Modern Physics. Phys 222

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Contents

LECTURE 1
Introduction. Geometry.

- Contact info.
- Book.
- Grading.
- Homeworks (deadlines, collaborations, mistakes, etc.)
- Exams.
- Language.
- Course content and philosophy. Questions: profound vs. stupid.

What do we know?

- Calculus (derivatives, integrals, partial derivatives, Taylor expansion, integration over a path, Fourier transformation.)
- Linear algebra (vectors, matrices, eigen values, eigen vectors.)
- Complex variables.
- Mechanics.
- Electrodynamics.
- Geometry.

Geometry

- What is the sum of all angles in a triangle? Why?
- What is distance?
- Metric tensor.
- A story of an ant on a sphere. Sum of the angles in a triangle. The number π.

What is a straight line?

- Length of a curve as a functional.
- Functional, variations, Extremum.
- Straight line in Euclidean space in Cartesian coordinates.

LECTURE 2
Mechanics.

- Calculus.
- Home work solutions
- Geometry
  - System of coordinates.
  - Metric tensor in Cartesian coordinate system: $(dl)^2 = (dx)^2 + (dy)^2$.
  - Straight line in Euclidean space in Cartesian coordinates: $y = ax + b$.
  - Metric tensor in polar coordinates $(dl)^2 = (dr)^2 + r^2(d\phi)^2$.
  - Straight line in Euclidean space in Polar coordinates $r = \frac{a}{\cos(\phi-\phi_0)}$.
  - Change of variables in the integral: under the change $x = r(\phi)\cos(\phi)$, $y = r(\phi)\sin(\phi)$ we have
  $$L = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{r^2 + (r')^2} d\phi.$$
  - Metric tensor on a sphere $(dl)^2 = R^2(d\theta)^2 + R^2(d\phi)^2 \sin^2\theta$.
  - "Straight" line on a sphere.
  - What is our space?
- Topology.
  - Number of vertices V, edges E, and faces F.
  - Compute $V + F - E$ for several polyhedral.
  - $V + F - E$ as invariant.
  - A face must have no holes.
  - Continuum limit.
  - $V + F - E$ for torus.
  - $V + F - E = 2 - 2g$
  - A story of an ant.
  - What does it have to do with physics?

LECTURE 3
Galilean invariance. Newton laws. Work. Conservative forces.

Mechanics

- Galilean invariance.
- Time reversal.
- Newton laws.
- Work.
- Conservative forces.

LECTURE 4
Conservation laws.

- Galilean invariance in increments.

$$dx' = dx + V dt$$
$$dt' = dt.$$

- Motion with constant acceleration in $1D$.

$$v = v_0 + at$$
$$x = x_0 + v_0 t + \frac{at^2}{2}$$

These are correct only(!!!) for the case of constant acceleration.

Energy

- Calculus of many variables.
- Work.
- Conservative forces.

$$\oint \vec{F} \cdot d\vec{r} = 0, \qquad \vec{F} = -\frac{\partial U}{\partial \vec{r}}.$$

- Non-uniqueness of U.
- Energy. $E = \frac{m\vec{v}^2}{2} + U(\vec{r})$.
- Time translation invariance. Energy conservation.
- Translation invariance. Momentum conservation.

LECTURE 5
Homework. Motion in $1D$.

- Homeworks.
- Non-uniqueness of U.
- Energy. $E = \frac{m\vec{v}^2}{2} + U(\vec{r})$.
- Time translation invariance. Energy conservation.
- Translation invariance. Momentum conservation.
- Conservative forces in $1D$.
- Energy conservation. Motion in $1D$. $E = \frac{mv^2}{2} + U(x)$ so $U(x) < E$. Let's initial conditions be $x(t_0) = x_0$ and $v(t_0) = v_0$, the $E = \frac{mv_0^2}{2} + U(x_0)$

$$t - t_0 = \pm\sqrt{\frac{m}{2}} \int_{x_0}^{x(t)} \frac{dx}{\sqrt{E - U(x)}}$$

- Oscillator.

LECTURE 6
Hamiltonian. Lagrangian.

- Motion in $1D$ in arbitrary potential – picture. Period.
- Energy conservation. Full vs. partial derivatives.
- Hamiltonian (velocity).
- Phase space. Phase space trajectories.
- Energy as a value of a Hamiltonian on a trajectory.
- Definition of functionals. Examples.

LECTURE 7
Lagrangian. Oscillations.

- Hamilton principle. Action. Minimal action.
- Lagrangian.
- Euler-Lagrange equation.
- Examples.

LECTURE 8
Oscillations with friction.

- Homework.

8.1. Euler formula

$$e^{i\phi} = \cos(\phi) + i\sin(\phi).$$

which also mean

$$\cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2}, \qquad \sin(\phi) = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

and

$$e^{i\pi} = -1.$$

8.2. Oscillators.

- Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$$

- $$m\ddot{x} = -kx, \qquad ml\ddot{\phi} = -mg\sin\phi \approx -mg\phi, \qquad -L\ddot{Q} = \frac{Q}{C},$$

  All of these equation have the same form

$$\ddot{x} = -\omega_0^2 x, \qquad \omega_0^2 = \begin{cases} k/m \\ g/l \\ 1/LC \end{cases}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- The solution

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) = C\sin(\omega_0 t + \phi), \qquad B = x_0, \qquad \omega_0 A = v_0.$$

- Oscillates forever: $C = \sqrt{A^2 + B^2}$ — amplitude; $\phi = \tan^{-1}(A/B)$ — phase.

8.3. Oscillations with friction.

- Oscillations with friction:

$$m\ddot{x} = -kx - 2\alpha\dot{x}, \qquad -L\ddot{Q} = \frac{Q}{C} + R\dot{Q},$$

- The sign of α.
- Consider

$$\ddot{x} = -\omega_0^2 x - 2\gamma\dot{x}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

  This is a linear equation with constant coefficients. We look for the solution in the form $x = \Re C e^{i\omega t}$, where ω and C are complex constants.

$$\omega^2 - 2i\gamma\omega - \omega_0^2 = 0, \qquad \omega = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

- Two solutions, two independent constants.
- Two cases: $\gamma < \omega_0$ and $\gamma > \omega_0$.
- In the first case (underdamping):

$$x = e^{-\gamma t}\Re\left[C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right] = C e^{-\gamma t}\sin\left(\Omega t + \phi\right), \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}$$

  Decaying oscillations. Shifted frequency.
- In the second case (overdamping):

$$x = A e^{-\Gamma_- t} + B e^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}, \qquad \Gamma_+ > \Gamma_- > 0$$

- For the initial conditions $x(t=0) = x_0$ and $v(t=0) = 0$ we find $A = x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}$, $B = -x_0\frac{\Gamma_-}{\Gamma_+ - \Gamma_-}$. For $t \to \infty$ the B term can be dropped as $\Gamma_+ > \Gamma_-$, then $x(t) \approx x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}e^{-\Gamma_- t}$.
- At $\gamma \to \infty$, $\Gamma_- \to \frac{\omega_0^2}{2\gamma} \to 0$. The motion is arrested. The example is an oscillator in honey.

LECTURE 9
Oscillations with external force. Resonance.

9.1. Comments on dissipation.

- Time reversibility. A need for a large subsystem.
- Locality in time.

9.2. Resonance

- Let's add an external force:
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$
- The full solution is the sum of the solution of the homogeneous equation with any solution of the inhomogeneous one. This full solution will depend on two arbitrary constants. These constants are determined by the initial conditions.
- Let's assume, that $f(t)$ is not decaying with time. The solution of the inhomogeneous equation also will not decay in time, while any solution of the homogeneous equation will decay. So in a long time $t \gg 1/\gamma$ The solution of the homogeneous equation can be neglected. In particular this means that the asymptotic of the solution does not depend on the initial conditions.
- Let's now assume that the force $f(t)$ is periodic. with some period. It then can be represented by a Fourier series. As the equation is linear the solution will also be a series, where each term corresponds to a force with a single frequency. So we need to solve
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$
where f is the force's amplitude.
- Let's look at the solution in the form $x = f\Im C e^{i\Omega_f t}$, and use $\sin(\Omega_f t) = \Im e^{i\Omega_f t}$. We then get
$$C = \frac{1}{\omega_0^2 - \Omega_f^2 + 2i\gamma\Omega_f} = |C|e^{-i\phi},$$
$$|C| = \frac{1}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}}, \quad \tan\phi = \frac{2\gamma\Omega_f}{\omega_0^2 - \Omega_f^2}$$
$$x(t) = f\Im|C|e^{i\Omega_f t - i\phi} = f|C|\sin\left(\Omega_f t - \phi\right),$$

- Resonance frequency:
$$\Omega_f^r = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\Omega^2 - \gamma^2},$$
where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ is the frequency of the damped oscillator.
- Phase changes sign at $\Omega_f^\phi = \omega_0 > \Omega_f^r$. Importance of the phase – phase shift.
- To analyze resonant response we analyze $|C|^2$.
- The most interesting case $\gamma \ll \omega_0$, then the response $|C|^2$ has a very sharp peak at $\Omega_f \approx \omega_0$:
$$|C|^2 = \frac{1}{(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2} \approx \frac{1}{4\omega_0^2}\frac{1}{(\Omega_f - \omega_0)^2 + \gamma^2},$$
so that the peak is very symmetric.
- $|C|^2_{\mathrm{max}} \approx \frac{1}{4\gamma^2\omega_0^2}$.
- to find HWHM we need to solve $(\Omega_f - \omega_0)^2 + \gamma^2 = 2\gamma^2$, so HWHM $= \gamma$, and FWHM $= 2\gamma$.
- Q factor (quality factor). The good measure of the quality of an oscillator is $Q = \omega_0/\mathrm{FWHM} = \omega_0/2\gamma$. (decay time) $= 1/\gamma$, period $= 2\pi/\omega_0$, so $Q = \pi\frac{\text{decay time}}{\text{period}}$.
- For a grandfather's wall clock $Q \approx 100$, for the quartz watch $Q \sim 10^4$.

