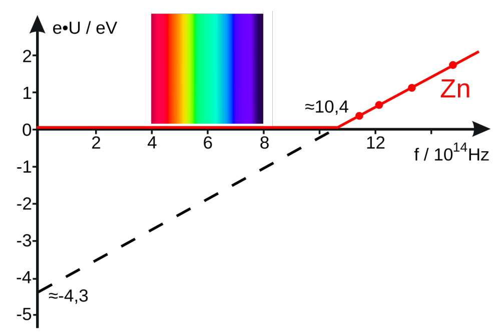
$$u(f, T) = \frac{8\pi h f^3}{c^3} \frac{1}{e^{hf/k_B T} - 1}.$$

where $h = 6.6 \times 10^{-34} J \cdot s$. Often used $\hbar = \frac{h}{2\pi}$.

Planck's formula suggests that light consists of particles with energy $\epsilon = hf = \hbar\omega$ each.

- Photo-electric effect.





  - One shines light on a metal plate. The metal plate is on one plate of a parallel plate capacitor, see figure. We can apply voltage to the plates of the capacitor and measure current. We also can control the frequency/wavelength and intensity of the light.
  - If no light shines, there will be no current.
  - If we shine the light, the light may kick some electrons out of the metal plate. Then if we apply the negative terminal of the battery (opposite to what is show on the figure) then to the plate with the metal, then if electrons are kicked out by the light we will measure the current. The current is proportional to the number of electrons per second "kicked out" by the light.
  - If we now apply the positive battery terminal to the metal plate only the electrons which have large enough kinetic energy to overcome the electric field inside the capacitor will reach the upper plate. This way measuring the current we will be able to measure the kinetic energy of the "kicked out" electrons.
  - Classically the light is a wave, hence the energy of the "kicked out" electrons should only depend on intensity. One can think of it as a wave in the ocean

coming to a send beach. It hits the send and the velocities of the send particles will depend on how big the wave is (or how large the intensity is). The number of flying send particle at a given waves' intensity, should depend on how often the waves hit the beach — on frequency of the waves.

- However, the experiment shows exactly the opposite: the energy of the knocked out electrons depends only on frequency, while the number of knocked out electrons (per unit time) depends on intensity.

$$E = \hbar\omega - \mathcal{A}.$$

With the same constant $\hbar$ as in the Planck's formula!

- The threshold energy $\mathcal{A}$ does not depend on light and only depends on the material.
- The explanation, due to Einstein, is that the light is a flux of light particles. Each particle has the energy $\hbar\omega$. The intensity of the light is how many light particles crosses a given cross-section in a given time. An electron absorbs the light particle and acquires the energy $\hbar\omega$. It has to overcome the crystal attraction, it loses the energy $\mathcal{A}$ by doing so. So it emerges outside with the energy $E = \hbar\omega - \mathcal{A}$. The number of kicked out electrons (if they can overcome the attraction) depends on how many light particles hit the material per unit time — the light intensity.
- ARPES.

• Compton scattering (X ray of large energy, electrons are free). θ is the angle of the scattered light.

$$\lambda' - \lambda = \lambda_e(1 - \cos\theta), \qquad \lambda_e \approx 2.4 \times 10^{-12} m$$

Where does the length λ_e come from?

For e.-m. wave from the Maxwell equations we can find, that the momentum flux is energy flux divided by c. If we consider light with a wave length λ and frequency $f = c/\lambda$ as a stream of particles with energy $\epsilon = hf = hc/\lambda$, as photoelectric effect and black body radiation suggest, then the momentum of each particle is $p = \hbar\omega/c = h/\lambda$. Then momentum and energy conservation laws give

$$\begin{aligned} \text{momentum, parallel component:} \quad & \tfrac{h}{\lambda} = \tfrac{h}{\lambda'}\cos(\theta) + p_e\cos(\alpha) \\ \text{momentum, perpendicular component:} \quad & 0 = \tfrac{h}{\lambda'}\sin(\theta) - p_e\sin(\alpha) \\ \text{energy:} \quad & \tfrac{ch}{\lambda} = \tfrac{ch}{\lambda'} + \tfrac{p_e^2}{2m} \end{aligned}$$

Expressing p_e^2 from the first two equations and using it in the third we get

$$\frac{1}{2m}\left(\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}\cos(\theta)\right) = \frac{ch}{\lambda} - \frac{ch}{\lambda'}.$$

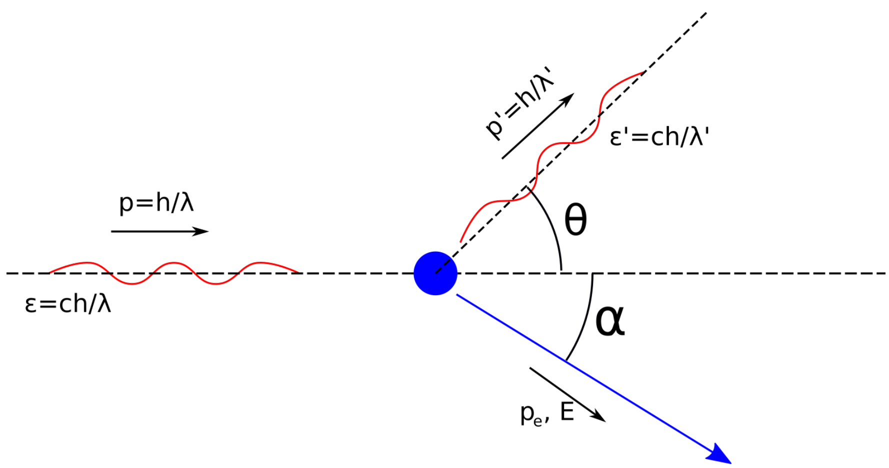
In the case $\lambda \approx \lambda'$ this simplifies to

$$\lambda' - \lambda = \frac{h}{cm}(1 - \cos\theta), \qquad \text{so} \quad \lambda_e = \frac{h}{cm}.$$

Surprisingly many of the puzzling experimental results can be explained by considering particles as waves and waves as particles.

LECTURE 23
Beginnings of the Quantum Mechanics.

- Survey.



- What we learned from Photo-electric effect.
  - Light is a stream of particles.
  - The energy of each particle depends only on frequency

$$\epsilon = \hbar\omega, \qquad \epsilon = hf.$$

  - Intensity of light is how many particles cross a given area during a given time – the current of particles.
  - The particles move with the speed of light.
  - If it is a particle, then what is its momentum?
  - This questions could be answered in two ways, first by computing the momentum flux of the light directly from the Maxwell equations; second using special theory of relativity. The second way is simpler, so we use it.
  - As $\epsilon = \frac{m_0c^2}{\sqrt{1-v^2/c^2}}$, the only possibility for a particle to move with the speed of light is to have $m_0 = 0$. (then we have $0/0$ and should think of it as a limit and take it properly.)
  - This corresponds to the fact that the light cannot be stationary – there are no static solutions of the Maxwell equations in vacuum.
  - Then $\epsilon = \sqrt{m_0^2c^4 + p^2c^2} = pc$.

  - So a light particle — a photon — has the momentum

$$p = \frac{\hbar\omega}{c}, \qquad p = \frac{hf}{c}.$$

  - ARPES

- Compton scattering experiment (X ray of large energy, electrons are free). θ is the angle of the scattered light.

$$\lambda' - \lambda = \lambda_e(1 - \cos\theta), \qquad \lambda_e \approx 2.4 \times 10^{-12} m$$

From the classical point of view this is a very strange result, because it does not depend on the intensity of the X-rays. Classically, electron is accelerated by the electric field of light. The electric field is larger, the larger the intensity (intensity is proportional to the electric field squared). So the momentum and the energy of the scattered electron should depend on the intensity, but the energy the electron acquires is the energy the light loses, so the change of the light frequency/wavelength should depend on the intensity. How come the experiment does not show this?

- This question can be reformulated as: Where does the length λ_e come from?
- Let's consider light with a wave length λ and frequency $f = c/\lambda$ as a stream of particles with energy $\epsilon = hf = hc/\lambda$, as photo-electric effect and black body radiation suggest, than the momentum of each particle is $p = \hbar\omega/c = h/\lambda$. Then momentum and energy conservation laws give

$$\begin{aligned} \text{momentum, parallel component:} \quad & \tfrac{h}{\lambda} = \tfrac{h}{\lambda'}\cos(\theta) + p_e\cos(\alpha) \\ \text{momentum, perpendicular component:} \quad & 0 = \tfrac{h}{\lambda'}\sin(\theta) - p_e\sin(\alpha) \\ \text{energy:} \quad & \tfrac{ch}{\lambda} = \tfrac{ch}{\lambda'} + \tfrac{p_e^2}{2m} \end{aligned}$$

Expressing p_e^2 from the first two equations and using it in the third we get

$$\frac{1}{2m}\left(\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}\cos(\theta)\right) = \frac{ch}{\lambda} - \frac{ch}{\lambda'}.$$

In the case $\lambda \approx \lambda'$ this simplifies to

$$\lambda' - \lambda = \frac{h}{cm}(1 - \cos\theta), \qquad \text{so} \quad \lambda_e = \frac{h}{cm}.$$

This λ_e has the same value as the experimental one. Again, there is h in this result. There is no h in classical physics.

23.1. Bohr atom.

We want to consider how atoms emit the light. Remember, the light is a bunch of particles, it is also a wave.

- Consider a electron moving around a center — nucleus — with the opposite charge. (Mass of the nucleus is much larger than the mass of the electron.) There are two conserved quantities: energy E, and angular momentum L. We want to express energy in terms of angular momentum.
  - For a circular orbit we have

$$\frac{ke^2}{r^2} = \frac{mv^2}{r}, \qquad mv^2 = \frac{ke^2}{r}$$

– The angular momentum

$$L = mvr, \qquad \text{or} \qquad L^2 = m^2v^2r^2 = ke^2mr, \qquad r = \frac{L^2}{mke^2}, \qquad v = \frac{L}{mr} = \frac{ke^2}{L}$$

– Energy and frequency then are:

$$E = \frac{mv^2}{2} - \frac{ke^2}{r} = -\frac{1}{2}\frac{k^2e^4m}{L^2}, \qquad \omega = \frac{v}{r} = \frac{mk^2e^4}{L^3}.$$

- According to Maxwell the frequency of light emitted by a hydrogen atom must equal to the frequency of the rotation of the electron — this is “light as a wave” picture.
- The energy of the emitted “Einstein photon” $\hbar\omega$ must be equal to the difference in the energies of the electron — the total energy is conserved! This is the “light as a particle” picture.
- Assume that the change of the electron’s energy is small.

$$dE = \frac{dE}{dL}dL = \frac{k^2e^4m}{L^3}dL = \omega dL$$

(in fact $\omega = \dot{\phi} = \frac{\partial H(L,\phi)}{\partial L}$ – Hamiltonian equation.)
- This change of energy dE must be equal to the energy of the emitted photon $\hbar\omega$. We then have

$$\hbar\omega = \omega dL, \qquad dL = \hbar.$$

- Then

$$L = \hbar n + L_0, \qquad n = 1, 2 \ldots$$

Assuming $L_0 = 0$ we get

$$L = \hbar n, \qquad n = 1, 2 \ldots$$

These are the only “allowed” values of angular momentum. Using these we can compute

$$E_n = -\frac{1}{2}\frac{k^2e^4m}{\hbar^2}\frac{1}{n^2} = -\frac{13.6}{n^2}\text{eV}, \qquad r_n = \frac{\hbar^2}{mke^2}n^2 = a_Bn^2, \quad a_B = 0.0529\text{nm}.$$

These are the only “allowed” values of energy, and sizes of the atom.

From this picture it is not clear why there is only a discrete set of “allowed” energies for an electron. The calculation only states that in order for the “light as a wave” and “light as a particle” pictures to be consistent an atom must have only a discrete set of “allowed” energies. Simultaneously, this picture of an atom explains the two puzzles: the stability of the atom — there is a minimal “allowed” energy; the discrete atomic spectra — the light is emitted when an electron transitions from one “state” to another with lower energy

$$\hbar\omega = E_n - E_{k<n}.$$

So this spectrum is discrete and describes the experimental observation very well.