9.3. Response.

- Response. The main quantity of interest. What is "property"?
- The equation
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t).$$
The LHS is time translation invariant!
- Multiply by $e^{i\omega t}$ and integrate over time. Denote
$$x_\omega = \int x(t)e^{i\omega t}dt.$$
Then we have
$$\left(-\omega^2 - 2i\gamma\omega + \omega_0^2\right)x_\omega = \int f(t)e^{i\omega t}dt, \qquad x_\omega = -\frac{\int f(t')e^{i\omega t'}dt'}{\omega^2 + 2i\gamma\omega - \omega_0^2}$$
- The inverse Fourier transform gives
$$x(t) = \int \frac{d\omega}{2\pi}e^{-i\omega t}x_\omega = -\int f(t')dt' \int \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \int \chi(t-t')f(t')dt'.$$
- Where the response function is ($\gamma < \omega_0$)
$$\chi(t) = -\int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2 + 2i\gamma\omega - \omega_0^2} = \begin{cases} e^{-\gamma t}\frac{\sin(t\sqrt{\omega_0^2-\gamma^2})}{\sqrt{\omega_0^2-\gamma^2}} & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases}, \qquad \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$
- Causality principle. Poles in the lower half of the complex ω plane. True for any (linear) response function. The importance of $\gamma > 0$ condition.

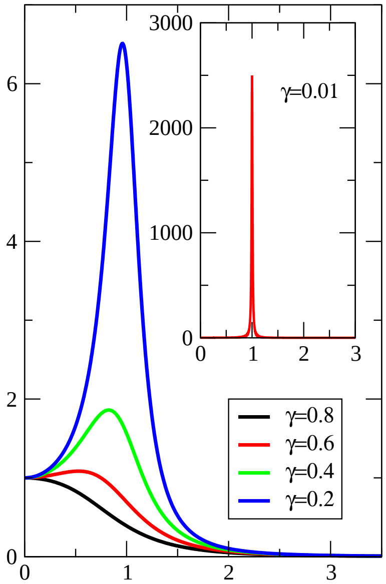


Figure 1. Resonant response. For insert $Q = 50$.

LECTURE 10
Spontaneous symmetry braking.

10.1. Spontaneous symmetry braking.

A bead on a vertical rotating hoop.

- Potential energy:
$$U(\theta) = mgR(1-\cos\theta).$$
- Kinetic energy:
$$K = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta.$$
- The Lagrangian.
$$L = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta - mgR(1-\cos\theta).$$
- The Lagrangian can be written as
$$L = \frac{m}{2}R^2\dot{\theta}^2 - U_{eff}(\theta),$$
where the "effective" potential energy is
$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1-\cos\theta).$$
- Equation of motion.
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}.$$
or
$$R\ddot{\theta} = (\Omega^2 R\cos\theta - g)\sin\theta = -\frac{1}{mR}\frac{\partial U_{eff}(\theta)}{\partial\theta}.$$
There are four equilibrium points: $\frac{\partial U_{eff}}{\partial\theta} = 0$
$$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R}$$
- Critical Ω_c. The second two equilibriums are possible only if
$$\frac{g}{\Omega^2 R} < 1, \qquad \Omega > \Omega_c = \sqrt{g/R}.$$

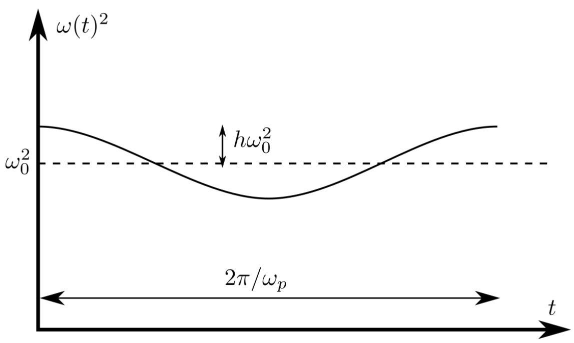
- Effective potential energy for $\Omega \sim \Omega_c$. Assuming $\Omega \sim \Omega_c$ we are interested only in small θ. So
$$U_{eff}(\theta) \approx \frac{1}{2} m R^2 (\Omega_c^2 - \Omega^2)\theta^2 + \frac{3}{4!} m R^2 \Omega_c^2 \theta^4$$
$$U_{eff}(\theta) \approx m R^2 \Omega_c (\Omega_c - \Omega)\theta^2 + \frac{3}{4!} m R^2 \Omega_c^2 \theta^4$$
- Spontaneous symmetry breaking. Plot the function $U_{eff}(\theta)$ for $\Omega < \Omega_c$, $\Omega = \Omega_c$, and $\Omega > \Omega_c$. Discuss universality.
- Small oscillations around $\theta = 0$, $\Omega < \Omega_c$
$$m R^2 \ddot{\theta} = -m R^2 (\Omega_c^2 - \Omega^2)\theta, \qquad \omega = \sqrt{\Omega_c^2 - \Omega^2} \approx \sqrt{2\Omega_c(\Omega_c - \Omega)}.$$
- Small oscillations around θ_0, $\Omega > \Omega_c$.
$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mrR(1 - \cos\theta),$$
$$\frac{\partial U_{eff}}{\partial \theta} = -mR(\Omega^2 R\cos\theta - g)\sin\theta, \qquad \frac{\partial^2 U_{eff}}{\partial \theta^2} = mR^2\Omega^2\sin^2\theta - mR\cos\theta(\Omega^2 R\cos\theta - g)$$
$$\left.\frac{\partial U_{eff}}{\partial \theta}\right|_{\theta=\theta_0} = 0, \qquad \left.\frac{\partial^2 U_{eff}}{\partial \theta^2}\right|_{\theta=\theta_0} = mR^2(\Omega^2 - \Omega_c^4/\Omega^2) \approx 2mR^2(\Omega^2 - \Omega_c^2) \approx 4mR^2\Omega_c(\Omega - \Omega_c)$$
So the Tylor expansion gives
$$U_{eff}(\theta \sim \theta_0) \approx \text{const} + \frac{1}{2} 4\Omega_c m R^2 (\Omega - \Omega_c)(\theta - \theta_0)^2$$
The frequency of small oscillations then is
$$\omega = 2\sqrt{\Omega_c(\Omega - \Omega_c)}.$$
- The effective potential energy for small θ and $|\Omega - \Omega_c|$
$$U_{eff}(\theta) = \frac{1}{2} a(\Omega_c - \Omega)\theta^2 + \frac{1}{4} b\theta^4.$$
- θ_0 for the stable equilibrium is given by $\partial U_{eff}/\partial\theta = 0$
$$\theta_0 = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \sqrt{\frac{a}{b}(\Omega - \Omega_c)} & \text{for} \quad \Omega > \Omega_c \end{cases}$$
Plot $\theta_0(\Omega)$. Non-analytic behavior at Ω_c.
- Response: how θ_0 responses to a small change in Ω.
$$\frac{\partial \theta_0}{\partial \Omega} = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \frac{1}{2}\sqrt{\frac{a}{b}}\frac{1}{\sqrt{(\Omega - \Omega_c)}} & \text{for} \quad \Omega > \Omega_c \end{cases}$$
Plot $\frac{\partial \theta_0}{\partial \Omega}$ vs Ω. The response *diverges* at Ω_c.

LECTURE 11
Oscillations with time dependent parameters.

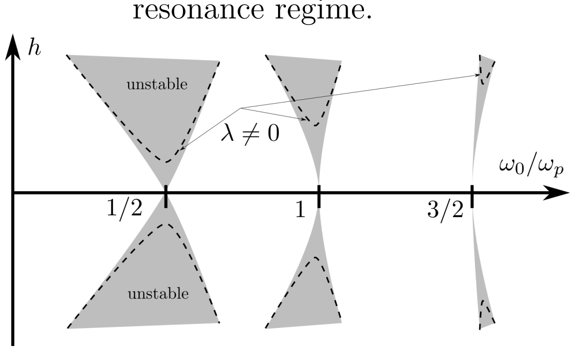
- Homework.

11.1. Oscillations with time dependent parameters.

$$\ddot{x} = -\omega^2(t)x, \qquad \omega^2(t) = \omega_0^2(1 + h\cos(\omega_p t)), \quad h \ll 1$$



- $\omega_p \gg \omega_0$ — Kapitza pendulum. (demo) Criteria: $\overline{\left(\dot{\xi}\right)^2} > gl$
- $\omega_p \sim \omega_0$ — parametric resonance ($\omega_p = \frac{2}{n}\omega_0$)
  Different from the usual resonance:
  - If the initial conditions $x(t=0) = 0$, $\dot{x}(t=0) = 0$, then $x(t) = 0$.
  - Frequency ω_p is a fraction of ω_0.
  - At finite dissipation one must have a finite amplitude h in order to get to the resonance regime.



- Foucault pendulum as an example of slow change of the parameter $\Delta\phi$ =solid angle of the path. (quantum: Berry phase 1984; classical: Hannay angle 1985.)

LECTURE 12
Waves.

12.1. Waves.

- Waves. Ripples. Sound waves. Light waves. Amplitude, phase.
- Linearity. Superposition.
- Interference.
- Wave front. Rays.
- Snell's Law.
- Green's picture.
- Diffraction.
- Resonator.
- Wave in a loop.
- Difference between waves and particles.
- Doppler effect. The source of frequency f_s is moving towards the stationary observer with velocity v_s. The observer hears the frequency f_o:
  $$f_o = f_s \frac{c}{c - v_s}.$$
  Discuss the role of the medium.
- Anderson localization.

LECTURE 13
Currents

- Current. Mass current. General current.
- Current density: vector.
- Charge/mass conservation:

$$\dot{\rho} + \nabla \vec{j} = 0$$

- Voltage. Current.
- Capacitor. Inductance.
- Resistor. Ohm's law.

$$V = IR, \qquad \vec{j} = \sigma \vec{E}.$$

- Kirchhoff's law.
- Phasor diagrams.

$$V_L = i\omega L I_L, \qquad V_C = -i\frac{1}{\omega C} I_C, \qquad V_R = R I_R$$

LECTURE 14
Electrodynamics.

- Homework.

$$\vec{j} = en\vec{v}.$$

Electrodynamics.

- Lorenz force.

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}.$$

  - Problem with Lorenz force.
  - Force on a piece of wire.
  - Cyclotron radius, cyclotron frequency.
- Notations: boundary of area Ω is denoted $\partial\Omega$.

LECTURE 15
Electrodynamics.

Electrodynamics.

- Flux of a vector field.
- Current as a flux of current density field.
- Gauss's theorem,
$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \int_{\Omega} \mathrm{div}\vec{E} dV$$
- Gauss's Law,
$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_{\Omega} \rho dV$$
- Local form of the Gauss's Law
$$\mathrm{div}\vec{E} = \frac{\rho}{\epsilon_0}.$$
  - Charged sphere.
  - Charged plane.
  - Electric field of a charged wire.
- Gauss law for magnetic field.
- Circulation of a vector field.

LECTURE 16
Maxwell Equations.

- Homework
- Notation
$$\text{curl}\vec{A} \equiv \nabla \times \vec{A}, \qquad \text{div}\vec{A} \equiv \nabla \cdot \vec{A} \qquad \text{grad}U \equiv \nabla U.$$
- So far we know

$$\text{Gauss's law:} \qquad \oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \tfrac{1}{\epsilon_0} \int_\Omega \rho dV, \qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\text{Gauss's law magnetic:} \qquad \oint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0, \qquad \nabla \cdot \vec{B} = 0$$

- Circulation of a vector field.
$$\int_{\partial\Sigma} \vec{A} \cdot d\vec{l} = \int_\Sigma \nabla \times \vec{A} \cdot d\vec{s}.$$
  - Orientation of Σ.
  - Independence of $\int_\Sigma \nabla \times \vec{A} \cdot d\vec{s}$ of Σ. Consider Σ_1 and Σ_2. The flux through $\Sigma_1 \cup \Sigma_2$ is $\Phi_{\Sigma_1 \cup \Sigma_2} = \Phi_{\Sigma_2} - \Phi_{\Sigma_1}$
  $$\Phi_{\Sigma_2} - \Phi_{\Sigma_1} = \Phi_{\Sigma_1 \cup \Sigma_2} = \int_{\Sigma_1 \cup \Sigma_2} \nabla \times \vec{A} \cdot d\vec{s} = \int_\Omega \nabla \cdot \nabla \times \vec{A} dV = 0.$$
  - Example of a circulation: work of a force vector field over a closed path.
  $$\mathcal{A} = \int_{\partial\Sigma} \vec{F} \cdot d\vec{l} = \int_\Sigma \nabla \times \vec{F} \cdot d\vec{s}.$$
  if the force is a potential force, then $\vec{F} = \nabla U$ and
  $$\mathcal{A} = \int_\Sigma \nabla \times \nabla U \cdot d\vec{s} = 0.$$
- Faraday's Law, Circulation of Electric field. (zero in statics)
$$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_\Sigma \vec{B} \cdot d\vec{S}.$$
- Faraday's Law is independent of Σ — it only depends on $\partial\Sigma$.
$$\Phi_{\Sigma_1} - \Phi_{\Sigma_2} = \Phi_{\Sigma_1 \cup \Sigma_2} = \int_{\Sigma_1 \cup \Sigma_2} \vec{B} \cdot d\vec{S} = \int_\Omega \nabla \cdot \vec{B} dV = 0.$$
- Ampere's Law, Circulation of Magnetic field.
$$\oint_{\partial\Sigma} \vec{B} \cdot d\vec{l} = \mu_0 \int_\Sigma \vec{j} \cdot d\vec{S}$$

- Problem with the Ampere's Law. As written it depends on Σ. Consider $\int_{\Sigma_1}\vec{j}\cdot d\vec{S}-\int_{\Sigma_2}\vec{j}\cdot d\vec{S}=\int_{\Sigma_1\cup\Sigma_2}\vec{j}\cdot d\vec{S}=\int_{\Omega}\nabla\cdot\vec{j}=-\frac{d}{dt}\int_{\Omega}\rho dV=-\epsilon_0\frac{d}{dt}\int_{\Omega}\nabla\cdot\vec{E}dV=$ $-\epsilon_0\frac{d}{dt}\int_{\Sigma_1\cup\Sigma_2}\vec{E}\cdot d\vec{S}=-\epsilon_0\frac{d}{dt}\int_{\Sigma_1}\vec{E}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma_2}\vec{E}\cdot d\vec{S}$

  We see, that
$$\int_{\Sigma_1}\vec{j}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma_1}\vec{E}\cdot d\vec{S}=\int_{\Sigma_2}\vec{j}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma_2}\vec{E}\cdot d\vec{S}$$
  So that the combination $\int_{\Sigma}\vec{j}\cdot d\vec{S}+\epsilon_0\frac{d}{dt}\int_{\Sigma}\vec{E}\cdot d\vec{S}$ is independent of Σ. If there is no electric field, then it is the same as just $\int_{\Sigma}\vec{j}\cdot d\vec{S}$. So we should write
- Ampere's law, corrected.
$$\oint_{\partial\Sigma}\vec{B}\cdot d\vec{l}=\mu_0\int_{\Sigma}\vec{j}\cdot d\vec{S}+\mu_0\epsilon_0\frac{d}{dt}\int_{\Sigma}\vec{E}\cdot d\vec{S}.$$

Full set of Maxwell equations:

Gauss's law:	$\oint_{\partial\Omega}\vec{E}\cdot d\vec{S}=\frac{1}{\epsilon_0}\int_{\Omega}\rho dV,$	$\nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0}$
Gauss's law magnetic:	$\oint_{\partial\Omega}\vec{B}\cdot d\vec{S}=0,$	$\nabla\cdot\vec{B}=0$
Faraday's law:	$\oint_{\partial\Sigma}\vec{E}\cdot d\vec{l}=-\frac{d}{dt}\int_{\Sigma}\vec{B}\cdot d\vec{S},$	$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$
Ampere's law:	$\oint_{\partial\Sigma}\vec{B}\cdot d\vec{l}=\mu_0\int_{\Sigma}\vec{j}\cdot d\vec{S}+\mu_0\epsilon_0\frac{d}{dt}\int_{\Sigma}\vec{E}\cdot d\vec{S},$	$\nabla\times\vec{B}-\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{j}$

In addition we should supply

- Initial conditions.
- Boundary conditions.
- "Material law". Plasmons.

Consequences:

- Coulomb law.
- Charge conservation – Gauss's and Ampere's laws.

LECTURE 17
Gauge invariance. Let there be light!

17.1. Maxwell equations.

Full set of Maxwell equations:

Gauss's law:	$\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_\Omega \rho dV,$	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Gauss's law magnetic:	$\oint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0,$	$\nabla \cdot \vec{B} = 0$
Faraday's law:	$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{l} + \frac{d}{dt} \int_\Sigma \vec{B} \cdot d\vec{S} = 0,$	$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
Ampere's law:	$\oint_{\partial\Sigma} \vec{B} \cdot d\vec{l} - \mu_0\epsilon_0 \frac{d}{dt} \int_\Sigma \vec{E} \cdot d\vec{S} = \mu_0 \int_\Sigma \vec{j} \cdot d\vec{S},$	$\nabla \times \vec{B} - \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$

In addition we should supply

- Initial conditions.
- Boundary conditions.
- "Material law". Plasmons.

Consequences:

- Coulomb law.
- Charge conservation – Gauss's and Ampere's laws.

Analysis of the equations.

- Units. From Faraday's law $\frac{[E]}{[l]} = \frac{[B]}{[t]}$, or $[E] = \frac{[l]}{[t]}[B]$. From Ampere's law $\frac{[B]}{[l]} = [\mu_0\epsilon_0]\frac{[E]}{[t]}$. So $\frac{1}{[\mu_0\epsilon_0]} = \frac{[l]^2}{[t]^2}$ – units of the square of the velocity.
- We have 8 equations for only 6 unknown functions $\vec{E}$ and $\vec{B}$.
- The equations impose two constraints.
- The first constraint is that the charge is conserved $\text{div}\vec{j} + \partial\rho/\partial t = 0$. It comes from Gauss's and Ampere's laws.
- The second one is trivial and comes from Gauss's magnetic and Faraday's laws. (If we had magnetic charges, this constraint would give us the conservation of magnetic charge.)

17.2. Gauge fields.

- Solve magnetic Gauss's and Faraday's laws (they are constraints)

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}.$$

- The fields ϕ and $\vec{A}$ are called potential and vector potential respectively.

If we express $\vec{E}$ and $\vec{B}$ through the gauge fields $\vec{A}$ and ϕ the magnetic Gauss's law and the Faraday's law are automatically satisfied (notice, that these the laws that have zeros on RHS) The other two laws can be written as ($\Delta \equiv \nabla^2$.)

$$-\Delta\phi - \frac{\partial \mathrm{div}\vec{A}}{\partial t} = \frac{\rho}{\epsilon_0}$$

$$-\Delta\vec{A} + \vec{\nabla}\mathrm{div}\vec{A} + \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

17.3. Gauge invariance.

- Gauge transformation, for any $f(\vec{r}, t)$ the transformation

$$\vec{A} \to \vec{A} + \nabla f, \qquad \phi \to \phi - \frac{\partial f}{\partial t}$$

  does not change $\vec{E}$ and $\vec{B}$. But $\vec{E}$ and $\vec{B}$ are the only physically observable fields. So no matter what physical property we comute the result must be invariant under these gauge transformations.