- Still the question remains: what is the nature of the electrons (and hence all other particles) that would lead to the above result.

23.2. de Brolie's idea.

- According to Bohr
$$L = pr = n\hbar, \qquad \text{or} \qquad 2\pi r p = nh.$$
If we now assume that the electron is a wave with the wavelength $\lambda = \frac{h}{p}$, then the Bohr quantization rule becomes
$$\frac{2\pi r}{\lambda} = n,$$
which is the condition for the constructive interference.
- Particles as waves.

LECTURE 24
Particles as waves. The Schrödinger equation.

- Homework.
- Survey, Evaluations.

de Brolie's idea was that a particle is a wave.

$$\lambda = \frac{h}{p}, \qquad \omega = \frac{E}{\hbar}$$

- Double slit experiment. Interpretation as probability. Interference – superposition and square of the wave.

Wave of what? This question is asked very often. Somehow, when we talk about particles no one asks "particle of what?" For example, if I say "electron is a particle" no one asks "particle of what?" however, if I say electron is a wave the standard question is "wave of what?". The reason for this is clear, when we say something is a particle we imply certain properties – trajectories or time evolution, momentum, interactions with other elements etc. This intuitive understanding of what to expect from "a particle" makes the question "particle of what?" irrelevant. So the question "wave of what?" in fact means "what properties does this wave have?", or "what do we measure?" and "How does it evolve with time?" and "How does it interact with other elements?". These questions are non-trivial and are the central question of the quantum mechanics.

The question of what to measure we will address later. In this lecture we find how this wave evolves with time.

The question we ask is if we know the wave at the initial time and we have a description of our system what will be that wave at a later time? (Notice, that this is exactly the same as with particles: if we have initial conditions and have a description of our system what will be the position of the particle at a later time?)

Time evolution of a wave should be described by a wave equation. The major tool for finding this wave equation is the realization, that classical mechanics works well and our new description of the world should not contradict the classical mechanics where classical mechanics works well.

24.1. The wave equation.

An oscillator.

$$\ddot{f} + \omega^2 f = 0$$

There are two linearly independent solutions

$$f_1(t) = \cos(\omega t) \qquad \text{and} \qquad f_2(t) = \sin(\omega t).$$

Any linear combination of these is also a solution. In particular

$$f(t) = \cos(\omega t) + i\sin(\omega t) = e^{i\omega t}$$

is a solution. This solution has the property that

$$|f|^2 = 1$$

at all times.

We can look at this oscillator as a zero dimensional wave. In $1D$ it will become

$$\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = 0$$

The simplest solutions are

$$f_\pm(x,t) = e^{i\omega t \pm i\omega x/v}$$

for any ω.

- Both solutions describe the waves propagating with the velocity v.
- The velocity does not depend on ω.
- The period and the wavelength of the both waves are

$$T = \frac{2\pi}{\omega}, \qquad \lambda = \frac{2\pi v}{\omega} = Tv.$$

- f_+ propagates to the left, f_- propagates to the right.
- Both solutions have the property that

$$|f_\pm|^2 = 1$$

  at all times and everywhere in space.

The wave equation can be written as

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right) f = 0$$

Looking at each factor separately we see that

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f_- = 0$$

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right) f_+ = 0$$

So the equation

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f = 0$$

describes a wave propagating to the right only.

24.2. Schrödinger equation.

The propagation of the electromagnetic wave of frequency ω and wavelength λ is given by $e^{ikx-i\omega t} = e^{2\pi ix/\lambda - i\omega t}$. For the el.-m. wave the velocity is always c, so $\lambda\omega/2\pi = c$. For matter wave we do not have such restriction. However, for the both el.-m. and matter waves we have $p = 2\pi\hbar/\lambda$ and $E = \hbar\omega$, so we write

$$\Psi(x,t) = e^{ipx/\hbar - iEt/\hbar}$$

For a classical particle we must have $E = \frac{p^2}{2m}$, the wave Ψ then must satisfy the following equation

$$\left[i\hbar\frac{\partial}{\partial t} - \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\right]\Psi = 0$$

Or

$$i\hbar\frac{\partial \Psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi.$$

Let's look at the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. If we act on a wave function by this operator we get $\hat{p}\Psi = p\Psi$. So this is an operator of momentum. Using this notation we get

$$i\hbar\frac{\partial \Psi}{\partial t} = \frac{\hat{p}^2}{2m}\Psi.$$

Comparing this to the Hamiltonian for the free moving particle $H = \frac{p^2}{2m}$, one can write

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x).$$

The operator $\hat{H}$ is called the Hamiltonian operator. The above equation is the Srödinger equation.

24.3. Wave function.

- Interpretation. Probability Density Amplitude.
- Srödinger equation is linear in Ψ and homogeneous, so Ψ is defined up to a multiplicative factor. Normalization.

LECTURE 25
Wave function. Time independent Schrödinger equation.

In the last lecture we discussed the time evolution of the wave function. It is given by the Schrödinger equation (we will only discuss $1D$ case.)

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x), \qquad \int |\Psi(x,t)|^2 dx = 1.$$

Interpreting $|\Psi(x,t)|^2$ as probability density, we see that the last equation above is the statement that you have 1 particle somewhere at all times.

- NORMALIZATION!! You must normalize the wave functions. Always.
- Particles as waves.
- Heisenberg uncertainty principle: $\Delta x \Delta p \geq \hbar/2$.
  - We try to localize a free particle withing an interval Δx.
  - The boundary conditions demand $p\Delta x = 2\pi\hbar$.
  - We see, that we cannot localize a particle without changing its momentum!
- Waves as particles: Notice, that the wave function $e^{\frac{i}{\hbar}(px-Et)}$ can be written as $e^{\frac{i}{\hbar}\int(p\dot{x}-E)dt} = e^{\frac{i}{\hbar}S}$, where S is the classical Action.
- To classical. If we take $\hbar \to 0$, then only the stationary point (minimum) of the Action will contribute, so the trajectory of the classical particle is the one which is given by the minimum of the Action! This is the Hamilton principle!