Gauge symmetry (gauge freedom) allows us to chose any gauge we want. This choice is done by imposing an additional constraint on the fields ϕ and $\vec{A}$.
There are many particularly useful gauges:

Coulomb gauge. This gauge is given by the following gauge fixing condition

$$\mathrm{div}\vec{A} = 0.$$

The Maxwell equations then become

$$-\Delta\phi = \frac{\rho}{\epsilon_0}$$

$$-\Delta\vec{A} + \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

Lorenz gauge. This gauge is given by the following gauge fixing condition

$$\mathrm{div}\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} = 0.$$

The Maxwell equations then become

$$-\Delta\phi + \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\epsilon_0}$$

$$-\Delta\vec{A} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j}$$

Notice, that both equations in this gauge can be written as

$$\left(-\Delta + \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\phi\\ \vec{A}\end{pmatrix} = \begin{pmatrix}\rho/\epsilon_0\\ \mu_0\vec{j}\end{pmatrix}.$$

Also notice, that the combination $1/\sqrt{\epsilon_0\mu_0}$ has units of velocity.

17.4. Biot-Savart law.

In particular, if we are looking for the static solutions, meaning that neither ρ nor $\vec{j}$ depend on time and there is no EM waves around then we can use the Coulomb gauge and write ($\partial_t\phi = 0$ and $\partial_t\vec{A} = 0$).

$$-\Delta\phi = \frac{\rho}{\epsilon_0}$$
$$-\Delta\vec{A} = \mu_0\vec{j}$$

Notice, that the equations look exactly the same. We know that the solution of the first equation for the point like charge is given by the Coulomb potential

$$d\phi = \frac{1}{4\pi\epsilon_0}\frac{\rho dV}{R}$$

So the solution of the second equation (for the "point like" current) must be

$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{j}dV}{R} = \frac{\mu_0}{4\pi}\frac{\vec{j}dSdl}{R} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}}{R}$$

Taking the curl of this we find Biot-Savart law (named after Jean-Baptiste Biot and Félix Savart who discovered this relationship in 1820.)

$$d\vec{B} = -\frac{\mu_0}{4\pi}\frac{I\vec{R}\times d\vec{l}}{R^3}.$$

So for any static distribution of charges and currents we can find the electric and magnetic fields using the Coulomb and Biot-Savart laws

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\rho dV\vec{R}}{R^3}$$
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{dV\vec{j}\times\vec{R}}{R^3}$$

17.5. Light.

- Maxwell equations in vacuum — no static solutions.
- Wave equation.
- General solution of the wave equation.
- Speed of light.

LECTURE 18
Electromagnetic waves. Speed of light.

Exam. Homework.

- Maxwell equations in vacuum — no static solutions.

Gauss's law:	$\nabla \cdot \vec{E} = 0$
Gauss's law magnetic:	$\nabla \cdot \vec{B} = 0$
Faraday's law:	$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
Ampere's law:	$\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$

  Acting by $\nabla \times$ on Faraday's law and using $\nabla \times \nabla \times \vec{E} = \nabla \text{div} \vec{E} - \Delta \vec{E}$ and the Gauss and Ampere's laws we get

$$\Delta \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

- Wave equation, $1D$.

$$\frac{\partial^2 \vec{E}}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

  - Boundary condition: $\vec{E}(t, x \to \pm\infty) \to 0$.
  - Initial condition: $\vec{E}(t = 0, x) = \vec{E}_0(x)$. – it can be thought as a boundary condition in time.
- General solution of the wave equation.

$$\vec{E}(x, t) = \vec{E}_0(x \pm ct).$$

- Speed of light.
- Problem with the speed of light.
- Idea of Ether. Michelson-Morley experiment.
- Galilean transformation. Transformations that leave the Newton' equation invariant:

$$dx = dx' + V dt', \qquad dt = dt'$$

- Lorenz transformation. Transformations that leave the wave equation invariant. Look for the transformation in the form

$$dx = A dt' + B dx', \qquad dt = C dt' + D dx'$$

then

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = B\frac{\partial}{\partial x} + D\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = A\frac{\partial}{\partial x} + C\frac{\partial}{\partial t}$$

So that

$$\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \left(B^2 - \frac{1}{c^2}A^2\right)\frac{\partial^2}{\partial x^2} + \left(D^2 - \frac{1}{c^2}C^2\right)\frac{\partial^2}{\partial t^2} + 2\left(BD - \frac{1}{c^2}AC\right)\frac{\partial^2}{\partial x\partial t}$$

In order for the wave equation not to change its form we must have

$$B^2 - \frac{1}{c^2}A^2 = 1, \qquad D^2 - \frac{1}{c^2}C^2 = -\frac{1}{c^2}, \qquad BD - \frac{1}{c^2}AC = 0$$

We have three equation with four unknowns. The solution depends on one parameter γ and can be written as

$$dx = \frac{\gamma c dt'}{\sqrt{1-\gamma^2}} + \frac{dx'}{\sqrt{1-\gamma^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-\gamma^2}} + \frac{\gamma dx'}{\sqrt{1-\gamma^2}},$$

This is called Lorentz transformation.

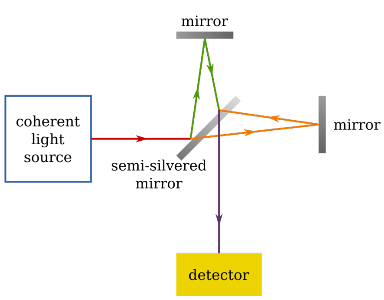


Figure 1. A Michelson interferometer uses the same principle as the original experiment. But it uses a laser for a light source.

LECTURE 19
Special theory of relativity.

- Lorenz transformation. Transformations that leave the wave equation invariant.

$$dx = \frac{\gamma c dt'}{\sqrt{1-\gamma^2}} + \frac{dx'}{\sqrt{1-\gamma^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-\gamma^2}} + \frac{\gamma dx'}{\sqrt{1-\gamma^2}},$$

  Comparing to the Galileo transformation we find that $\gamma = V/c$

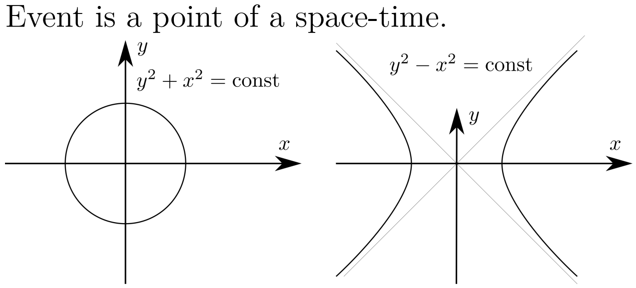
$$dx = \frac{V dt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{V dx'/c}{\sqrt{1-V^2/c^2}},$$

- These transformations tell us that our space-time has a very different structure than what was thought before.
- Lorenz transformation is the transformation that leaves the interval $ds^2 = c^2dt^2 - dx^2$ invariant.

$$dx = \frac{V dt'}{\sqrt{1-V^2/c^2}} + \frac{dx'}{\sqrt{1-V^2/c^2}}, \qquad cdt = \frac{cdt'}{\sqrt{1-V^2/c^2}} + \frac{V dx'/c}{\sqrt{1-V^2/c^2}},$$

- $ds^2 = c^2dt^2 - dx^2$ — metric of space-time!
- GPS, LHC.

Event is a point of a space-time.



Lorenz transformation is a “rotation” of the space-time.

- Events that are simultaneous in one frame of reference are not necessarily simultaneous in another (In contrast to Galilean transformation.)
- Velocity transformation: $v' = dx'/dt'$, $v = dx/dt$.

$$v = \frac{V + v'}{1 + \frac{Vv'}{c^2}}.$$

  if $v' = c$, then $v = c$.

- Time change. The experiment is the following: A person in the moving (primed) frame is staying put, so $dx' = 0$, and measures the time interval dt' so the time interval dt in the frame of reference at rest is:
$$dt = \frac{dt'}{\sqrt{1 - V^2/c^2}}$$
- Twin's paradox.
- Length change. The experiment is the following: a stick in the moving frame of reference is measured by a person in the same (moving) frame of reference (so the stick is not moving with respect to this person) The result is dx'. The length of this stick is now measured in the frame of references at rest. In order to do that the researcher must note the positions of the ends of the stick at the same moment of time in his frame! so for his measurement $dt = 0$. It then means that $cdt' = -\frac{V}{c}dx'$, and
$$dx = \frac{-V^2 dx'/c^2 + 1}{\sqrt{1 - V^2/c^2}} dx' = dx'\sqrt{1 - V^2/c^2}.$$

LECTURE 20
Special theory of relativity. General theory of relativity.

The speed of light is the same for every observer. However, different observers see this light differently.

- Doppler effect.

The light source S' moves with respect to the observer S with velocity V directly away. In the frame S' the distance between two wave fronts is $dx' = c/f'$, the time between them is $dt' = 1/f'$. In the frame S we then have

$$dx = \frac{V/f'}{\sqrt{1 - V^2/c^2}} + \frac{c/f'}{\sqrt{1 - V^2/c^2}}, \qquad cdt = \frac{c/f'}{\sqrt{1 - V^2/c^2}} + \frac{V/f'}{\sqrt{1 - V^2/c^2}}.$$

First we notice, that $cdt = dx$ as it must be – the speed of light is the same for both observers.(It also means that the Minkowskii interval $(ds)^2 = (cdt)^2 - (dx)^2 = 0$ for light. So the light is the straight line in Minkowskii space.) Second, we notice, that

$$f = \frac{1}{dt} = \sqrt{\frac{c - v}{c + v}} f'.$$

This is Doppler effect.

- Red shift.
- Blue shift.
- Velocity of the stars in the galaxy
- Hable constant.
- Universe expansion.
- Distance to the stars.
- Look into the past.

Dynamics.