25.1. Wave function.

- Interpretation. Probability Density Amplitude.
- Measurables as operator averages. Given a wave function $\Psi(x,t)$ – it must be normalized $\int_{-\infty}^{\infty} |\Psi|^2 dx = 1$.
  - Coordinate:

$$\bar{x}(t) = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 x dx = \int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) dx \int_{-\infty}^{\infty} = \Psi^*(x,t)\hat{x}\Psi(x,t)dx,$$

where $\hat{x}$ is an operator of coordinate — it just multiplies the function it acts on by x: $\hat{x}\Psi(x,t) \equiv x\Psi(x,t)$.

- Momentum. Momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, so the average momentum is
$$\bar{p} = \int_{-\infty}^{\infty} \Psi^* \hat{p} \Psi dx.$$
It will always be real! despite the fact that there is i in the definition of $\hat{p}$.
- Energy
$$\bar{E} = \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi dx.$$
- Any other measurable $\mathcal{O}$:
$$\bar{\mathcal{O}} = \int_{-\infty}^{\infty} \Psi^* \hat{\mathcal{O}} \Psi dx$$
- Noise. The good (but not the only) measure of Quantum mechanical noise in the measurement $\hat{\mathcal{O}}$ is
$$(\Delta\mathcal{O})^2 = \overline{\left(\hat{\mathcal{O}} - \bar{\mathcal{O}}\right)^2} = \overline{\mathcal{O}^2} - \bar{\mathcal{O}}^2$$

- What operator average is measured by a given experiment is always the first question in analysis of the experiment in quantum mechanics.

25.2. Time independent Schrödinger equation.

If the Hamiltonian does not depend on time, then we can look for the solution of the Schrödinger equation
$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$
in the form
$$\Psi(x,t) = e^{-iEt/\hbar}\psi(x),$$
Then we have
$$\hat{H}\psi = E\psi.$$
This is a second order (remember, $\hat{H}$ has $\hat{p}^2$ inside), linear homogeneous differential equation. For any E it has two linearly independent solutions. However, if we are looking for the solutions that satisfy the normalization condition $\int \psi^*\psi dx = 1$, then we find that such solutions exist only for real E and in many cases only for a discrete set of E.

- Energy as an eigen-value of the Hamiltonian.
  - There is a set of functions $\psi_n(x)$ and corresponding set of numbers E_n, such that
$$\hat{H}\psi_n(x) = E_n\psi_n(x), \qquad \int \psi_n^*(x)\psi_n(x)dx = 1.$$
  - From linear algebra we know that if $\hat{H}$ is hermitian `https://en.wikipedia.org/wiki/Hermitian_matrix`, then all E_n are real, and
$$\int \psi_n^*(x)\psi_{n'}(x)dx = \delta_{n,n'}.$$
  - The functions $\psi_n(x)$ are called states.
- Quantum numbers = enumeration of the eigen functions.
- Eigen functions = Complete basis in the space of functions.
- Bra-ket notations $|\psi_n\rangle \equiv \psi_n(x)$, and $\langle\psi_n| \equiv \psi^*(x)$.
- Normalization. $\langle\psi_{n'}|\psi_n\rangle = \int_{-\infty}^{\infty} \psi_{n'}^*(x)\psi_n(x)dx = \delta_{n,n'}$.

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$$

If at initial time we have $\Psi(x,0)$, then we can write

$$\Psi(x,0) = \sum_n a_n\psi_n(x), \qquad \text{or} \qquad |\Psi(t=0)\rangle = \sum_n a_n|\psi_n\rangle, \qquad \text{or} \qquad a_n = \langle\psi_n|\Psi(t=0)\rangle$$

The time evolution of an eigen function is simple

$$|\psi_n\rangle \to |\psi_n\rangle e^{-iE_nt/\hbar}$$

so

$$|\Psi(t)\rangle = \sum_n a_n e^{-iE_nt/\hbar}|\psi_n\rangle.$$

We see, that if the Hamiltonian does not depend on time the set of eigenvalues and eigenfunctions of the Hamiltonian operator solves the problem — we can compute the wave function at all times.

In order to compute a quantum mechanical average for some operator $\hat{\mathcal{O}}$ at arbitrary time we can use

$$\langle\Psi(x,t)|\hat{\mathcal{O}}|\Psi(x,t)\rangle = \sum_n e^{iE_nt/\hbar}a_n^*\langle\psi_n|\hat{\mathcal{O}}\sum_m a_m e^{-iE_mt/\hbar}|\psi_m\rangle = \sum_n\sum_m e^{i(E_n-E_m)t}a_n^*\langle\psi_n|\hat{\mathcal{O}}|\psi_m\rangle a_m$$

Similar to the matrix manipulations. Numbers $\langle\psi_n|\hat{\mathcal{O}}|\psi_m\rangle$ are called matrix elements of the operator $\hat{\mathcal{O}}$.

- Linear combinations. Basis. Quantum numbers.
- Spectrum. Discrete and continuous spectrum.
- Ground state, excited states. Transitions. Perturbations.

LECTURE 26
Discrete spectrum. Classically prohibited region. Tunneling.

In this lecture we consider several quantum mechanical problems. All of them are $1D$ and are defined by the potential. So classically all forces are conservative, then knowing the potential energy $U(x)$ we can relatively easily solve these problem. Moreover, just by looking at the form of the potential energy we can figure out how the classical motion will look like. Quantum mechanics has some surprises even for these simple problems.

26.1. Particle in the infinite square well potential.

- The potential:

$$U(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{for } x < 0 \text{ and } x > L \end{cases}$$

- Classical picture: any energy, particle is localized within the well.
- The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi.$$

has the solution $k^2 = \frac{2mE}{\hbar^2}$

$$\psi(x) = C \begin{cases} 0, & & & \text{for } x < 0 \\ \sin(kx), & \text{or} & \cos(kx), & \text{for } 0 < x < L \\ 0, & & & \text{for } L < x \end{cases}$$

- Boundary conditions: the wave function must be continuous:

$$\psi(x=0) = 0, \qquad \psi(x=L) = 0.$$

so the solutions in $0 < x < L$ are

$$\psi(x) = C\sin(k_n x), \qquad k_n L = \pi n, \quad n = 1, 2\ldots$$

- Normalization constant C is found from

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = |C|^2 \int_0^L \sin^2\left(\frac{\pi n x}{L}\right) dx = |C|^2 \frac{L}{2}, \qquad C = \sqrt{\frac{2}{L}}.$$

- Energy spectrum is discrete:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2.$$

- Particle is localized within the well, but it can only have a discrete (infinite) set of energies. The set is very dense in the classical limit.

26.2. Particle in the finite square well potential.

- Consider a potential

$$U(x) = \begin{cases} 0 & \text{for } |x| < L \\ U_0 & \text{for } |x| > L \end{cases}$$

- I am interested only in solutions for $E < U_0$.
- Classical: the particle can have any energy $0 < E < U_0$; the particle is completely localized in $-L < x < L$ region.
- The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi.$$

- The normalizable solutions are

$$\psi(x) = \begin{cases} A_- e^{\kappa x}, & & & \text{for } x < -L \\ \sin(kx), & \text{or} & \cos(kx), & \text{for } -L < x < L \\ A_+ e^{-\kappa x}, & & & \text{for } L < x \end{cases}, \qquad k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad \kappa = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$

- Symmetry. As the Hamiltonian is symmetric with respect to $x \to -x$ the solutions are either symmetric $\psi(-x) = \psi(x)$ or antisymmetric $\psi(-x) = -\psi(x)$. These symmetric and antisymmetric solutions are

$$\psi_s(x) = \begin{cases} Ae^{\kappa x} & \text{for } x < -L \\ \cos(kx) & \text{for } -L < x < L \\ Ae^{-\kappa x} & \text{for } x > L \end{cases}, \qquad \psi_a(x) = \begin{cases} -Ae^{\kappa x} & \text{for } x < -L \\ \sin(kx) & \text{for } -L < x < L \\ Ae^{-\kappa x} & \text{for } x > L \end{cases},$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad \kappa = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} = \sqrt{k_u^2 - k^2}, \qquad k_u = \sqrt{\frac{2mU_0}{\hbar^2}}.$$

- Matching the solutions at $x = L$.
  - The wave function ψ must be continuous.

  $$\psi(x = L - \epsilon) = \psi(x = L + \epsilon), \quad \text{at the limit } \epsilon \to 0.$$

  - In addition, let's integrate the above equation over x from $L - \epsilon$ to $L + \epsilon$. We have

  $$-\frac{\hbar^2}{2m}\left(\psi'(L+\epsilon) - \psi'(L-\epsilon)\right) + \int_{L-\epsilon}^{L+\epsilon} U(x)\psi(x)dx = E\int_{L-\epsilon}^{L+\epsilon} \psi(x)dx.$$

  Taking a limit $\epsilon \to 0$ we have

  $$\psi'(L+0) = \psi'(L-0)$$

  So ψ' must also be continuous at the points $x = \pm L$ (and thus everywhere).

- So we need to match the value of ψ and ψ' from both sides for $x = L$, so we have (left column for the symmetric, right for antisymmetric)

$$\begin{aligned} Ae^{-\kappa L} &= \cos(kL) & \qquad Ae^{-\kappa L} &= \sin(kL) \\ -\kappa Ae^{-\kappa L} &= -k\sin(kL) & \qquad -\kappa Ae^{-\kappa L} &= k\cos(kL) \end{aligned}$$

  Dividing the equation we get

$$k\tan(kL) = \kappa \qquad k\cot(kL) = -\kappa,$$

  which can be written as

$$\cos(kL) = \frac{k}{k_u}, \qquad \sin(kL) = -\frac{k}{k_u},$$

  where

$$k_u = \sqrt{\frac{2mU_0}{\hbar^2}}.$$

  These equations have a discrete set of solutions. No matter how small U_0 is there is always at least one symmetric localized solution!
- Unlike classical case the particle can be found outside the well.
- The transition in behavior at $E \approx U_0$ is not as sharp in Quantum mechanics.

26.3. Tunneling.

- Transition through a square potential bump.

$$U(x) = \begin{cases} 0 & \text{for } x < 0 \\ U_0 & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{cases}.$$

- We are interested at energies $0 < E < U_0$.
- In classical mechanics the particle coming from the left is simply reflected back.
- In quantum mechanics we look for the solution in the form

$$\psi(x) = \begin{cases} e^{ipx/\hbar} + Re^{-ipx/\hbar} & \text{for } x < 0 \\ A_+e^{\kappa x/\hbar} + A_-e^{-\kappa x/\hbar} & \text{for } 0 < x < L \\ Te^{ipx/\hbar} & \text{for } x > L \end{cases},$$

  where R and T are reflection and transition amplitudes respectively and

$$\frac{p^2}{2m} = E, \qquad \frac{\kappa^2}{2m} = U_0 - E$$

- At the points $x = 0$ and $x = L$ we must match the value of the wave function and its derivatives from the left and the right. So we have four linear conditions/equations and four unknowns T, R, A_+, and A_-!
- The answer is

$$|T|^2 = \frac{4p^2\kappa^2}{(p^2+\kappa^2)^2\sinh^2(\kappa L/\hbar) + 4p^2\kappa^2}, \qquad |R|^2 = 1 - |T|^2.$$

- Using the definitions of p and κ it can be written

$$|T|^2 = \frac{4E(U_0 - E)}{4E(U_0 - E) + U_0^2\sinh^2\left(\sqrt{\frac{U_0 - E}{\epsilon}}\right)}, \qquad \epsilon = \frac{\hbar^2}{2mL^2}$$

- Limits of large $L \gg \hbar/\kappa$ and $\kappa \gg p$ (or $U_0 \gg E$).

$$\kappa \approx \sqrt{2mU_0}, \qquad |T|^2 \approx \frac{16E}{U_0} e^{-\kappa L/\hbar}$$

- This is under barrier transition — tunneling. (There is also over barrier reflection.)