- Energy and momentum.

$$dE = F dx, \quad dp = F dt, \quad ds^2 = c^2 dt^2 - dx^2 = (c^2 dp^2 - dE^2)/F^2$$

so $E^2 - c^2 p^2 =$ const must be invariant under the Lorenz transformation. For small p compare to $E = p^2/2m_0$ we find

$$E^2 = c^2 p^2 + m_0^2 c^4,$$

where m_0 – mass at rest. In particular for $p = 0$ we have $E = m_0 c^2$ – energy at rest.

- Momentum and velocity.

  Energy as a function of momentum is Hamiltonian, so

$$\dot{x} = \frac{\partial E(p)}{\partial p}, \qquad \dot{p} = -\frac{\partial E(p)}{\partial x}$$

The first equation gives $v = \dot{x}$:

$$v = \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}}, \quad \text{or} \quad p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}.$$

One can also say, that

$$p = mv, \qquad m = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

- Energy and velocity.

  Using p in $E(p)$ we find

$$E = c^2p^2 + m_0^2c^4 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} = mc^2.$$

- Nuclear energy $E = mc^2$.
- Space-time metric.
- Black holes.
- Gravitational lensing.
- Gravitation waves.

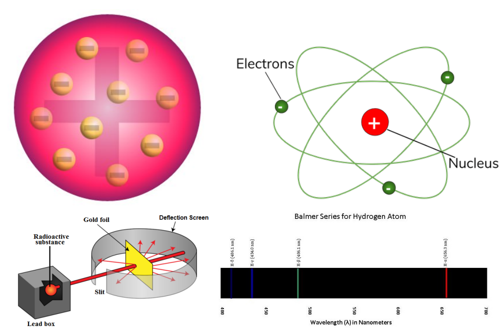
LECTURE 21
Problems with classical theory.

- Homework.

Waves vs stream of particles

- Common features:
  - Energy flux. Amount of energy which crosses a unit area per unit time.
  - Momentum flux. Amount of momentum which crosses a unit area per unit time
- Difference
  - Waves: diffraction and interference, or, in one word, phase.
  - Particles: number of particles.

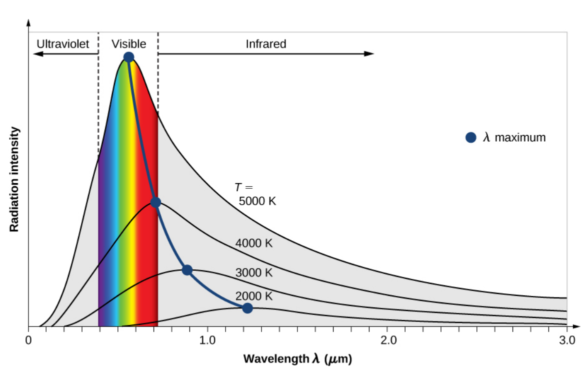
Particles are waves.



- Atom stability.
  - Pudding model.
  - Rutherford experiment.
- Atomic spectra.

Waves are particles.

- Black body radiation.



Observations:

- Goes to zero when $\lambda \to 0$.
- Has a maximum! Nothing in the Maxwell equations can give a scale!

$$\lambda_{\max} T = 2.898 \times 10^{-3} m \cdot K.$$

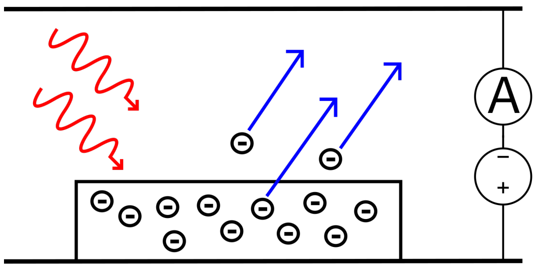
Planck's formula:

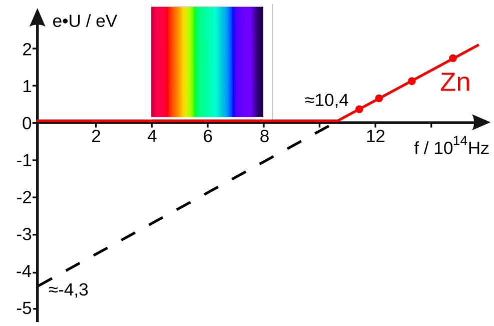
$$u(f,T) = \frac{8\pi h f^3}{c^3} \frac{1}{e^{hf/k_B T} - 1}.$$

where $h = 6.6 \times 10^{-34} J \cdot s$. Often used $\hbar = \frac{h}{2\pi}$.

Planck's formula suggests that light consists of particles with energy $\epsilon = hf = \hbar\omega$ each.

- Photo-electric effect.





If light is wave, then the energy of electrons should only depend on intensity. However, the experiment shows, that the energy of the knocked out electrons depends only on frequency, while the number of knocked out electrons (per unit time) depends on intensity.

$$E = \hbar\omega - \mathcal{A}.$$

With the same constant $\hbar$ as in the Planck's formula!

- Compton scattering (X ray of large energy, electrons are free). θ is the angle of the scattered light.

$$\lambda' - \lambda = \lambda_e(1 - \cos\theta), \qquad \lambda_e \approx 2.4 \times 10^{-12} m$$

Where does the length λ_e come from?

For e.-m. wave from the Maxwell equations we can find, that the momentum flux is energy flux divided by c. If we consider light with a wave length λ and frequency $f = c/\lambda$ as a stream of particles with energy $\epsilon = hf = hc/\lambda$, as Photo Effect suggests, then the momentum of each particle is $p = \hbar\omega/c = h/\lambda$. Then momentum and energy conservation laws give

$$\begin{aligned} \text{momentum, parallel component:} &\quad \tfrac{h}{\lambda} = \tfrac{h}{\lambda'}\cos(\theta) + p_e\cos(\alpha) \\ \text{momentum, perpendicular component:} &\quad 0 = \tfrac{h}{\lambda'}\sin(\theta) - p_e\sin(\alpha) \\ \text{energy:} &\quad \tfrac{ch}{\lambda} = \tfrac{ch}{\lambda'} + \tfrac{p_e^2}{2m} \end{aligned}$$

Expressing p_e^2 from the first two equations and using it in the third we get

$$\frac{1}{2m}\left(\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}\cos(\theta)\right) = \frac{ch}{\lambda} - \frac{ch}{\lambda'}.$$

In the case $\lambda \approx \lambda'$ this simplifies to

$$\lambda' - \lambda = \frac{h}{cm}(1 - \cos\theta), \qquad \text{so} \quad \lambda_e = \frac{h}{cm}.$$

Surprisingly many of the puzzling experimental results can be explained by considering particles as waves and waves as particles.

LECTURE 22
Beginnings of the Quantum Mechanics.

- Photo-electric effect.
- Compton scattering. θ is the angle of the scattered light.

$$\lambda' - \lambda = \frac{h}{cm}(1 - \cos\theta)$$

There is no h in classical (not quantum) physics

$$h = 6.62607015 \times 10^{-34} J \cdot s, \qquad \hbar = \frac{h}{2\pi}$$

- Energy and momentum of a photon

$$\epsilon = hf = \hbar\omega, \qquad p = \frac{\epsilon}{c} = \frac{h}{\lambda}$$

- Bohr atom.
  - According to Maxwell the frequency of light emitted by a hydrogen atom must equal to the frequency of the rotation of the electron.
  - The energy of the emitted "Einstein" photon $\hbar\omega$ must be the difference in the energies of the electron.
  - An electron on an orbit has an energy

$$E = \frac{mv^2}{2} - \frac{ke^2}{r}.$$

  - For a circular orbit we have

$$\frac{ke^2}{r^2} = \frac{mv^2}{r}$$

so that

$$\frac{mv^2}{2} = \frac{1}{2}\frac{ke^2}{r}, \qquad E = -\frac{1}{2}\frac{ke^2}{r}.$$

Also

$$L = mvr, \qquad \text{or} \qquad L^2 = m^2v^2r^2 = ke^2mr \qquad \text{and} \qquad r = \frac{L^2}{mke^2}$$

and

$$mvr = L \qquad \text{so} \qquad v = \frac{L}{mr} = \frac{ke^2}{L}$$

So, finally we have everything in terms of the angular momentum L

$$E=-\frac{1}{2}\frac{k^2e^4m}{L^2},\qquad r=\frac{L^2}{mke^2},\qquad v=\frac{ke^2}{L},\qquad \omega=\frac{v}{r}=\frac{mk^2e^4}{L^3}$$

– Assume that the change of the electron's energy is small.

$$dE=\frac{dE}{dL}dL=\frac{k^2e^4m}{L^3}dL=\omega dL$$

(in fact $\omega=\dot{\phi}=\frac{\partial H(L,\phi)}{\partial L}$ – Hamiltonian equation.)

– This change of energy dE must be equal to the energy of the emitted photon $\hbar\omega$. We then have

$$\hbar\omega=\omega dL,\qquad dL=\hbar.$$

– Then

$$L=\hbar n+L_0,\qquad n=1,2\ldots$$

Assuming $L_0=0$ we get

$$L=\hbar n,\qquad n=1,2\ldots$$

and

$$E_n=-\frac{1}{2}\frac{k^2e^4m}{\hbar^2}\frac{1}{n^2}=-\frac{13.6}{n^2}\mathrm{eV},\qquad r_n=\frac{\hbar^2}{mke^2}n^2=a_Bn^2,\quad a_B=0.0529\mathrm{nm}.$$

- de Brolie's idea. According to Bohr

$$L=pr=n\hbar,\qquad \text{or}\qquad 2\pi rp=nh.$$

If we now assume that the electron is a wave with the wavelength $\lambda=\frac{h}{p}$, then the Bohr quantization rule becomes

$$\frac{2\pi r}{\lambda}=n,$$

which is the condition for the constructive interference.
- Particles as waves.
- Wave packet.
- Uncertainty for waves.
- Double slit experiment. Wave of probability.
- Wave function as probability density amplitude.