26.4. Particle in the δ-function attractive potential. Optional.

- I want to consider a potential

$$U(x) = -U_0 \delta(x).$$

- I am interested only in localized state, so $E < 0$.
- The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' - U_0\delta(x)\psi = -|E|\psi$$

- Let's integrate this equation over x from $-\epsilon$ to ϵ, we get

$$-\frac{\hbar^2}{2m}\left(\psi'(\epsilon) - \psi'(-\epsilon)\right) - U_0\psi(0) = -|E| \int_{\epsilon}^{\epsilon} \psi(x) dx.$$

Taking the limit $\epsilon \to 0$ we see that

$$\psi'(+0) - \psi'(-0) = -\frac{2mU_0}{\hbar^2}\psi(0)$$

So the function ψ' must have a jump (discontinuity at $x = 0$)
- The solutions are

$$\psi = \begin{cases} Ae^{\kappa x} & \text{for } x < 0 \\ Ae^{-\kappa x} & \text{for } x > 0 \end{cases}, \tag{26.1}$$

where

$$\kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$$

- Then

$$\psi'(+0) = -\kappa A, \qquad \psi'(-0) = \kappa A, \qquad \psi(0) = A$$

- Using the condition for matching the derivatives we get

$$2\kappa = \frac{2mU_0}{\hbar^2}, \qquad |E| = \frac{U_0^2}{2m\hbar^2}$$

- Although the potential is very short range the particle can be found in the finite region $-1/\kappa < x < 1/\kappa$, or

$$-\frac{\hbar^2}{mU_0} < x < \frac{\hbar^2}{mU_0}.$$

LECTURE 27
Wave function. Wave packet.

- Homework.

Srödinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x).$$

Momentum operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}.$$

27.1. A bit of math.

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}}dx = \alpha\sqrt{2\pi}.$$

It can be derived as the following: Let's denote

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx.$$

And consider I^2

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \iint_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \int_0^{2\pi} d\phi \int_0^{\infty} e^{-r^2}rdr = \pi\int_0^{\infty} e^{-r^2}d(r^2) = \pi$$

So we have

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

27.2. A particle as a wave packet.

Let's consider a free particle $U(x) = 0$.

- A wave $e^{ipx/\hbar}$ has a definite momentum p, but is everywhere $-\infty < x < \infty$.
- A particle is localized in space.
- Srödinger equation is linear in Ψ.

A particle must be represented by wave packet.

We know how a plane wave evolves with time

$$e^{ipx/\hbar} \to e^{ipx/\hbar - iE_pt/\hbar}.$$

How does the wave packet — the particle — evolves with time?

27.2.1. Evolution of a wave packet.

Let's assume that we know that at initial time $t = 0$ the wave function is given by $\Psi(x, 0)$, we want to know what will be the wave function at time t.

In order to do that we need to present $\Psi(x, 0)$ as a collection of a plane waves — the wave packet.

$$\Psi(x,0) = \int_{-\infty}^{\infty} a_p e^{ipx/\hbar} \frac{dp}{2\pi\hbar}, \qquad a_p = \int_{-\infty}^{\infty} \Psi(x,0) e^{-ipx/\hbar} dx$$

After a time t a wave $e^{ipx/\hbar}$ becomes $e^{ipx/\hbar - iE_p t/\hbar}$. So

$$\Psi(x,t) = \int a_p e^{ipx/\hbar - iE_p t/\hbar} \frac{dp}{2\pi\hbar}.$$

Let's see how it works for a classical free particle $E_p = \frac{p^2}{2m}$.

27.2.1.1. *Wave packet spreading.*

Let's assume, that we have started with the initial wave-function $\Psi(x, 0) = Ce^{-x^2/4\alpha^2}$, and $|\Psi(x, 0)|^2 = C^2 e^{-x^2/2\alpha^2}$, so that $\Delta x = \alpha$. First we must compute C from the normalization condition

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx = |C|^2 \int_{-\infty}^{\infty} e^{-x^2/2\alpha^2} dx = |C|^2 \sqrt{2\pi}\alpha$$

then

$$a_p = \int_{-\infty}^{\infty} \Psi(x,0) e^{-ipx/\hbar} dx = C \int_{-\infty}^{\infty} e^{-x^2/4\alpha^2 - ipx/\hbar} dx = C \int_{-\infty}^{\infty} e^{-\frac{1}{4\alpha^2}\left(x^2 + 2ipx\frac{2\alpha^2}{\hbar} - p^2\frac{4\alpha^4}{\hbar^2}\right) - p^2\frac{\alpha^2}{\hbar^2}} dx =$$
$$Ce^{-p^2\frac{\alpha^2}{\hbar^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{4\alpha^2}\left(x + 2ip\frac{\alpha^2}{\hbar}\right)^2} dx = 2C\alpha\sqrt{\pi} e^{-p^2\frac{\alpha^2}{\hbar^2}}$$

So that according to the prescription

$$\Psi(x,t) = \int_{-\infty}^{\infty} a_p e^{ipx/\hbar - iE_p t/\hbar} \frac{dp}{2\pi\hbar} = \int_{-\infty}^{\infty} C\alpha\sqrt{2\pi} e^{-p^2\frac{\alpha^2}{\hbar^2} + ipx/\hbar - p^2\frac{it}{2m\hbar}} \frac{dp}{2\pi\hbar} =$$
$$C\alpha\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-p^2\left(\frac{\alpha^2}{\hbar^2} + it/2m\hbar\right) + ipx/\hbar} \frac{dp}{2\pi\hbar} = C\alpha\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4\left(\frac{4\alpha^2}{\hbar^2} + \frac{2it}{m\hbar}\right)^{-1}} + ipx/\hbar} \frac{dp}{2\pi\hbar} =$$
$$2C\alpha\frac{1}{\hbar}\left(\frac{4\alpha^2}{\hbar^2} + \frac{2it}{m\hbar}\right)^{-1/2} e^{-\frac{x^2}{4\hbar^2\left(\frac{\alpha^2}{\hbar^2} + \frac{it}{2m\hbar}\right)}} = \frac{C}{\sqrt{1 + \frac{it\hbar}{2m\alpha^2}}} e^{-\frac{x^2}{4\left(\alpha^2 + \frac{it\hbar}{2m}\right)}}$$

So we see that

$$|\Psi(x,t)|^2 = \frac{C^2}{\sqrt{1 + \left(\frac{t\hbar}{2m\alpha^2}\right)^2}} e^{-\frac{x^2}{2\left(\alpha^2 + \left(\frac{t\hbar}{2m\alpha}\right)^2\right)}}$$

So we see, that the particle is still at the center on average, but

$$\Delta x(t) = \sqrt{[\Delta x(0)]^2 + \left[\frac{t\hbar}{2m\Delta x(0)}\right]^2}$$

We now can compute how much time it would take for a $1g$ marble initially localized with a precision $0.1mm$ to disperse so that $\Delta x(t) = 10\Delta x(0)$. The answer is $t \approx 2 \times 10^{24} s$ – by far longer than the life-time of our Universe.

27.2.1.2. *Group velocity.*

Let's construct a wave packet with a momentum p_0 on average at $t = 0$. We want this packet to be very sharply peaked at p_0.

$$\Psi(x,0) = C\int_{-\infty}^{\infty} e^{-\frac{(p-p_0)^2}{4\alpha^2}} e^{ipx/\hbar} dp$$

where we assume that the $\alpha \sim \Delta p$ is small.

At time t the wave packet will be

$$\Psi(x,t) = C\int_{-\infty}^{\infty} e^{-\frac{(p-p_0)^2}{4\alpha^2}} e^{ipx/\hbar - iE_p t/\hbar} dp$$

As α is small, only $p \sim p_0$ contribute to the integral, so we can write

$$\Psi(x,t) \approx C e^{ip_0 x/\hbar - iE_{p_0} t/\hbar} \int_{-\infty}^{\infty} e^{-(p-p_0)^2\left(\frac{1}{4\alpha^2} + i\frac{1}{\hbar}\frac{\partial^2 E_p}{\partial p_0^2} t\right) + \frac{i}{\hbar}(p-p_0)\left(x - \frac{\partial E}{\partial p_0} t\right)} dp$$

So we see, that

$$|\Psi(x,t)|^2 = f\left(x - \frac{\partial E}{\partial p_0} t, t\right)$$

So we see, that the wave packet is moving with the "group" velocity

$$v = \frac{\partial E}{\partial p_0},$$

as it should according to the Hamiltonian equations.

27.3. Relativistic quantum mechanics.

- Srödinger equation is not Lorenz invariant – it is non-relativistic.
- The relativistic quantum mechanics is described by Dirac equation `https://en.wikipedia.org/wiki/Paul_Dirac`.
- Every particle has an antiparticle.

LECTURE 28
Band structure. Tunneling. Density of states.

28.1. Particle in two far away potential wells.

- The Hamiltonian is
$$\hat{H} = \frac{\hat{p}^2}{2m} + U_L(x) + U_R(x), \qquad U_{L,R} = U(x \pm l/2), \qquad l \gg \frac{\hbar^2}{mU_0}$$
- The condition $l \gg \frac{\hbar^2}{mU_0}$ means that the distance between the wells are much larger than the spread of the wave function.
- If the two wells are far away from each other, then the overlap of the wave functions is small.
- Let's define two functions $|\psi_L\rangle$ and $|\psi_R\rangle$
$$\left(\frac{\hat{p}^2}{2m} + U_L\right)|\psi_L\rangle = E_0|\psi_L\rangle, \qquad \langle\psi_L|\psi_L\rangle = 1$$
$$\left(\frac{\hat{p}^2}{2m} + U_R\right)|\psi_R\rangle = E_0|\psi_R\rangle, \qquad \langle\psi_R|\psi_R\rangle = 1$$
We also notice, that
$$|\langle\psi_R|\psi_L\rangle| \ll 1.$$
- Let's look for the solution in the form
$$|\psi\rangle = a_L|\psi_L\rangle + a_R|\psi_R\rangle.$$
- The Schrödinger equation now reads.
$$a_L E|\psi_L\rangle + a_R E|\psi_R\rangle = a_L\hat{H}|\psi_L\rangle + a_R\hat{H}|\psi_R\rangle.$$
- Multiplying this equation by $\langle\psi_L|$ and $\langle\psi_R|$ we get
$$Ea_L = \left(E_0 + \langle\psi_L|U_R|\psi_L\rangle\right)a_L + \langle\psi_L|U_L|\psi_R\rangle a_R$$
$$Ea_R = \left(E_0 + \langle\psi_R|U_L|\psi_R\rangle\right)a_R + \langle\psi_R|U_R|\psi_L\rangle a_L.$$
- We expect $E \approx E_0$, so we ignored the terms of the kind $(E - E_0)\langle\psi_L|\psi_R\rangle$, as they are of the second order.

- Introducing $\tilde{E}_0 = E_0 + \langle\psi_L|U_R|\psi_L\rangle$, $-\Delta = \langle\psi_R|U_R|\psi_L\rangle$, and a vector $\begin{pmatrix} a_L \\ a_R \end{pmatrix}$ we have

$$E \begin{pmatrix} a_L \\ a_R \end{pmatrix} = \begin{pmatrix} \tilde{E}_0 & -\Delta \\ -\Delta & \tilde{E}_0 \end{pmatrix} \begin{pmatrix} a_L \\ a_R \end{pmatrix}.$$

- So E is just an eigenvalue of the simple 2×2 matrix. The result is

$$E_\pm = \tilde{E}_0 \pm \Delta.$$

- A single degenerate energy level is split in two levels: symmetric and antisymmetric combinations.
- Interaction splits degeneracy.
- In the symmetric potential the ground state is always symmetric.