LECTURE 23
Particles as waves.

de Brolie's idea was that a particle is a wave.

- Double slit experiment. Wave of probability.
- Wave function as probability density amplitude.

Its propagation then should be described by a wave equation.

What does it mean to describe? Time evolution!

23.1. The wave equation.

An oscillator.

$$\ddot{f} + \omega^2 f = 0$$

There are two linearly independent solutions

$$f_1(t) = \cos(\omega t) \qquad \text{and} \qquad f_2(t) = \sin(\omega t).$$

Any linear combination of these is also a solution. In particular

$$f(t) = \cos(\omega t) + i \sin(\omega t) = e^{i\omega t}$$

is a solution. This solution has the property that

$$|f|^2 = 1$$

at all times.

We can look at this oscillator as a zero dimensional wave. In $1D$ it will become

$$\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = 0$$

The simplest solutions are

$$f_{\pm}(x, t) = e^{i\omega t \pm i\omega x/v}$$

for any ω.

- Both solutions describe the waves propagating with the velocity v.
- The velocity does not depend on ω.
- The period and the wavelength of the both waves are

$$T = \frac{2\pi}{\omega}, \qquad \lambda = \frac{2\pi v}{\omega} = Tv.$$

- f_+ propagates to the left, f_- propagates to the right.

- Both solutions have the property that
$$|f_\pm|^2 = 1$$
at all times and everywhere in space.

The wave equation can be written as
$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right) f = 0$$
Looking at each factor separately we see that
$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f_- = 0$$
$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right) f_+ = 0$$
So the equation
$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f = 0$$
describes a wave propagating to the right only.

23.2. Schrödinger equation.

The propagation of the electromagnetic wave of frequency ω and wavelength λ is given by $e^{ikx-i\omega t} = e^{2\pi i x/\lambda - i\omega t}$. For the el.-m. wave the velocity is always c, so $\lambda\omega/2\pi = c$. For matter wave we do not have such restriction. However, for the both el.-m. and matter waves we have $p = 2\pi\hbar/\lambda$ and $E = \hbar\omega$, so we write
$$\Psi(x,t) = e^{ipx/\hbar - iEt/\hbar}$$
For a classical particle we must have $E = \frac{p^2}{2m}$, the wave Ψ then must satisfy the following equation
$$\left[i\hbar\frac{\partial}{\partial t} - \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\right]\Psi = 0$$
Or
$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi.$$
Let's look at the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. If we act on a wave function by this operator we get $\hat{p}\Psi = p\Psi$. So this is an operator of momentum. Using this notation we get
$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{\hat{p}^2}{2m}\Psi.$$
Comparing this to the Hamiltonian for the free moving particle $H = \frac{p^2}{2m}$, one can write
$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x).$$
The operator $\hat{H}$ is called the Hamiltonian operator. The above equation is the Srödinger equation.

23.3. Wave function.

- Interpretation. Probability Density Amplitude.
- Srödinger equation is linear in Ψ and homogeneous, so Ψ is defined up to a multiplicative factor. Normalization.

LECTURE 24
Wave function. Wave packet.

- Homework.

Srödinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \qquad \hat{H} = \frac{\hat{p}^2}{2m} + U(x).$$

Momentum operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}.$$

24.1. A particle as a wave packet.

Let's consider a free particle $U(x) = 0$.

- A wave $e^{ipx/\hbar}$ has a definite momentum p, but is everywhere $-\infty < x < \infty$.
- A particle is localized in space.
- Srödinger equation is linear in Ψ.

A particle must be represented by wave packet.

We know how a plane wave evolves with time

$$e^{ipx/\hbar} \to e^{ipx/\hbar - iE_p t/\hbar}.$$

How does the wave packet — the particle — evolves with time?

24.1.1. Evolution of a wave packet.

Let's assume that we know that at initial time $t = 0$ the wave function is given by $\Psi(x, 0)$, we want to know what will be the wave function at time t.

In order to do that we need to present $\Psi(x, 0)$ as a collection of a plane waves — the wave packet.

$$\Psi(x, 0) = \int a_p e^{ipx/\hbar}\frac{dp}{2\pi\hbar}, \qquad a_p = \int \Psi(x, 0)e^{-ipx/\hbar}dx$$

After a time t a wave $e^{ipx/\hbar}$ becomes $e^{ipx/\hbar - iE_p t/\hbar}$. So

$$\Psi(x, t) = \int a_p e^{ipx/\hbar - iE_p t/\hbar}\frac{dp}{2\pi\hbar}.$$

Let's see how it works for a classical free particle $E_p = \frac{p^2}{2m}$.

24.1.1.1. *Wave packet spreading.*

Let's assume, that we have started with the initial wave-function $\Psi(x,0) = Ce^{-x^2/4\alpha^2}$, and $|\Psi(x,0)| = C^2 e^{-x^2/2\alpha^2}$, so that $\Delta x = \alpha$. First we must compute C from the normalization condition

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2\, dx = |C|^2 \int_{-\infty}^{\infty} e^{-x^2/2\alpha^2} dx = |C|^2 \sqrt{2\pi}\alpha$$

then

$$a_p = \int \Psi(x,0) e^{-ipx/\hbar} dx = C \int e^{-x^2/4\alpha^2 - ipx/\hbar} dx = C \int e^{-\frac{1}{4\alpha^2}\left(x^2 + 2ipx\frac{2\alpha^2}{\hbar} - p^2\frac{4\alpha^4}{\hbar^2}\right) - p^2\frac{\alpha^2}{\hbar^2}} dx =$$

$$Ce^{-p^2\frac{\alpha^2}{\hbar^2}} \int e^{-\frac{1}{4\alpha^2}\left(x + 2ip\frac{\alpha^2}{\hbar}\right)^2} dx = 2C\alpha\sqrt{\pi} e^{-p^2\frac{\alpha^2}{\hbar^2}}$$

So that according to the prescription

$$\Psi(x,t) = \int a_p e^{ipx/\hbar - iE_p t/\hbar} \frac{dp}{2\pi\hbar} = \int C\alpha\sqrt{2\pi} e^{-p^2\frac{\alpha^2}{\hbar^2} + ipx/\hbar - p^2\frac{it}{2m\hbar}} \frac{dp}{2\pi\hbar} =$$

$$\int C\alpha\sqrt{2\pi} e^{-p^2\left(\frac{\alpha^2}{\hbar^2} + it/2m\hbar\right) + ipx/\hbar} \frac{dp}{2\pi\hbar} = C\alpha\sqrt{2\pi} \int e^{-\frac{p^2}{4\left(\frac{4\alpha^2}{\hbar^2} + \frac{2it}{m\hbar}\right)^{-1}} + ipx/\hbar} \frac{dp}{2\pi\hbar} =$$

$$2C\alpha\frac{1}{\hbar}\left(\frac{4\alpha^2}{\hbar^2} + \frac{2it}{m\hbar}\right)^{-1/2} e^{-\frac{x^2}{4\hbar^2\left(\frac{\alpha^2}{\hbar^2} + \frac{it}{2m\hbar}\right)}} = C\frac{1}{\sqrt{1 + \frac{it\hbar}{2m\alpha^2}}} e^{-\frac{x^2}{4\left(\alpha^2 + \frac{it\hbar}{2m}\right)}}$$

So we see that

$$|\Psi(x,t)|^2 = \frac{C^2}{\sqrt{1 + \left(\frac{t\hbar}{2m\alpha^2}\right)^2}} e^{-\frac{x^2}{2\left(\alpha^2 + \left(\frac{t\hbar}{2m\alpha}\right)^2\right)}}$$

So we see, that the particle is still at the center on average, but

$$\Delta x(t) = \sqrt{[\Delta x(0)]^2 + \left[\frac{t\hbar}{2m\Delta x(0)}\right]^2}$$

We now can compute how much time it would take for a $1g$ marble initially localized with a precision $0.1mm$ to disperse so that $\Delta x(t) = 10\Delta x(0)$. The answer is $t \approx 2 \times 10^{24} s$ – by far longer than the life-time of our Universe.

24.1.1.2. *Group velocity.*

Let's construct a wave packet with a momentum p_0 on average at $t = 0$. We want this packet to be very sharply peaked at p_0.

$$\Psi(x,0) = \int e^{-\frac{(p-p_0)^2}{4\alpha^2}} e^{ipx/\hbar} dp$$

where we assume that the $\alpha \sim \Delta p$ is small.

At time t the wave packet will be

$$\Psi(x,t) = \int e^{-\frac{(p-p_0)^2}{4\alpha^2}} e^{ipx/\hbar - iE_p t/\hbar} dp$$

As α is small, only $p \sim p_0$ contribute to the integral, so we can write

$$\Psi(x,t) \approx e^{ip_0 x/\hbar - iE_{p_0} t/\hbar} \int e^{-(p-p_0)^2\left(\frac{1}{4\alpha^2} + i\frac{1}{\hbar}\frac{\partial^2 E_p}{\partial p_0^2} t\right) + \frac{i}{\hbar}(p-p_0)\left(x - \frac{\partial E}{\partial p_0} t\right)} dp$$

So we see, that

$$|\Psi(x,t)|^2 = f\left(x - \frac{\partial E}{\partial p_0}t, t\right)$$

So we see, that the wave packet is moving with the "group" velocity

$$v = \frac{\partial E}{\partial p_0},$$

as it should according to the Hamiltonian equations.

LECTURE 25
Wave function. Time independent Schrödinger equation.

- Particles as waves.
- Heisenberg uncertainty principle: $\Delta x \Delta p \geq \hbar/2$.
- Waves as particles: $e^{\frac{i}{\hbar}S}$.
- To classical.

25.1. Wave function.

- Interpretation. Probability Density Amplitude.
- Measurables as operator averages. Given a wave function $\Psi(x,t)$ – it must be normalized $\int_{-\infty}^{\infty} |\Psi|^2 dx = 1$.
  - Coordinate:
  $$\bar{x} = \int_{-\infty}^{\infty} |\Psi|^2 x dx = \int_{-\infty}^{\infty} \Psi^* x \Psi dx \int_{-\infty}^{\infty} \Psi^* \hat{x} \Psi dx.$$
  - Momentum. Momentum operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$:
  $$\bar{p} = \int_{-\infty}^{\infty} \Psi^* \hat{p} \Psi dx.$$
  (I will always be real!)
  - Energy
  $$\bar{E} = \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi dx.$$
  - Any other measurable $\mathcal{O}$:
  $$\bar{\mathcal{O}} = \int_{-\infty}^{\infty} \Psi^* \hat{\mathcal{O}} \Psi dx$$
  - Noise. The good (but not the only) measure of Quantum mechanical noise in the measurement $\hat{\mathcal{O}}$ is
  $$(\Delta \mathcal{O})^2 = \overline{\left(\hat{\mathcal{O}} - \bar{\mathcal{O}}\right)^2} = \overline{\mathcal{O}^2} - \bar{\mathcal{O}}^2$$

25.2. Time independent Schrödinger equation.

If the Hamiltonian does not depend on time, then we can look for the solution of the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

in the form

$$\Psi(x,t) = e^{-iEt/\hbar}\psi(x),$$

Then we have

$$\hat{H}\psi = E\psi.$$

This is a second order differential equation. For any E it has two linearly independent solutions. However, if we are looking for the solutions that satisfy the normalization condition $\int\psi^*\psi dx = 1$, then we find that such solutions exist only for real E and in many cases only for a discrete set of E.

- Energy as an eigen-value of the Hamiltonian.
- Quantum numbers = enumeration of the eigen functions.
- Eigen functions = Basis in the space of functions.
- Bra-ket notations.
- Normalization. $\langle\psi_{n'}|\psi_n\rangle = \delta_{n,n'}$.

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$$

If at initial time we have $\Psi(x,0)$, then we can write

$$\Psi(x,0) = \sum_n a_n\psi_n(x), \qquad \text{or} \qquad |\Psi(t=0)\rangle = \sum_n a_n|\psi_n\rangle, \qquad \text{or} \qquad a_n = \langle\psi_n|\Psi(t=0)\rangle$$

The time evolution of an eigen function is simple

$$|\psi_n\rangle \to |\psi_n\rangle e^{-iE_nt/\hbar}$$

so

$$|\Psi(t)\rangle = \sum_n a_n e^{-iE_nt/\hbar}|\psi_n\rangle.$$

We see, that if the Hamiltonian does not depend on time the set of eigenvalues and eigenfunctions of the Hamiltonian operator solves the problem — we can compute the wave function at all times.

In order to compute a quantum mechanical average for some operator $\hat{\mathcal{O}}$ at arbitrary time we can use

$$\langle\Psi(x,t)|\hat{\mathcal{O}}|\Psi(x,t)\rangle = \sum_n e^{iE_nt/\hbar}a_n^*\langle\psi_n|\hat{\mathcal{O}}\sum_m a_m e^{-iE_mt/\hbar}|\psi_m\rangle = \sum_n\sum_m e^{i(E_n-E_m)t}a_n^*\langle\psi_n|\hat{\mathcal{O}}|\psi_m\rangle a_m$$

Similar to the matrix manipulations. Numbers $\langle\psi_n|\hat{\mathcal{O}}|\psi_m\rangle$ are called matrix elements of the operator $\hat{\mathcal{O}}$.

- Linear combinations. Basis. Quantum numbers.
- Spectrum. Discrete and continuous spectrum.
- Ground state, excited states. Transitions. Perturbations.

LECTURE 26
Discrete spectrum. Classically prohibited region.

26.1. Particle in the infinite square well potential.

- The potential:

$$U(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{for } x < 0 \text{ and } x > L \end{cases}$$

- Classical picture: any energy, particle is localized within the well.
- The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi.$$

has the solution $k^2 = \frac{2mE}{\hbar^2}$

$$\psi(x) = C \begin{cases} 0, & & \text{for } x < 0 \\ \sin(kx), \quad \text{or} & \cos(kx), & \text{for } 0 < x < L \\ 0, & & \text{for } L < x \end{cases}$$

- Boundary conditions: the wave function must be continuous:

$$\psi(x = 0) = 0, \qquad \psi(x = L) = 0.$$

so the solutions in $0 < x < L$ are

$$\psi(x) = C \sin(k_n x), \qquad k_n L = \pi n, \quad n = 1, 2 \ldots$$

- Energy spectrum is discrete:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2.$$

- Particle is localized within the well, but it can only have a discrete (infinite) set of energies. The set is very dense in the classical limit.

26.2. Particle in the finite square well potential.

- Consider a potential

$$U(x) = \begin{cases} 0 & \text{for } |x| < L \\ U_0 & \text{for } |x| > L \end{cases}$$

- I am interested only in solutions for $E < U_0$.
- Classical: the particle can have any energy $0 < E < U_0$; the particle is completely localized in $-L < x < L$ region.
- The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi.$$

- The normalizable solutions are

$$\psi(x) = \begin{cases} A_- e^{\kappa x}, & & \text{for } x < -L \\ \sin(kx), \quad \text{or} & \cos(kx), & \text{for } -L < x < L \\ A_+ e^{-\kappa x}, & & \text{for } L < x \end{cases}, \qquad k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad \kappa = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$

- Symmetry. As the Hamiltonian is symmetric with respect to $x \to -x$ the solutions are either symmetric $\psi(-x) = \psi(x)$ or antisymmetric $\psi(-x) = -\psi(x)$. These symmetric and antisymmetric solutions are

$$\psi_s(x) = \begin{cases} Ae^{\kappa x} & \text{for } x < -L \\ \cos(kx) & \text{for } -L < x < L \\ Ae^{-\kappa x} & \text{for } x > L \end{cases}, \qquad \psi_a(x) = \begin{cases} -Ae^{\kappa x} & \text{for } x < -L \\ \sin(kx) & \text{for } -L < x < L \\ Ae^{-\kappa x} & \text{for } x > L \end{cases},$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad \kappa = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} = \sqrt{k_u^2 - k^2}, \qquad k_u = \sqrt{\frac{2mU_0}{\hbar^2}}.$$

- Matching the solutions at $x = L$.
  - The wave function ψ must be continuous.

$$\psi(x = L - \epsilon) = \psi(x = L + \epsilon), \quad \text{at the limit } \epsilon \to 0.$$

  - In addition, let's integrate the above equation over x from $L - \epsilon$ to $L + \epsilon$. We have

$$-\frac{\hbar^2}{2m}\left(\psi'(L+\epsilon) - \psi'(L-\epsilon)\right) + \int_{L-\epsilon}^{L+\epsilon} U(x)\psi(x)dx = E\int_{L-\epsilon}^{L+\epsilon}\psi(x)dx.$$

    Taking a limit $\epsilon \to 0$ we have

$$\psi'(L+0) = \psi'(L-0)$$

    So ψ' must also be continuous at the points $x = \pm L$ (and thus everywhere).
- So we need to match the value of ψ and ψ' from both sides for $x = L$, so we have (left column for the symmetric, right for antisymmetric)

$$\begin{aligned} Ae^{-\kappa L} &= \cos(kL) & \qquad Ae^{-\kappa L} &= \sin(kL) \\ -\kappa Ae^{-\kappa L} &= -k\sin(kL) & \qquad -\kappa Ae^{-\kappa L} &= k\cos(kL) \end{aligned}$$

  Dividing the equation we get

$$k\tan(kL) = \kappa \qquad k\cot(kL) = -\kappa,$$

which can be written as

$$\cos(kL) = \frac{k}{k_u}, \qquad \sin(kL) = -\frac{k}{k_u},$$

where

$$k_u = \sqrt{\frac{2mU_0}{\hbar^2}}.$$

These equations have a discrete set of solutions. No matter how small U_0 is there is always at least one symmetric localized solution!

- Unlike classical case the particle can be found outside the well.
- The transition in behavior at $E \approx U_0$ is not as sharp in Quantum mechanics.

26.3. Particle in the δ-function attractive potential.

- I want to consider a potential

$$U(x) = -U_0\delta(x).$$

- I am interested only in localized state, so $E < 0$.
- The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' - U_0\delta(x)\psi = -|E|\psi$$

- Let's integrate this equation over x from $-\epsilon$ to ϵ, we get

$$-\frac{\hbar^2}{2m}\left(\psi'(\epsilon) - \psi'(-\epsilon)\right) - U_0\psi(0) = -|E|\int_\epsilon^\epsilon \psi(x)dx.$$

Taking the limit $\epsilon \to 0$ we see that

$$\psi'(+0) - \psi'(-0) = -\frac{2mU_0}{\hbar^2}\psi(0)$$

So the function ψ' must have a jump (discontinuity at $x = 0$)

- The solutions are

$$\psi = \begin{cases} Ae^{\kappa x} & \text{for } x < 0 \\ Ae^{-\kappa x} & \text{for } x > 0 \end{cases}, \tag{26.1}$$

where

$$\kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$$

- Then

$$\psi'(+0) = -\kappa A, \qquad \psi'(-0) = \kappa A, \qquad \psi(0) = A$$

- Using the condition for matching the derivatives we get

$$2\kappa = \frac{2mU_0}{\hbar^2}, \qquad |E| = \frac{U_0^2}{2m\hbar^2}$$

- Although the potential is very short range the particle can be found in the finite region $-1/\kappa < x < 1/\kappa$, or

$$-\frac{\hbar^2}{mU_0} < x < \frac{\hbar^2}{mU_0}.$$

LECTURE 27
Band structure. Tunneling. Density of states.