28.2. Strong periodic potential. (Tight binding model.)

- The potential is

$$U(x) = \sum_{n=-\infty}^{\infty} U(x - nl).$$

- We again assume that l is much grater than the spread of a wave function for a single well.

$$\left(\frac{\hat{p}^2}{2m} + U(x)\right)|\psi(x)\rangle = E_0|\psi\rangle$$

- We look at the solution in the form

$$|\psi\rangle = \sum_{n=-\infty}^{\infty} a_n|\psi(x - nl)\rangle$$

- We then have

$$\begin{pmatrix} . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & 0 & 0 & . \\ . & 0 & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & 0 & . \\ . & 0 & 0 & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} . \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \\ a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ . \end{pmatrix} = E \begin{pmatrix} . \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \\ a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ . \end{pmatrix}.$$

or

$$-\Delta a_{n-1} + \tilde{E}_0 a_n - \Delta a_{n+1} = E a_n$$

- We look for the solution in the form $a_n = ae^{ipln/\hbar}$, so

$$-\Delta e^{ipl(n-1)/\hbar} + \tilde{E}_0 e^{ipln/\hbar} - \Delta e^{ipl(n-1)/\hbar} = Ee^{ipln/\hbar},$$

which gives

$$E(k) = \tilde{E}_0 - 2\Delta\cos(pl/\hbar), \qquad -\pi\hbar/l < p < \pi\hbar/l.$$

So a single energy level is split into a band.

- p is quasi-momentum. In particular, for small p

$$E(k) \approx \tilde{E}_0 - 2\Delta + \frac{p^2}{2(\hbar^2/2l^2\Delta)}.$$

So it behaves as a normal particle with the "effective" mass $m^* = \hbar^2/2l^2\Delta$.

28.3. Density of states.

- Density of states: Discrete spectrum to continuous.
- Tunneling current as a measure of the density of states (STM).

LECTURE 29
Commutators. Quantum harmonic oscillator.

- Homework.

Quantum harmonic oscillator.

- Hermitian operators. Observables.
- x as an operator.
- $[\hat{p}, \hat{x}] = -i\hbar$.
- Hamiltonian for a harmonic oscillator $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} = \frac{\hat{p}^2}{2m} + m\omega^2\frac{\hat{x}^2}{2}$.
- Operators $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ and $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$.
- $[\hat{a}, \hat{a}^\dagger] = 1$, and $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right)$.
- The Schrödinger equation $\hat{H}|\psi\rangle = E|\psi\rangle$ becomes

$$\hbar\omega\hat{a}^\dagger\hat{a}|\psi\rangle = (E - \hbar\omega/2)\,|\psi\rangle$$

- A function $|0\rangle$ such that $\hat{a}|0\rangle = 0$ and $\langle 0|0\rangle = 1$ exists.

$$|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}, \qquad E_0 = \frac{1}{2}\hbar\omega$$

- Consider a function/state $|1\rangle = \hat{a}^\dagger|0\rangle$. Let's act on it by an operator $\hbar\omega\hat{a}^\dagger\hat{a}$

$$\hbar\omega\hat{a}^\dagger\hat{a}|1\rangle = \hbar\omega\hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\rangle = \hbar\omega\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a} + 1\right)|0\rangle = \hbar\omega\hat{a}^\dagger\hat{a}^\dagger\hat{a}|0\rangle + \hbar\omega\hat{a}^\dagger|0\rangle = \hbar\omega\hat{a}^\dagger|0\rangle = \hbar\omega|1\rangle.$$

  So we see, that the function $|1\rangle$ is an eigen function of our Hamiltonian and

$$E_1 = \hbar\omega + \frac{1}{2}\hbar\omega.$$

- Normalization

$$\langle 1|1\rangle = \langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \langle 0|1 + \hat{a}^\dagger\hat{a}|0\rangle = \langle 0|0\rangle = 1$$

- For a state $|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle$ we have

$$\hbar\omega\hat{a}^\dagger\hat{a}|n\rangle = n\hbar\omega|n\rangle, \qquad \langle n|n\rangle = 1,$$

  so

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

- Also $\langle n|m\rangle = 0$, for $n \neq m$, and

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

- $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$, and $\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$, so

$$\langle n|\hat{x}|n\rangle = 0, \qquad \langle n|\hat{p}|n\rangle = 0$$

  and

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\hat{a}^\dagger\hat{a}{+}\hat{a}\hat{a}^\dagger|n\rangle = \frac{\hbar}{2m\omega}\langle n|2\hat{a}^\dagger\hat{a}{+}1|n\rangle = (n{+}1/2)\frac{\hbar}{m\omega}, \qquad \langle n|\hat{p}^2|n\rangle = (n{+}1/2)m\omega\hbar$$

- Coherent states. For any α we construct a state:

$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle, \qquad \langle\alpha|\alpha\rangle = 1, \qquad \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2.$$

  This set of such states is overcomplete $\langle\alpha|\alpha'\rangle \neq 0$, for $\alpha \neq \alpha'$. The time evolution of these states describes the motion of a particle.

LECTURE 30
Quantum mechanics in $3D$. Many-particle states. Identical particles.

- Double well potential.
- Periodic potential.
$$-\frac{\hbar^2}{2m}\psi''(x) + U(x)\psi(x) = E\psi(x)$$
or
$$\psi''(x) = -\left(\frac{2mE}{\hbar^2} - \frac{2m}{\hbar^2}U(x)\right)\psi(x).$$
By changing the notations $\psi \to x$, $x \to t$, and $\frac{2mE}{\hbar^2} = \omega^2$ this equation is
$$\ddot{x} = -\left(\omega^2 - \frac{2m}{\hbar^2}U(t)\right)x = -\Omega^2(t)x.$$
This is an oscillator with parameters periodically depending on time – parametric resonance. The difference is that the wave function must be normalizable.
- Bloch theorem `https://en.wikipedia.org/wiki/Bloch_wave`. Band structure!
- Quantum mechanics in $3D$.
- Many-particle states.
- Identical particles.
- Bosons. Bose-Einstein condensate, superfluidity.
- Electrons as fermions.
- Metals and insulators. Response to the electric field.
- Semiconductors. Electrons and holes.
- LED.
- Lasers.
- Fermi-surface. Superconductivity.
- Inner Life of the Cell: `https://youtu.be/FzcTgrxMzZk`
- Closing remarks. More is different. Simple rules — complex behavior.