27.1. Particle in two far away potential wells.

- The Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + U_L(x) + U_R(x), \qquad U_{L,R} = U(x \pm l/2), \qquad l \gg \frac{\hbar^2}{mU_0}$$

- The condition $l \gg \frac{\hbar^2}{mU_0}$ means that the distance between the wells are much larger than the spread of the wave function.
- If the two wells are far away from each other, then the overlap of the wave functions is small.
- Let's define two functions $|\psi_L\rangle$ and $|\psi_R\rangle$

$$\left(\frac{\hat{p}^2}{2m} + U_L\right)|\psi_L\rangle = E_0|\psi_L\rangle, \qquad \langle\psi_L|\psi_L\rangle = 1$$
$$\left(\frac{\hat{p}^2}{2m} + U_R\right)|\psi_R\rangle = E_0|\psi_R\rangle, \qquad \langle\psi_R|\psi_R\rangle = 1$$

  We also notice, that

$$|\langle\psi_R|\psi_L\rangle| \ll 1.$$

- Let's look for the solution in the form

$$|\psi\rangle = a_L|\psi_L\rangle + a_R|\psi_R\rangle.$$

- The Schrödinger equation now reads.

$$a_L E|\psi_L\rangle + a_R E|\psi_R\rangle = a_L\hat{H}|\psi_L\rangle + a_R\hat{H}|\psi_R\rangle.$$

- Multiplying this equation by $\langle\psi_L|$ and $\langle\psi_R|$ we get

$$Ea_L = (E_0 + \langle\psi_L|U_R|\psi_L\rangle)\, a_L + \langle\psi_L|U_L|\psi_R\rangle a_R$$
$$Ea_R = (E_0 + \langle\psi_R|U_L|\psi_R\rangle)\, a_R + \langle\psi_R|U_R|\psi_L\rangle a_L.$$

- We expect $E \approx E_0$, so we ignored the terms of the kind $(E - E_0)\langle\psi_L|\psi_R\rangle$, as they are of the second order.

- Introducing $\tilde{E}_0 = E_0 + \langle\psi_L|U_R|\psi_L\rangle$, $-\Delta = \langle\psi_R|U_R|\psi_L\rangle$, and a vector $\begin{pmatrix} a_L \\ a_R \end{pmatrix}$ we have
$$E\begin{pmatrix} a_L \\ a_R \end{pmatrix} = \begin{pmatrix} \tilde{E}_0 & -\Delta \\ -\Delta & \tilde{E}_0 \end{pmatrix}\begin{pmatrix} a_L \\ a_R \end{pmatrix}.$$
- So E is just an eigenvalue of the simple 2×2 matrix. The result is
$$E_\pm = \tilde{E}_0 \pm \Delta.$$
- A single degenerate energy level is split in two levels: symmetric and antisymmetric combinations.
- Interaction splits degeneracy.
- In the symmetric potential the ground state is always symmetric.

27.2. Strong periodic potential. (Tight binding model.)

- The potential is
$$U(x) = \sum_{n=-\infty}^{\infty} U(x-nl).$$
- We again assume that l is much grater than the spread of a wave function for a single well.
$$\left(\frac{\hat{p}^2}{2m} + U(x)\right)|\psi(x)\rangle = E_0|\psi\rangle$$
- We look at the solution in the form
$$|\psi\rangle = \sum_{n=-\infty}^{\infty} a_n|\psi(x-nl)\rangle$$
- We then have
$$\begin{pmatrix} . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & 0 & 0 & . \\ . & 0 & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & 0 & . \\ . & 0 & 0 & 0 & -\Delta & \tilde{E}_0 & -\Delta & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}\begin{pmatrix} . \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \\ a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ . \end{pmatrix} = E\begin{pmatrix} . \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \\ a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ . \end{pmatrix}.$$
or
$$-\Delta a_{n-1} + \tilde{E}_0 a_n - \Delta a_{n+1} = E a_n$$
- We look for the solution in the form $a_n = ae^{ipln/\hbar}$, so
$$-\Delta e^{ipl(n-1)/\hbar} + \tilde{E}_0 e^{ipln/\hbar} - \Delta e^{ipl(n-1)/\hbar} = Ee^{ipln/\hbar},$$
which gives
$$E(k) = \tilde{E}_0 - 2\Delta\cos(pl/\hbar), \qquad -\pi\hbar/l < p < \pi\hbar/l.$$
So a single energy level is split into a band.

- p is quasi-momentum. In particular, for small p

$$E(k) \approx \tilde{E}_0 - 2\Delta + \frac{p^2}{2(\hbar^2/2l^2\Delta)}.$$

So it behaves as a normal particle with the "effective" mass $m^* = \hbar^2/2l^2\Delta$.

27.3. Tunneling.

- Transition trough a square potential bump.

$$U(x) = \begin{cases} 0 & \text{for } x < 0 \\ U_0 & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{cases}.$$

- We are interested at energies $0 < E < U_0$.
- In classical mechanics the particle coming from the left is simply reflected back.
- In quantum mechanics we look for the solution in the form

$$\psi(x) = \begin{cases} e^{ipx/\hbar} + Re^{-ipx/\hbar} & \text{for } x < 0 \\ A_+ e^{\kappa x/\hbar} + A_- e^{-\kappa x/\hbar} & \text{for } 0 < x < L \\ Te^{ipx/\hbar} & \text{for } x > L \end{cases},$$

where R and T are reflection and transition amplitudes respectively and

$$\frac{p^2}{2m} = E, \qquad \frac{\kappa^2}{2m} = U_0 - E$$

- At the points $x = 0$ and $x = L$ we must match the value of the wave function and its derivatives from the left and the right. So we have four linear conditions/equations and four unknowns T, R, A_+, and A_-!
- The answer is

$$|T|^2 = \frac{4p^2\kappa^2}{(p^2+\kappa^2)^2 \sinh^2(\kappa L/\hbar) + 4p^2\kappa^2}, \qquad |R|^2 = 1 - |T|^2.$$

- Using the definitions of p and κ it can be written

$$|T|^2 = \frac{4E(U_0 - E)}{4E(U_0 - E) + U_0^2 \sinh^2\left(\sqrt{\frac{U_0 - E}{\epsilon}}\right)}, \qquad \epsilon = \frac{\hbar^2}{2mL^2}$$

- Limits of large $L \gg \hbar/\kappa$ and $\kappa \gg p$ (or $U_0 \gg E$).
- This is under barrier transition — tunneling. (There is also over barrier reflection.)

27.4. Density of states.

- Density of states: Discrete spectrum to continuous.
- Tunneling current as a measure of the density of states (STM).

LECTURE 28
Commutators. Quantum harmonic oscillator.

- Homework.

Quantum harmonic oscillator.

- Hermitian operators. Observables.
- x as an operator.
- $[\hat{p}, \hat{x}] = -i\hbar$.
- Hamiltonian for a harmonic oscillator $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} = \frac{\hat{p}^2}{2m} + m\omega^2\frac{\hat{x}^2}{2}$.
- Operators $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ and $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$.
- $[\hat{a}, \hat{a}^\dagger] = 1$, and $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right)$.
- The Schrödinger equation $\hat{H}|\psi\rangle = E|\psi\rangle$ becomes

$$\hbar\omega\hat{a}^\dagger\hat{a}|\psi\rangle = (E - \hbar\omega/2)\,|\psi\rangle$$

- A function $|0\rangle$ such that $\hat{a}|0\rangle = 0$ and $\langle 0|0\rangle = 1$ exists.

$$|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}, \qquad E_0 = \frac{1}{2}\hbar\omega$$

- Consider a function/state $|1\rangle = \hat{a}^\dagger|0\rangle$. Let's act on it by an operator $\hbar\omega\hat{a}^\dagger\hat{a}$

$$\hbar\omega\hat{a}^\dagger\hat{a}|1\rangle = \hbar\omega\hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\rangle = \hbar\omega\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a} + 1\right)|0\rangle = \hbar\omega\hat{a}^\dagger\hat{a}^\dagger\hat{a}|0\rangle + \hbar\omega\hat{a}^\dagger|0\rangle = \hbar\omega\hat{a}^\dagger|0\rangle = \hbar\omega|1\rangle.$$

So we see, that the function $|1\rangle$ is an eigen function of our Hamiltonian and

$$E_1 = \hbar\omega + \frac{1}{2}\hbar\omega.$$

- Normalization

$$\langle 1|1\rangle = \langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \langle 0|1 + \hat{a}^\dagger\hat{a}|0\rangle = \langle 0|0\rangle = 1$$

- For a state $|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle$ we have

$$\hbar\omega\hat{a}^\dagger\hat{a}|n\rangle = n\hbar\omega|n\rangle, \qquad \langle n|n\rangle = 1,$$

so

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

- Also $\langle n|m\rangle = 0$, for $n \neq m$, and

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

- $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$, and $\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$, so
$$\langle n|\hat{x}|n\rangle = 0, \qquad \langle n|\hat{p}|n\rangle = 0$$
and
$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger|n\rangle = \frac{\hbar}{2m\omega}\langle n|2\hat{a}^\dagger\hat{a}+1|n\rangle = (n+1/2)\frac{\hbar}{m\omega}, \qquad \langle n|\hat{p}^2|n\rangle = (n+1/2)m\omega\hbar$$
- Coherent states. For any α we construct a state:
$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle, \qquad \langle\alpha|\alpha\rangle = 1, \qquad \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2.$$
This set of such states is overcomplete $\langle\alpha|\alpha'\rangle \neq 0$, for $\alpha \neq \alpha'$. The time evolution of these states describes the motion of a particle.

LECTURE 29
Quantum mechanics in $3D$. Many-particle states. Identical particles.

Homework.

- Quantum mechanics in $3D$.
- Many-particle states.
- Identical particles.
- Bosons. Bose-Einstein condensate, superfluidity.
- Electrons as fermions.
- Metals and insulators. Response to the electric field.
- Semiconductors. Electrons and holes.
- Fermi-surface. Superconductivity.
- Closing remarks. More is different. Simple rules — complex behavior.
- `https://youtu.be/FzcTgrxMzZk`